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On rings of real valued clopen continuous functions

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Abstract

Among variant kinds of strong continuity in the literature, the clopen continuity or cl-supercontinuity (i.e., inverse image of every open set is a union of clopen sets) is considered in this paper. We investigate and study the ring $C_s(X)$ of all real valued clopen continuous functions on a topological space X. It is shown that every $f \in C_s(X)$ is constant on each quasi-component in X and using this fact we show that $C_s(X) \cong C(Y)$, where Y is a zero-dimensional s-quotient space of X. Whenever X is locally connected, we observe that $C_s(X) \cong C(Y)$, where Y is a discrete space. Maximal ideals of $C_s(X)$ are characterized in terms of quasi-components in X and it turns out that X is mildly compact (every clopen cover has a finite subcover) if and only if every maximal ideal of $C_s(X)$ is fixed. It is shown that the socle of $C_s(X)$ is an essential ideal if and only if the union of all open quasi-components in X is s-dense. Finally the counterparts of some familiar spaces, such as P_s-spaces, almost P_s-spaces, s-basically and s-extremally disconnected spaces are defined and some algebraic characterizations of them are given via the ring $C_s(X)$.

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1. Introduction

If X and Y are topological spaces, then a function $f : X \to Y$ is said to be *clopen continuous* [9] or *cl-supercontinuous* [10] if for every $x \in X$ and for each open set V in Y containing $f(x)$, there exists a clopen (closed and open) set U in X containing x such that $f(U) \subseteq V$. Since this is a strong form of continuity, let us rename "clopen continuous" as *strongly continuous* and for brevity write *s-continuous*. If A is a subset of X, then s-interior of A denotes the set of all $x \in A$ such that A contains a clopen set containing x and we denote it by $\text{int}_s A$. A subset G of X is said to be s-open if $G = \text{int}_s G$. In fact a set is s-open if and only if it is a union of clopen sets. The set of all $x \in X$ such that every clopen set containing x intersects A is called s-closure of A and it is denoted by $\text{cl}_s A$. Similarly a set H is called s-closed if $H = \text{cl}_s H$ and a set is s-closed if and only if it is an intersection of clopen sets. A bijection function $\theta : X \to Y$ is called *s-homeomorphism* [10, under the name of cl-homeomorphism] if both θ and θ^{-1} are s-continuous. If such function from X onto Y exists, we say that X and Y are *s-homeomorphic* and we write $X \cong_s Y$.

We denote by $C(X)$ the ring of all real-valued continuous functions on a space X and by $C_s(X)$ the set of all real valued s-continuous functions on X. It is easy to see that $C_s(X)$ is a ring and in fact it is a subring of $C(X)$. For each $f \in C(X)$, the *zero-set* of f, denoted by $Z(f)$, is the set of zeros of f and $X \setminus Z(f)$ is the *cozero-set* of f. The set of all zero-sets in X is denoted by $Z(X)$ and we also denote by $Z_s(X)$ the set of all zero-sets $Z(f)$, where $f \in C_s(X)$. Clearly $Z_s(X) \subseteq Z(X)$. If $Z \in Z_s(X)$, then it is the inverse image of the closed subset $\{0\}$ of $\mathbb{R}$ under an element of $C_s(X)$ and this implies that every zero-set in $Z_s(X)$ is s-closed. Hence every cozero-set $X \setminus Z(f)$, where $f \in C_s(X)$, is s-open. The converse is not necessarily true. For instance let S be an uncountable space in which all points are isolated except for a distinguish point s. Neighborhoods of s are considered to be those sets containing s with countable complement, see Problem 4N in [6]. Since $\{s\} = \bigcap_{s \neq a \in S}(S \setminus \{a\})$, the singleton $\{s\}$ is s-closed but it is not a zero-set. It is well-known that a space X is a completely regular Hausdorff space if and only if $Z(X)$ is a base for closed subsets of X, if and only if the set of all cozero-sets is a base for open subsets of X, see Theorem 3.2 in [6]. Whenever X is zero-dimensional (i.e., a T_1 space with a base consisting of clopen sets), then clearly $C(X)$ and $C_s(X)$ coincide, see also Remark 2.3 in [10]. If X is a completely regular Hausdorff space, the converse is also true, we cite this fact as a lemma for later use.

