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Fixed points and coupled fixed points in partially ordered ν-generalized metric spaces

Mortaza Abtahi [a], Zoran Kadelburg [b] and Stojan Radenović [c]

[a] School of Mathematics and Computer Sciences, Damghan University, P.O.B. 36715-364, Damghan, Iran (abtahi@du.ac.ir, mortaza.abtahi@gmail.com)
[b] University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Serbia (kadelbur@matf.bg.ac.rs)
[c] University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Beograd, Serbia (radens@beotel.rs)

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Abstract

New fixed point and coupled fixed point theorems in partially ordered ν-generalized metric spaces are presented. Since the product of two ν-generalized metric spaces is not in general a ν-generalized metric space, a different approach is needed than in the case of standard metric spaces.

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1. Introduction and Preliminaries

Throughout the paper, the sets of integers, nonnegative integers, and positive integers are denoted, respectively, by $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{N}$; the sets of real numbers and nonnegative real numbers are denoted, respectively, by $\mathbb{R}$ and $\mathbb{R}^+$.

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1.1. **ν-generalized metric spaces.** There are lots of works done on fixed point theory by weakening the requirements of the Banach contraction principle. One direction of these generalizations was introduced by Branciari in [6], where the triangle inequality was replaced by a so-called *polygonal inequality*. In what follows, we briefly recall concepts of ν-generalized metric spaces. See also [3, 8, 11, 19, 20].

Definition 1.1 (Branciari [6])**.** Let E be a nonempty set and $\nu \in \mathbb{N}$. A mapping $d_\nu : E \times E \to \mathbb{R}^+$ is called a *ν-generalized metric* and the pair (E, d_ν) is called a *ν-generalized metric space* if the following hold:

(1) $d_\nu(x, y) = 0$ if and only if $x = y$;
(2) $d_\nu(x, y) = d_\nu(y, x)$, for all $x, y \in E$;
(3) $d_\nu(x, y) \leq d_\nu(x, z_1) + d_\nu(z_1, z_2) + \cdots + d_\nu(z_\nu, y)$, for each set $\{x, z_1, \ldots, z_\nu, y\}$ of $\nu + 2$ distinct elements of E.

Clearly, (E, d_ν) is a metric space if $\nu = 1$, i.e., it is a 1-generalized metric space. It is shown in [19], that the topology of a ν-generalized metric space may be non-compatible.

Definition 1.2 ([3])**.** Let (E, d_ν) be a ν-generalized metric space. Given $k \in \mathbb{N}$, a sequence $\{x_n\}$ in E is said to be *k-Cauchy* if

$$\lim_{n\to\infty} \sup\{d_\nu(x_n, x_{n+1+mk}) : m \in \mathbb{Z}^+\} = 0.$$

The sequence $\{x_n\}$ is called *Cauchy* if it is 1-Cauchy.

Cauchy sequences in ν-generalized metric spaces were investigated in [3, 6, 20].

Proposition 1.3 ([3, 20])**.** *Let $\{x_n\}$ be a ν-Cauchy sequence with distinct terms in (E, d_ν). If ν is odd, or if ν is even and $d_\nu(x_n, x_{n+2}) \to 0$ as $n \to \infty$, then $\{x_n\}$ is Cauchy.*

As shown in [16] and [18], a sequence in a 2-generalized metric space may converge to more than one point and a convergent sequence may not be a Cauchy sequence. It is said [3, 20] that a sequence $\{x_n\}$ in E *converges to x in the strong sense* if $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x. [18, Example 1.1] shows that there exist convergent sequences that do not converge in the strong sense. The completeness of ν-generalized metric spaces is investigated in [3].

Proposition 1.4 ([20])**.** *Let $\{x_n\}$ and $\{y_n\}$ be sequences in (E, d_ν) that converge to x and y in the strong sense, respectively. Then $d_\nu(x, y) = \lim_{n\to\infty} d_\nu(x_n, y_n)$.*

Branciari proved in [6] a generalization of the Banach contraction principle. Since, as was already said, a ν-generalized metric space does not necessarily have a compatible topology, his proof needed some corrections, see [9, 16, 18, 19, 21]. Proofs of Kannan's [10] and Ćirić's [7] fixed point theorems in ν-generalized metric spaces appear in [20]. The analogue of Proinov's result from [15], as an ultimate generalization of the Banach contraction principle in the setting of ν-generalized metric spaces, was proved in [2].

1.2. **Partially ordered spaces and coupled fixed points.** Nieto and Rodríguez-López initiated in [14] the use of another enrichment of metric space structure by using additional partial order. A lot of researchers obtained several results in such structures. Among them, Bhaskar and Lakshmikantham started in [5] investigation of so-called coupled fixed points. They proved the existence of coupled fixed points for contractive mappings in partially ordered complete metric spaces. These and similar results were later obtained by different methods; see, e.g. [4, 12, 13, 17].

Assume that $(E, \preceq)$ is a partially ordered set and that $F : E \times E \to E$ is a mapping. The notions of a *coupled fixed point* of F and the (strict) *mixed monotone property* has become standard, so we omit them here. Given a pair (x, y) of elements in E, the *Picard iterates* $\{F^n(x, y)\}$ and $\{F^n(y, x)\}$ are defined, inductively, as follows. Let $F^0(x, y) = x$, $F^0(y, x) = y$, and then, for $n \in \mathbb{Z}^+$,

$$\begin{aligned} F^{n+1}(x,y) &= F\big(F^n(x,y), F^n(y,x)\big), \\ F^{n+1}(y,x) &= F\big(F^n(y,x), F^n(x,y)\big). \end{aligned} \tag{1.1}$$

In 2012, Berinde and Păcurar [4] presented more general coupled fixed point theorems in partially ordered metric spaces $(E, \preceq, d)$.

Theorem 1.5 ([4]). *Let $(E, \preceq, d)$ be a complete partially ordered metric space, and $F : E \times E \to E$ be a generalized symmetric Meir-Keeler type mapping, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that, for $x \succeq u$ and $y \preceq v$,*

$$\begin{aligned} \epsilon \le d(x,u) + d(y,v) < \epsilon + \delta \Longrightarrow \\ d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) < \epsilon. \end{aligned}$$

Suppose that

(i) *the mapping F is continuous and has the mixed strict monotone property,*

(ii) *there exist $x_0, y_0 \in E$ such that*

$$x_0 \preceq F(x_0, y_0), \quad y_0 \succeq F(y_0, x_0), \quad \text{or} \quad x_0 \succeq F(x_0, y_0), \quad y_0 \preceq F(y_0, x_0). \tag{1.2}$$

Then F has a coupled fixed point.

1.3. **Fixed points of monotone contractions.** Fixed point theorems of Ćirić-Matkowski and Proinov types for monotone contractions in partially ordered ν-generalized metric spaces can be deduced from a sequence of lemmas and propositions, similarly as it has been done in the setting of (ν-generalized) metric spaces in [1] and [2]. Hence, we just state the respective formulations, omitting the proofs.

Theorem 1.6. *Let $(E, \preceq, d_\nu)$ be a complete partially ordered ν-generalized metric space and $T : E \to E$ be a monotone contraction of Ćirić-Matkowski type, i.e.,*

(1) *the mapping T is nondecreasing,*

(2) *$d_\nu(Tx, Ty) < d_\nu(x, y)$, for $x \prec y$ (that is $x \preceq y$ and $x \neq y$),*

(3) *for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\big(x \preceq y, \epsilon < d_\nu(x,y) < \epsilon + \delta\big) \Longrightarrow d_\nu(Tx,Ty) \le \epsilon.$$

Then T *has a fixed point provided there exists* $x_0 \in E$ *such that* $x_0 \preceq Tx_0$*. Moreover, for any* $x \in E$ *with* $x \preceq Tx$*, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *in the strong sense.*

We also have a Proinov type fixed point theorem.

Theorem 1.7. *Suppose that* $(E, \preceq, d_\nu)$ *is a complete partially ordered* ν*-generalized metric space, and that* $T : E \to E$ *is sequentially continuous and asymptotically regular, i.e.,*

$$\lim_{n\to\infty} \big(d_\nu(T^n x, T^{n+1}x) + d_\nu(T^n x, T^{n+2}x)\big) = 0, \quad x \in E.$$

For $\gamma > 0$*, define* $\mathbf{m}(x,y) = d_\nu(x,y) + \gamma\big(d_\nu(x,Tx) + d_\nu(y,Ty)\big)$*. Suppose that*

$$d_\nu(Tx,Ty) < \mathbf{m}(x,y), \quad x,y \in E, x \prec y,$$

and that, for any $\epsilon > 0$*, there exist* $\delta > 0$ *and* $N \in \mathbb{Z}^+$ *such that, for all* $x, y \in E$*,*

$$(1.3) \qquad x \preceq y,\ \mathbf{m}(T^N x, T^N y) < \delta + \epsilon \implies d_\nu(T^{N+1}x, T^{N+1}y) \le \epsilon.$$

Then T *has a fixed point provided there exists* $x_0 \in E$ *such that* $x_0 \preceq Tx_0$*. Moreover, for any* $x \in E$ *with* $x \preceq Tx$*, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *in the strong sense.*

Remark 1.8. Similarly as in various other known fixed point results in ordered spaces, the presented conditions are not sufficient to conclude that the fixed point is unique. An additional assumption is needed, and this can be either that arbitrary two elements of the fixed point set are comparable, or that there exists a third element, comparable with both of them. We do not go into details here, leaving them for the case of coupled fixed points in the next section.