Lemma 1.1. *Whenever X is zero-dimensional, then $C(X) = C_s(X)$ and if X is a completely regular Hausdorff space, the converse is also true.*

Proof. The first implication is obvious, see also Remark 2.3 in [10]. For the converse, as we have already mentioned the collection $\mathcal{C} = \{X \setminus Z(f) : f \in C(X)\}$ is a base for open sets in X. Now let $f \in C(X)$ and $x \in X \setminus Z(f)$. Since $f(x) \neq 0$ and $f \in C(X) = C_s(X)$, there exists a clopen set U in X containing x such that $f(y) \neq 0$, for each $y \in U$. Hence $U \subseteq X \setminus Z(f)$ which means that X has a base with clopen sets. Since X is T_1, it is zero-dimensional. □

We recall that a completely regular Hausdorff space X is a *P-space* if every G_δ-set or equivalently every zero-set in X is open and it is an *almost P-space* if every non-empty G_δ-set or equivalently every nonempty zero-set in X has a non-empty interior. Hence every P-space is an almost P-space but the converse fails, for instance, the one-point compactification of an uncountable discrete space is an almost P-space which is not a P-space, see Example 2 in [8] and problem 4K(1) in [6]. Basically (extremally) disconnected spaces are those spaces in which every cozero-set (open set) has an open closure. Clearly every extremally disconnected space is basically disconnected but not conversely, see Problem 4N in [6]. Several algebraic and topological characterizations of aforementioned spaces are given in [3], [4], [6] and [8].

An ideal I in a commutative ring is called a *z-ideal* if $M_a \subseteq I$ for each $a \in I$, where M_a is the intersection of all maximal ideals in the ring containing a. It is easy to see that an ideal I in $C(X)$ ($C_s(X)$) is a z-ideal if and only if whenever $f \in I$, $g \in C(X)$ ($g \in C_s(X)$) and $Z(f) \subseteq Z(g)$, then $g \in I$, see Problem 4A in [6]. A non-zero ideal in a ring is said to be *essential* if it intersects every non-zero ideal non-trivially. Intersection of all essential ideals in a ring is called the *socle* of the ring. A topological characterization of essential ideals in $C(X)$ and the socle of $C(X)$ are given in [2] and [7] respectively. An ideal I in $C(X)$ or $C_s(X)$ is said to be *fixed* if $\bigcap_{f\in I} Z(f) \neq \varnothing$, otherwise it is called *free*. The space βX is the *Stone-Čech compactification* of X and for any $p \in \beta X$, M^p (resp., O^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$ (resp., $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$).

The *component* C_x of a point x in a topological space X is the union of all connected subspaces of X which contain x. The *quasi-component* Q_x of x is the intersection of all clopen subsets of X which contain x. Clearly $C_x \subseteq Q_x$ for each $x \in X$ and the inclusion may be strict, see Example 6.1.24 in [5]. It is well-known that for any two distinct points x and y in a space X, either $Q_x = Q_y$ or $Q_x \cap Q_y = \varnothing$, see [5]. Components and quasi-components in a space X are closed and whenever X is locally connected, then they are also open, see Corollary 27.10 in [12]. The converse of this fact is not true in general, see the example which is given preceding Lemma 1.1. Whenever the components in a space X are the points, then X is called *totally disconnected*. Equivalently, X is totally disconnected if and only if the only non-empty connected subsets of X are the singleton sets.

2. Some properties of clopen continuous functions

Behaviour of s-continuous functions on quasi-components is investigated in this section. The results of this section play an important role in the next sections.

Proposition 2.1. *Let X and Y be topological spaces and $f : X \to Y$ be a s-continuous function. Then the following statements hold.*

1. *If $x \in X$, $y \in Y$ and $f(x) = y$, then $f(Q_x) \subseteq Q_y$.*
2. *If Y is a T_1-space, then f is constant on each quasi-component in X.*

(3) *Whenever f is one-to-one and Y is a T_1-space, then X is totally disconnected.*
(4) *If f is s-homeomorphism and $f(x) = y$, then $f(Q_x) = Q_y$. Moreover, if X or Y is T_1, then both X and Y are totally disconnected.*

Proof. (1) Let $z \in Q_x$ but $f(z) \notin Q_y$. Then there is a clopen set U in Y such that $f(z) \in U$ and $U \cap Q_y = \varnothing$ which implies $y \notin U$. Since f is s-continuous, there exists a clopen set G in X containing z such that $f(G) \subseteq U$. But $Q_x \subseteq G$ implies that $f(Q_x) \subseteq U$ and this yields that $f(x) = y \in U$, a contradiction.