1.4. **Outline.** The next (main) section is devoted to coupled fixed points, and is divided into three parts. In Subsection 2.1, we let E be a partially ordered metric space, and investigate the existence of coupled fixed points for different types of symmetric contractions on E. Our technique in this section involves considering induced metric and order on the set $\mathcal{E} = E \times E$ and reducing a symmetric contraction $F : E \times E \to E$ to a monotone contraction $T : \mathcal{E} \to \mathcal{E}$ and then applying results obtained in Section 1.3 to T. This technique appears in several papers. However, we will show that this method is not applicable in the case of partially ordered ν-generalized metric spaces (see further Example 2.2). Hence, in Subsection 2.2, we shall take a different approach to obtain coupled fixed point results in such spaces. Finally, in Subsection 2.3, we present a brief discussion of the uniqueness of coupled fixed points. We conclude by an illustrative example in the last subsection.

2. Coupled fixed points of symmetric contractions

In this section, we present coupled fixed point theorems for symmetric contractions on (ν-generalized) metric spaces. We start with the following definition of symmetric contractions of Ćirić-Matkowski type.

Definition 2.1. Let $(E, \preceq, d_\nu)$ be a partially ordered ν-generalized metric space. A mapping $F : E \times E \to E$ is called a *symmetric contraction of Ćirić-Matkowski type* if

(1) for $x \preceq u$ and $y \succeq v$, with $(x, y) \neq (u, v)$,

$$d_\nu(F(x,y), F(u,v)) + d_\nu(F(y,x), F(v,u)) < d_\nu(x,u) + d_\nu(y,v), \tag{2.1}$$

(2) for every $\epsilon > 0$, there exists $\delta > 0$ such that, for $x \preceq u$ and $y \succeq v$,

$$\begin{aligned} &\epsilon < d_\nu(x,u) + d_\nu(y,v) < \epsilon + \delta \Longrightarrow \\ &\qquad\qquad d_\nu(F(x,y), F(u,v)) + d_\nu(F(y,x), F(v,u)) \leq \epsilon. \end{aligned} \tag{2.2}$$

To avoid repetitive writings and simplify calculations, the following convention seems to be convenient.

Convention. Let $(E, \preceq, d_\nu)$ be a partially ordered (ν-generalized) metric space. Set $\mathcal{E} = E \times E$ and, for all elements $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$ of $\mathcal{E}$, define

$$\mathbf{x} \sqsubseteq \mathbf{u} \quad \text{if and only if} \quad x \preceq u \text{ and } v \preceq y.$$

Clearly, $(\mathcal{E}, \sqsubseteq)$ is a partially ordered set. Define $\rho_\nu : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ by

$$\rho_\nu(\mathbf{x}, \mathbf{u}) = d_\nu(x,u) + d_\nu(y,v), \quad \mathbf{x} = (x,y), \mathbf{u} = (u,v). \tag{2.3}$$

Obviously, if $(E, \preceq, d_\nu)$ is a (complete) metric space then $(\mathcal{E}, \sqsubseteq, \rho_\nu)$ is a (complete) metric space. In general, however, as the following example shows, if E is a ν-generalized metric space ($\nu \geq 2$) then $(\mathcal{E}, \rho_\nu)$ may fail to be a ν-generalized metric space.

Example 2.2 ([8, Example 4.2])**.** Let $E = \{a, b, c\}$ and define $d_\nu : E \times E \to \mathbb{R}^+$ by $d_\nu(a,b) = 4$, $d_\nu(a,c) = d_\nu(b,c) = 1$, and $d_\nu(x,x) = 0$, $d_\nu(x,y) = d_\nu(y,x)$ for all $x, y \in E$. Since four distinct points in E do not exist, the rectangular inequality is trivially satisfied. Hence, (E, d_ν) is a 2-generalized metric space, which is obviously not a metric space.