(2) Let $y \in Q_x$ and $f(x) \neq f(y)$. Then there exists an open set H in Y such that $f(x) \in H$ but $f(y) \notin H$. Since f is s-continuous, there exists a clopen set G in X containing x such that $f(G) \subseteq H$. But any clopen set containing x contains y as well, for $y \in Q_x$, so $f(y) \in H$, a contradiction.

(3) Using part (2), f is constant on Q_x for every $x \in X$. But f is one-to-one, so Q_x is singleton for each $x \in X$. This implies that every quasi-component in X, and thus every component of X is singleton, i.e., X is totally disconnected.

(4) It is evident by parts (1) and (3). □

Corollary 2.2. *For every $f \in C_s(X)$, the following statements hold.*

(1) *f is constant on each quasi-component in X.*
(2) *The zero-set $Z(f)$ is a union of quasi-components in X. In fact $Z(f) = \bigcup_{x \in Z(f)} Q_x$.*

A space X is called *ultra Hausdorff* [11] if for each pair of distinct points in X, there exists a clopen set in X containing one but not the other. Two disjoint subsets of a space are called s-completely separated if there is a function $f \in C_s(X)$ which separates them. Similar to the proof of Theorem 1.15 in [6], it is easy to see that two disjoint sets are s-completely separated if and only if they are contained in two disjoint members of $Z_s(X)$. By the following proposition, ultra Hausdorff spaces are characterized by quasi-components in the spaces and every s-closed set is s-completely separated from every quasi-component disjoint from it.

Proposition 2.3. *Let X be a topological space. Then the following statements hold.*

(1) *If A is a s-closed subset of X and $x \in X \setminus A$, then there exists $g \in C_s(X)$ such that $g(A) = \{1\}$ and $g(Q_x) = \{0\}$.*
(2) *X is ultra Hausdorff if and only if every quasi-component in X is singleton.*

Proof. (1) Since $X \setminus A$ is s-open, there exists a clopen set U containing x such that $U \cap A = \varnothing$. Now define an idempotent e with $Z(e) = U$, i.e., $e(U) = \{0\}$ and $e(X \setminus U) = \{1\}$. Clearly $e \in C_s(X)$, $e(Q_x) = 0$, since $Q_x \subseteq U$ and $e(A) = 1$.

(2) Whenever X is ultra Hausdorff and $x, y \in X$ are distinct, then there exists a clopen set U containing x but not y. This implies that $y \notin Q_x$, i.e. Q_x

is singleton. Conversely, let $x, y \in X$ be distinct points. Since $Q_x = \{x\}$, there is a clopen set U such that $x \in U$ and $y \notin U$, i.e., X is ultra Hausdorff. $\square$

3. $C_s(X)$ is a $C(Y)$

As an equivalent definition, a space X is zero-dimensional if and only if it is T_1 and for each point $x \in X$ and each closed subset A of X not containing x, there exists a clopen set G in X containing x such that $G \cap A = \varnothing$. Clearly every zero-dimensional space is Tychonoff. Whenever we consider the collection of all clopen subsets of (X, τ) as a base for a topology τ^* on X, then $C_s(X, \tau) = C(X, \tau^*)$, by Theorem 5.1 in [10]. But the space (X, τ^*) is not necessarily T_1 and so it may not be zero-dimensional. In the following theorem we show that $C_s(X)$ is in fact a $C(Y)$ for a zero-dimensional space Y which is also a s-quotient space of X.

Theorem 3.1. *For each topological space X, there exists a zero-dimensional space X_z such that $C_s(X) \cong C(X_z)$.*

Proof. For each $x \in X$, let Q_x be the quasi-component of x and consider the decomposition $X_z = \{Q_x : x \in X\}$. Take a topology τ on X_z so that $G \in \tau$ if and only if $\bigcup_{Q_x \in G} Q_x$ is s-open in X. To see that τ is a topology, clearly $X = \bigcup_{Q_x \in X_z} Q_x$ and $\varnothing = \bigcup_{Q_x \in \varnothing} Q_x$ imply that X_z and $\varnothing$ are open. Whenever H and K are open sets in X_z, then $\bigcup_{Q_x \in H \cap K} Q_x = (\bigcup_{Q_x \in H} Q_x) \bigcap (\bigcup_{Q_x \in K} Q_x)$ imply that $H \cap K$ is open in X_z. It is also easy to see that $\bigcup_\alpha H_\alpha$ is open in X_z for each open set H_α in X_z. The space X_z is Hausdorff, in fact if Q_x and Q_y are two distinct points in X_z, where $x, y \in X$, then $x \notin Q_y$ and since Q_y is s-closed, there is an idempotent $e \in C_s(X)$ such that $e(Q_y) = 0$ and $e(Q_x) = 1$, by part (1) of Proposition 2.3. If we set $H = \{Q_z : z \in Z(e)\}$, then $Z(e) = \bigcup_{Q_z \in H} Q_z$ implies that H is a clopen subset of X_z. Moreover $Q_y \in H$ but $Q_x \notin H$, i.e., X_z is ultra Hausdorff and hence it is hausdorff as well. To show that X_z is zero-dimensional, let H be a closed set in X_z and $Q_y \notin H$. Hence $T = \bigcup_{Q_x \in H} Q_x$ is a s-closed subset of X and $y \notin T$. Therefore by Proposition 2.3, there exists a clopen set U in X containing T but not containing y. Now $\bigcup_{z \in U} Q_z = U$ implies that $K = \{Q_z : z \in U\}$ is a clopen subset of X_z. Clearly $H \subseteq K$ and $Q_y \notin K$, hence X_z is zero-dimensional.

Now it remains to show that $C_s(X) \cong C(X_z)$. To this end, define $\varphi : C_s(X) \to C(X_z)$ by $\varphi(f) = f_z$ for each $f \in C_s(X)$, where f_z is defined by $f_z(Q_x) = f(x)$, for each $x \in X$. By Corollary 2.2, clearly φ is well-defined. Moreover $f_z \in C(X_z)$ for each $f \in C_s(X)$. In fact if $f_z(Q_x) = f(x) = c$, then for each $\varepsilon > 0$, there exists a clopen set G in X containing x such that $f(G) \subseteq (c-\varepsilon, c+\varepsilon)$, for $f \in C_s(X)$. Now we take the open set $H = \{Q_z : z \in G\}$ in X_z containing Q_x. Hence $f_z(H) = f(G) \subseteq (c - \varepsilon, c + \varepsilon)$ implies that $f_z \in C(X_z)$. Whenever $\varphi(f) = \varphi(g)$, $f, g \in C_s(X)$, then $f_z = g_z$ implies that $f(x) = f_z(Q_x) = g_z(Q_x) = g(x)$, for all $x \in X$. Hence $f = g$, i.e., φ is one-to-one. φ is also homomorphism, for $\varphi(f + g) = (f + g)_z$ and $(f + g)_z(Q_x) = (f+g)(x) = f(x)+g(x) = f_z(Q_x)+g_z(Q_x)$, for each $Q_x \in X_z$. This implies that

$\varphi(f+g) = \varphi(f)+\varphi(g)$, for all $f, g \in C_s(X)$ and hence φ is homomorphism. To complete the proof, we must show that φ is onto. To this end, let $g \in C(X_z)$. The function $f : X \to \mathbb{R}$ defined by $f(x) = g(Q_x)$, for all $x \in X$ is s-continuous. In fact, if $x \in X$, $f(x) = g(Q_x) = c$ and $\varepsilon > 0$ is given, then there is an open set H in X_z containing Q_x such that $g(H) \subseteq (c-\varepsilon, c+\varepsilon)$. Now it is enough to take the s-open subset $G = \bigcup_{Q_z \in H} Q_z$ of X. Clearly $x \in G$ and $f(G) \subseteq (c-\varepsilon, c+\varepsilon)$ which implies that $f \in C_s(X)$. By definitions of f and φ, it is clear that $\varphi(f) = g$ and we have thus shown that φ is onto. □

Corollary 3.2. *If every quasi-component in X is open, in particular, if X is locally connected, then $C_s(X) \cong C(Y)$ for a discrete space Y.*