Now, consider the mappings $\rho_+, \rho_{\max} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ defined by

$$\rho_+(\mathbf{x}, \mathbf{u}) = d_\nu(x,u) + d_\nu(y,v),$$
$$\rho_{\max}(\mathbf{x}, \mathbf{u}) = \max\{d_\nu(x,u), d_\nu(y,v)\},$$

where $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$. Then, for the quadrilateral $\{(a,b), (b,c), (a,c), (c,c)\}$ in $\mathcal{E}$, we have

$$\begin{aligned} \rho_+\big((a,b),(b,c)\big) &= 5 > 1 + 1 + 1 \\ &= \rho_+\big((a,b),(a,c)\big) + \rho_+\big((a,c),(c,c)\big) + \rho_+\big((c,c),(b,c)\big), \end{aligned}$$

and

$$\begin{aligned}\rho_{\max}\big((a,b),(b,c)\big) &= 4 > 1+1+1\\ &= \rho_{\max}\big((a,b),(a,c)\big) + \rho_{\max}\big((a,c),(c,c)\big) + \rho_{\max}\big((c,c),(b,c)\big).\end{aligned}$$

Hence, in both cases, rectangular inequality is not satisfied, and so $(\mathcal{E}, \rho_+)$ and $(\mathcal{E}, \rho_{\max})$ are not 2-generalized metric spaces.

The following notion of regularity for mappings $F : E \times E \to E$ is needed in this section.

Definition 2.3. Given $x, y \in E$, the mapping $F : E \times E \to E$ is called *asymptotically regular* at $\mathbf{x} = (x, y)$ if the Picard iterates $x_n = F^n(x, y)$ and $y_n = F^n(y, x)$, defined by (1.1), satisfy the following condition

$$\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1}) + \rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+2}) \to 0, \quad \mathbf{x}_n = (x_n, y_n). \tag{2.4}$$

Note that, if (E, d_ν) is a metric space, the summand $\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+2})$ in (2.4) is redundant.

In the case of metric spaces, coupled fixed point results are usually obtained by considering the induced space $(\mathcal{E}, \sqsubseteq, \rho_\nu)$ and reducing a symmetric contraction $F : E \times E \to E$ to a monotone contraction $T : \mathcal{E} \to \mathcal{E}$. This strategy, as Example 2.2 shows, does not work in the case of ν-generalized metric spaces. Hence, we shall take a different approach in this case.

2.1. **Coupled fixed points in partially ordered metric spaces.** In this section, we assume that $(E, \preceq, d)$ is a partially ordered metric space. Given a mapping $F : E \times E \to E$, define $T : \mathcal{E} \to \mathcal{E}$ by

$$T\mathbf{x} = (F(x, y), F(y, x)), \quad \mathbf{x} = (x, y). \tag{2.5}$$

The following properties are straightforward.

(i) If F is continuous then T is continuous.
(ii) If F is asymptotically regular in the sense of Definition 2.3, then T is asymptotically regular in the sense of Theorem 1.7.
(iii) If F has the mixed monotone property, then T is nondecreasing on $(\mathcal{E}, \sqsubseteq)$.
(iv) If F is a symmetric contraction of Ćirić-Matkowski type, then T is a monotone contraction of Ćirić-Matkowski type, in the sense of Theorem 1.6.
(v) F has a (unique) coupled fixed point if and only if T has a (unique) fixed point.

These properties along with the results in Section 1.3 yield the following coupled fixed point results.

Theorem 2.4. *Let $(E, \preceq, d)$ be a complete partially ordered metric space. Suppose that $F : E \times E \to E$ has the following properties.*

(i) *F is continuous and has the mixed strict monotone property,*

(ii) *F is a symmetric contraction of Ćirić-Matkowski type,*
(iii) *there exist $x_0, y_0 \in E$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$.*

Then F has a coupled fixed point.

Proof. Since F is continuous, T is continuous. Since F is a symmetric contraction of Ćirić-Matkowski type and has the mixed strict monotone property, T is a monotone contraction of Ćirić-Matkowski type. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, we get $\mathbf{x}_0 \sqsubseteq T\mathbf{x}_0$ with $\mathbf{x}_0 = (x_0, y_0)$. All conditions of Theorem 1.6 are satisfied. Hence T has a fixed point, which in turn implies that F has a coupled fixed point. □

Finally, we have the following Proinov type coupled fixed point theorem.

Theorem 2.5. *Suppose that $(E, \preceq, d)$ is a complete partially ordered metric space, and that $F : E \times E \to E$ satisfies conditions (i)-(ii) of Theorem 1.5. For $\gamma > 0$, define $\mathbf{m} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ by (here $\mathbf{x} = (x, y)$, $\mathbf{u} = (u, v)$)*

$$\begin{aligned} \mathbf{m}(\mathbf{x}, \mathbf{u}) = d(x, u) + d(y, v) &+ \gamma\big(d(x, F(x, y)) + d(y, F(y, x))\big) \\ &+ \gamma\big(d(u, F(u, v)) + d(v, F(v, u))\big). \end{aligned} \tag{2.6}$$

Suppose that

$$d\big(F(x, y), F(u, v)\big) + d\big(F(y, x), F(v, u)\big) < \mathbf{m}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x} \sqsubseteq \mathbf{u}, \mathbf{x} \neq \mathbf{u}, \tag{2.7}$$

and that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $\mathbf{x} \sqsubseteq \mathbf{u}$,

$$\mathbf{m}(\mathbf{x}, \mathbf{u}) < \delta + \epsilon \Longrightarrow d\big(F(x, y), F(u, v)\big) + d\big(F(y, x), F(v, u)\big) \leq \epsilon. \tag{2.8}$$

If F is asymptotically regular, then F has a coupled fixed point.

Proof. It is easily seen that

$$\mathbf{m}(\mathbf{x}, \mathbf{u}) = \rho_\nu(\mathbf{x}, \mathbf{u}) + \gamma(\rho_\nu(\mathbf{x}, T\mathbf{x}) + \rho_\nu(\mathbf{u}, T\mathbf{u})).$$

The assumptions of the theorem imply that

$$\rho_\nu(T\mathbf{x}, T\mathbf{u}) < \mathbf{m}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x} \sqsubseteq \mathbf{u}, \mathbf{x} \neq \mathbf{u}.$$

and that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $\mathbf{x} \sqsubseteq \mathbf{u}$,

$$\mathbf{m}(\mathbf{x}, \mathbf{u}) < \delta + \epsilon \Longrightarrow \rho_\nu(T\mathbf{x}, T\mathbf{u}) \leq \epsilon. \tag{2.9}$$

Hence T is a monotone contraction of Ćirić-Matkowski type on $(\mathcal{E}, \rho_\nu)$ that satisfies all conditions of Theorem 1.7. Hence T has a fixed point which, in turn, implies that that F has a coupled fixed point. □

2.2. **Coupled fixed points in partially ordered ν-generalized metric spaces.** In this subsection, we assume that $(E, \preceq, d_\nu)$ is a partially ordered ν-generalized metric space. As Example 2.2 shows, the induced space $(\mathcal{E}, \sqsubseteq, \rho_\nu)$ may not be a partially ordered ν-generalized metric space. Hence, we take a different approach to get coupled fixed point theorems. When we call a mapping $F : E \times E \to E$ continuous (since, in general, we do not have a topological structure in $\mathcal{E}$), we mean that $F(x_n, y_n) \to F(x, y)$ in E whenever $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $x_n \to x$ and $y_n \to y$.

Lemma 2.6. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a ν-generalized metric space (E, d_ν) and let $\mathbf{x}_n = (x_n, y_n)$. The following statements are equivalent.*

(i) *Both $\{x_n\}$ and $\{y_n\}$ are ν-Cauchy sequences.*

(ii) $\lim_{n\to\infty} \sup\{\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1+m\nu}) : m \in \mathbb{Z}^+\} = 0.$

Proof. This follows easily from Definition 1.2 and the following simple inequalities.

$$\begin{aligned}
\max\{\sup\{d_\nu(x_n, x_{n+1+m\nu}) : m \in \mathbb{Z}^+\}, &\sup\{d_\nu(y_n, y_{n+1+m\nu}) : m \in \mathbb{Z}^+\}\} \\
&\le \sup\{d_\nu(x_n, x_{n+1+m\nu}) + d_\nu(y_n, y_{n+1+m\nu}) : m \in \mathbb{Z}^+\} \\
&= \sup\{\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1+m\nu}) m \in \mathbb{Z}^+\} \\
&\le \sup\{d_\nu(x_n, x_{n+1+m\nu}) : m \in \mathbb{Z}^+\} + \sup\{d_\nu(y_n, y_{n+1+m\nu}) : m \in \mathbb{Z}^+\}.
\end{aligned}$$

□

Lemma 2.7. *Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in a ν-generalized metric space (E, d_ν) satisfying (2.4), each of which consists of mutually distinct elements. Suppose that, for every $\epsilon > 0$ and for any two subsequences $\{\mathbf{x}_{p_i}\}$ and $\{\mathbf{x}_{q_i}\}$, if $\limsup_{i\to\infty} \rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) \le \epsilon$ then, for some N,*

$$\rho_\nu(\mathbf{x}_{p_i+1}, \mathbf{x}_{q_i+1}) \le \epsilon \quad (i \ge N).$$

Then both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Proof. First, we show that both $\{x_n\}$ and $\{y_n\}$ are ν-Cauchy. Towards a contradiction, assume that, for example, $\{x_n\}$ is not ν-Cauchy. Then condition (2.6) of Lemma 2.6 fails to hold. Therefore, for some $\epsilon > 0$, we have

$$\forall k \ge 1,\ \exists n \ge k, \quad \sup\{\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1+m\nu}) : m \in \mathbb{Z}^+\} > \epsilon. \tag{2.10}$$