Proof. Whenever each quasi-component of X is open, then each set $\{Q_x\}$ is open in the space X_z, defined in the proof of Theorem 3.1. Therefore each Q_x is an isolated point in X_z, so $Y = X_z$ is discrete and $C_s(X) \cong C(Y)$. □

The space X_z defined in the proof of Theorem 3.1 is a s-continuous image of X. In fact, if we regard the natural function $\tau : X \to X_z$, with $\tau(x) = Q_x$ for each $x \in X$, then τ is continuous and $f_z \circ \tau = f$ or equivalently, $\varphi(f) \circ \tau = f$. In order to prove that τ is s-continuous at $x \in X$, let H be a clopen subset of X_z containing Q_x (in fact H is an element of a base of the zero-dimensional space X_z). Now it is enough to take $U = \bigcup_{Q \in H} Q$ which is s-open in X containing x and clearly $\tau(U) = H$, i.e., τ is continuous at x. In this case we may say naturally, like the notion of quotient space, that X_z is a *s-quotient* of X and the induced map τ is a *s-quotient map.* The equality $f_z \circ \tau = f$ is evident.

We conclude this section by the following proposition.

Proposition 3.3. *For two spaces X and Y, if $X \cong_s Y$, then $X_z \cong Y_z$.*

Proof. let $\sigma : X \to Y$ be a s-homeomorphism. By Proposition 2.1, if $\sigma(x) = y$, then $\sigma(Q_x) = Q_y$. In fact every quasi-component in Y is exactly the image of a unique quasi-component in X under σ. Define $\phi : X_z \to Y_z$ by $\phi(Q) = \sigma(Q)$ for each quasi-component Q in X. Clearly ϕ is a bijection map. Given a quasi-component Q_x in X, we show that ϕ is continuous at Q_x. Let H be an open subset of Y_z containing $\phi(Q_x) = \sigma(Q_x) = Q_y$ and take $V = \bigcup_{Q \in H} Q$. Hence by definition of open sets in Y_z, V is s-open in Y containing $\sigma(Q_x) = Q_y$. Since σ is onto, there exists an element of Q_x, say x without loss of generality, such that $\sigma(x) = y$. Therefore there exists a clopen set U in X containing x (and hence containing Q_x) such that $\sigma(U) \subseteq V$. Now we set $G = \{Q_z : z \in U\}$. Clearly G is open in X_z containing Q_x and $\phi(G) \subseteq H$. Similarly, ϕ^{-1} is also continuous and we are done. □

The converse of the Proposition 3.3 is not true. If we take the discrete space $X = \{a, b\}$ and the space $Y = (0,1) \cup (1,2)$ as a subspace of $\mathbb{R}$, then clearly $X \cong X_z$ and Y_z is a discrete space containing two elements $(0,1)$ and $(1,2)$. Hence $X_z \cong Y_z$, but X and Y are not s-homeomorphic.

4. Maximal ideals of $C_s(X)$

In this section, for each space X, we consider X_z and the isomorphism $\varphi : C_s(X) \to C(X_z)$ as defined in the proof of Theorem 3.1. First we show that φ takes fixed (free) ideals to fixed (free) ideals and using this, we transfer some well-known facts in the context of $C(X)$ to that of $C_s(X)$.

Lemma 4.1. *An ideal I in $C_s(X)$ is fixed if and only if $\varphi(I)$ is a fixed ideal in $C(X_z)$. In particular, φ takes fixed maximal ideals to fixed maximal ideals.*

Proof. If $g \in \varphi(I)$, then there exists $f \in I$ such that $g = \varphi(f) = f_z$. Now $f(x) = 0$ if and only if $g(Q_x) = f_z(Q_x) = f(x) = 0$. Therefore, $x \in \bigcap_{f\in I} Z(f)$ if and only if $Q_x \in \bigcap_{g\in\varphi(I)} Z(g) = \bigcap_{f\in I} Z(f_z)$. □

Theorem 4.2. *For a topological space X, the fixed maximal ideals in $C_s(X)$ are precisely the sets*

$$M_{Q_x} = \{f \in C_s(X) : Q_x \subseteq Z(f)\} \qquad x \in X.$$

The ideals M_{Q_x} are distinct for distinct Q_x. For each $x \in X$, $C_s(X)/M_{Q_x}$ is isomorphic with the real field $\mathbb{R}$.