Condition $\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1}) \to 0$ implies the existence of a sequence $\{k_i\}$ of positive integers such that $k_{i-1} < k_i$ and

$$\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1}) < \epsilon/i \quad (n \ge k_i). \tag{2.11}$$

For each k_i, by (2.10), there exist $n_i \ge k_i + 1$ and $m_i \ge 0$ such that $\rho_\nu(\mathbf{x}_{n_i}, \mathbf{x}_{n_i+1+m_i\nu}) > \epsilon$. By (2.11), $\rho_\nu(\mathbf{x}_{n_i}, \mathbf{x}_{n_i+1}) < \epsilon$. Hence, we must have $m_i \ge 1$. Let m_i be the smallest number with this property so that $\rho_\nu(\mathbf{x}_{n_i}, \mathbf{x}_{n_i+1+m_i\nu-\nu}) \le \epsilon$. Take $p_i = n_i - 1$ and $q_i = n_i + m_i\nu$. Then, for every $i \ge 1$, we get $q_i > p_i \ge k_i$, and

$$\rho_\nu(\mathbf{x}_{p_i+1}, \mathbf{x}_{q_i+1}) > \epsilon, \tag{2.12}$$

$$\rho_\nu(\mathbf{x}_{p_i+1}, \mathbf{x}_{q_i+1-\nu}) \le \epsilon. \tag{2.13}$$

Since both $\{x_n\}$ and $\{y_n\}$ consist of mutually different elements, property (3) in Definition 1.1 shows that, for every $i \ge 1$,

$$\begin{aligned}
d_\nu(x_{p_i}, x_{q_i}) \le d_\nu(x_{p_i}, x_{p_i+1}) &+ d_\nu(x_{p_i+1}, x_{q_i+1-\nu}) \\
&+ d_\nu(x_{q_i+1-\nu}, x_{q_i+2-\nu}) + \cdots + d_\nu(x_{q_i-1}, x_{q_i}), \\
d_\nu(y_{p_i}, y_{q_i}) \le d_\nu(y_{p_i}, y_{p_i+1}) &+ d_\nu(y_{p_i+1}, y_{q_i+1-\nu}) \\
&+ d_\nu(y_{q_i+1-\nu}, y_{q_i+2-\nu}) + \cdots + d_\nu(y_{q_i-1}, y_{q_i}).
\end{aligned}$$

The above two inequalities along with (2.11) and (2.13) imply that $\rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) \leq 2\nu\epsilon/i + \epsilon$, for all $i \geq 1$, from which we get $\limsup_{i\to\infty} \rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) \leq \epsilon$. This along with (2.12) violate our assumption. Hence both $\{x_n\}$ and $\{y_n\}$ are ν-Cauchy.

Finally, the assumption $\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+2}) \to 0$ as $n \to \infty$ implies that $d_\nu(x_n, x_{n+2}) \to 0$ and $d_\nu(y_n, y_{n+2}) \to 0$ as $n \to \infty$. Proposition 1.3 now shows that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. □

The following is a Ćirić-Matkowski type coupled fixed point theorem in partially ordered ν-generalized metric spaces.

Theorem 2.8. *Let $(E, \preceq, d_\nu)$ be a complete partially ordered ν-generalized metric space. If $F : E \times E \to E$ is a symmetric contraction of Ćirić-Matkowski type satisfying conditions (i)-(ii) of Theorem 1.5, then F has a coupled fixed point.*

Proof. Suppose that (1.2) holds for $\mathbf{x}_0 = (x_0, y_0)$ and let $\mathbf{x}_n = (x_n, y_n)$ be the Picard iterates of F at $\mathbf{x}_0$ defined by (1.1). Note that $\mathbf{x}_p \sqsubseteq \mathbf{x}_q$ if $p \leq q$. In fact, $\mathbf{x}_n = T^n\mathbf{x}_0$, $n \geq 1$, where T is defined by (2.5). An argument similar to that of [2, Theorem 3.4] shows that $\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1}) + \rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+2}) \to 0$.

Now, let $\epsilon > 0$ and assume that $\limsup_{i\to\infty} \rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) \leq \epsilon$. Since F is a symmetric contraction of Ćirić-Matkowski type, by (2.2), there exists $\delta > 0$ such that

$$p \leq q,\ \epsilon < \rho_\nu(\mathbf{x}_p, \mathbf{x}_q) < \delta + \epsilon \Longrightarrow \rho_\nu(\mathbf{x}_{p+1}, \mathbf{x}_{q+1}) \leq \epsilon.$$

By [1, Lemma 3.1], there is $N \in \mathbb{N}$, such that

$$\rho_\nu(\mathbf{x}_{p_i+1}, \mathbf{x}_{q_i+1}) \leq \epsilon \quad (i \geq N).$$

Now, Lemma 2.7 shows that the sequences $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Since E is complete, there exist x and y in E such that $x_n \to x$ and $y_n \to y$. Since F is continuous, we conclude that $F(x, y) = x$ and $F(y, x) = y$, so that (x, y) is a coupled fixed point. □

The following is a Proinov type coupled fixed point result in the setting of partially ordered ν-generalized metric spaces.

Theorem 2.9. *Let $(E, \preceq, d_\nu)$ be a complete partially ordered ν-generalized metric space. Given $F : E \times E \to E$, define $\mathbf{m} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ by (2.6), and assume that (2.8) hold. If F is asymptotically regular and satisfies conditions (i)-(ii) of Theorem 1.5, then F has a coupled fixed point.*

Proof. Suppose that (1.2) holds for $\mathbf{x}_0 = (x_0, y_0)$ and let $\mathbf{x}_n = (x_n, y_n)$ be the Picard iterates of F at $\mathbf{x}_0$ defined by (1.1). Note that $\mathbf{x}_p \sqsubseteq \mathbf{x}_q$ if $p \leq q$. Since F is asymptotically regular, we have

$$\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1}) + \rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+2}) \to 0. \tag{2.14}$$

Let $\{\mathbf{x}_{p_i}\}$ and $\{\mathbf{x}_{q_i}\}$ be two subsequences of $\{\mathbf{x}_n\}$. Then

$$\mathbf{m}(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) = \rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) + \gamma\big(\rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{p_{i+1}}) + \rho_\nu(\mathbf{x}_{q_i}, \mathbf{x}_{q_{i+1}})\big).$$

Since $\rho_\nu(\mathbf{x}_n, \mathbf{x}_{n+1}) \to 0$, we get

$$\limsup_{i\to\infty} \mathbf{m}(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) = \limsup_{i\to\infty} \rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}). \tag{2.15}$$

Note that by (2.14) we get $d_\nu(x_n, x_{n+1}) \to 0$ and $d_\nu(y_n, y_{n+1}) \to 0$. Also, by the condition (1.2) and the strict mixed monotone property of F, we have that the sequences $\{x_n\}$ and $\{y_n\}$ consist of mutually distinct terms.

Now, let $\epsilon > 0$ and assume that $\limsup_{i\to\infty} \rho_\nu(\mathbf{x}_{p_i}, \mathbf{x}_{q_i}) \leq \epsilon$. The equality in (2.15) implies that $\limsup_{i\to\infty} \mathbf{m}(x_{p_i}, x_{q_i}) \leq \epsilon$. By (2.8), there exists $\delta > 0$ such that, for $p \leq q$,

$$\mathbf{m}(\mathbf{x}_p, \mathbf{x}_q) < \delta + \epsilon \Longrightarrow \rho_\nu(\mathbf{x}_{p+1}, \mathbf{x}_{q+1}) \leq \epsilon.$$

By [1, Lemma 3.1], there is $N \in \mathbb{N}$, such that

$$\rho_\nu(\mathbf{x}_{p_i+1}, \mathbf{x}_{q_i+1}) \leq \epsilon \quad (i \geq N).$$

All conditions of Lemma 2.7 are fulfilled and so the sequences $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Since E is complete, there exist x and y in E such that $x_n \to x$ and $y_n \to y$. Since F is continuous, we conclude that $F(x, y) = x$ and $F(y, x) = y$, so that (x, y) is a coupled fixed point. □

2.3. **Uniqueness.** In order to obtain the uniqueness of coupled fixed point in the previous results, we need some additional assumption. We formulate it just in the case of Theorem 2.8.

Proposition 2.10. *Let $(E, \preceq, d_\nu)$ and F be as in Theorem 2.8 and let $CFix(F)$ be the set of coupled fixed points of F. Then any two comparable elements of $CFix(F)$ (in the sense of order $\sqsubseteq$) are equal. In particular, if all the elements of $CFix(F)$ are comparable, then this set reduces to a singleton.*

Proof. Suppose, to the contrary, that there exist two distinct coupled fixed points (x, y) and (u, v) of F which are comparable, e.g., $(x, y) \sqsubseteq (u, v)$ and $(x, y) \neq (u, v)$. Then by (2.1) we get that

$$d_\nu(x, y) + d_\nu(u, v) < d_\nu(x, y) + d_\nu(u, v),$$

a contradiction. □

2.4. **Illustrative examples.** The following is a very easy example illustrating a possible use of Theorem 2.8.

Example 2.11. Let (E, d_ν) be the space defined in Example 2.2. Introduce an order $\preceq$ on E by $a \preceq a$, $b \preceq b$, $b \preceq a$ and $c \preceq c$. Consider a mapping $F : E \times E \to E$ given by $F(x, x) = a$, for all $x \in E$, $F(a, b) = F(b, a) = a$, and $F(x, y) = c$ otherwise. It is easy to see that all conditions of Theorem 2.8 are satisfied. In particular, the only nontrivial case when conditions (2.1) and (2.2) have to be checked (i.e., when $x \preceq u$, $y \succeq v$ and $(x, y) \neq (u, v)$) is when $(x, y) = (a, a)$ and $(u, v) = (b, a)$. It is easily seen that both of them are then satisfied.

Example 2.12. Consider the following 2-generalized metric space, which is a slight modification of the space considered in [18, Example 1.1]. Let $E = A \cup B$, with $A = \{0, 2\}$, $B = (0, 1]$, and define $d_\nu : E \times E \to [0, +\infty)$ by

$$d_\nu(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \text{ and } (\{x, y\} \subseteq A \text{ or } \{x, y\} \subseteq B) \\ y, & \text{if } x \in A, y \in B \\ x, & \text{if } x \in B, y \in A. \end{cases}$$

Take the usual order $\leq$ on E. Then $(E, \leq, d_\nu)$ is a complete, partially ordered, 2-generalized metric space which is not a metric space. Note that if we define ρ_ν on $\mathcal{E} = E \times E$ by (2.3), then $(\mathcal{E}, \rho_\nu)$ is not a 2-generalized metric space. Indeed, if we take the points $(0, 0)$, $(b_1, 0)$, (b_1, b_2), $(2, 2)$ from $\mathcal{E}$ (here, $0 < b_1, b_2 \leq 1$), we have

$$\begin{aligned} \rho_\nu((0,0),(2,2)) &= d_\nu(0,2) + d_\nu(0,2) = 1 + 1 = 2, \\ \rho_\nu((0,0),(b_1,0)) &= d_\nu(0,b_1) + d_\nu(0,0) = b_1, \\ \rho_\nu((b_1,0),(b_1,b_2)) &= d_\nu(b_1,b_1) + d_\nu(0,b_2) = b_2, \\ \rho_\nu((b_1,b_2),(2,2)) &= d_\nu(b_1,2) + d_\nu(b_2,2) = b_1 + b_2. \end{aligned}$$

Hence, if $2b_1 + 2b_2 < 2$, we have

$$\rho_\nu((0,0),(2,2)) > \rho_\nu((0,0),(b_1,0)) + \rho_\nu((b_1,0),(b_1,b_2)) + \rho_\nu((b_1,b_2),(2,2)).$$

Consider now the mapping $F : E \times E \to E$ given by

$$F(x, y) = \begin{cases} \dfrac{x - y}{2}, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

The conditions (i)-(ii) of Theorem 1.5 are easy to check (for example, the second one is satisfied for $x_0 = 2$ and $y_0 = 0$). In order to check the condition (2.8), consider the mapping $\mathbf{m}$ given by

$$\begin{aligned} \mathbf{m}(\mathbf{x}, \mathbf{u}) &= d_\nu(x, u) + d_\nu(y, v) \\ &\quad + d_\nu(x, F(x, y)) + d_\nu(y, F(y, x)) + d_\nu(u, F(u, v)) + d_\nu(v, F(v, u)), \end{aligned}$$

for $\mathbf{x} = (x, y)$, $\mathbf{u} = (u, v)$ (i.e., take $\gamma = 1$). For $\mathbf{u} \sqsubseteq \mathbf{x}$ (i.e., $u \leq x$, $y \leq v$) and, for example, $1 \geq x > u > v > y > 0$ (other possible cases can be treated in a similar way), we have

$$\begin{aligned} \mathbf{m}(\mathbf{x}, \mathbf{u}) &= d_\nu(x, u) + d_\nu(y, v) + d_\nu\Big(x, \frac{x - y}{2}\Big) + d_\nu(y, 0) + d_\nu\Big(u, \frac{u - v}{2}\Big) + d_\nu(v, 0) \\ &= 1 + 1 + 1 + y + 1 + v = 4 + y + v, \end{aligned}$$

hence $\mathbf{m}(\mathbf{x}, \mathbf{u}) < \delta + \epsilon$ implies that $\epsilon > 4 + y + v - \delta > 1$ (if $\delta < 3$). On the other hand

$$d_\nu(F(x, y), F(u, v)) + d_\nu(F(y, x), F(v, u)) = d_\nu\Big(\frac{x - y}{2}, \frac{u - v}{2}\Big) + d_\nu(0, 0) = 1 < \epsilon,$$

and the condition (2.8) is satisfied.

Thus, all the conditions of Theorem 2.9 are fulfilled and we conclude that F has a (unique) coupled fixed point (which is $(0,0)$).

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