Proof. Using Lemma 4.1, fixed maximal ideals of $C_s(X)$ are precisely of the form $\varphi^{-1}(M_y)$, where $y \in X_z$ (i.e., $y = Q_x$ for some $x \in X$). Now $f \in \varphi^{-1}(M_y)$ if and only if $f_z \in M_y = M_{Q_x}$, or equivalently $Q_x \subseteq Z(f)$, by Theorem 3.1. Hence $\varphi^{-1}(M_y) = M_{Q_x}$. Whenever $Q_p \neq Q_q$ for $p, q \in X$, using Proposition 2.3, there exists $g \in C_s(X)$ such that $g(Q_p) = 0$ and $g(Q_q) = 1$ and this means that $g \in M_{Q_p} \setminus M_{Q_q}$. Finally $\sigma : C_s(X) \to \mathbb{R}$ with $\sigma(f) = f(x)$ (note that $f(y) = f(x)$, for all $y \in Q_x$ by Corollary 2.2), for all $f \in C_s(X)$ is a homomorphism with kernel M_{Q_x}, so $C_s(X)/M_{Q_x} \cong \mathbb{R}$. □

A space is said to be *mildly compact* [11] if every clopen cover of X has a finite subcover. Clearly every compact space is mildly compact but not conversely. For instance, consider the space $X = (0, 1) \cup (1, 2)$ as a subspace of $\mathbb{R}$. By the following proposition, for a space X, the compactness of X_z is equivalent to mildly compactness of X.

Proposition 4.3. *For a space X, the following statements are equivalent.*

1. *X is mildly compact.*
2. *X_z is compact.*
3. *Every ideal of $C_s(X)$ is fixed.*
4. *Every maximal ideal of $C_s(X)$ is fixed.*

Proof. A collection $\{H_\alpha : \alpha \in S\}$ is an open cover of X_z if and only if the collection $\{G_\alpha : \alpha \in S\}$, where $G_\alpha = \bigcup_{Q\in H_\alpha} Q$, is a s-open cover of X. This implies the equivalence of parts (1) and (2). The equivalence of third and fourth parts with part (1) is an immediate consequence of Lemma 4.1 and Theorem 4.11 in [6]. □

Using Theorem 3.1, and in view of the fact that there is a correspondence between elements of the space βX_z and the set of all maximal ideals of $C(X_z)$ by Theorem 7.3 in [6], all maximal ideals of $C_s(X)$, fixed or free, will be characterize by the following theorem for each space X.

Theorem 4.4. *For every space X, the maximal ideals of $C_s(X)$ are precisely of the form*

$$M^p = \{f \in C_s(X) : p \in cl_{\beta X_z} Z(f_z)\} \qquad p \in \beta X_z.$$

Remark 4.5. As in $C(X)$, for each maximal ideal M of $C_s(X)$, we define $O_M = \{f \in C_s(X) : fg = 0 \text{ for some } g \notin M\}$, see 7.12(b) in [6] and the argument preceding Theorem 2.12 in [1]. Whenever M is fixed, then $M = M_{Q_x}$ for some $x \in X$ by Theorem 4.2, and therefore we have $O_M = O_{Q_x} = \{f \in C_s(X) : Q_x \subseteq \text{int}_s Z(f)\}$. In fact, if $f \in O_M$, then $fg = 0$ for some $g \notin M_{Q_x}$. Hence $Q_x \subseteq X \setminus Z(g) \subseteq Z(f)$. But $X \setminus Z(g)$ is s-open, hence $Q_x \subseteq \text{int}_s Z(f)$, so $f \in O_{Q_x}$. Whenever $f \in O_{Q_x}$, then $Q_x \subseteq \text{int}_s Z(f)$. Since $\text{int}_s Z(f)$ is s-open, there exists a clopen set U containing Q_x contained in $\text{int}_s Z(f)$. Now if we take an idempotent e such that $Z(e) = U$, then $Q_x \subseteq Z(e) \subseteq \text{int}_s Z(f)$. Therefore $1 - e \notin M_{Q_x}$ and $(1-e)f = 0$ which implies that $f \in O_{Q_x} = O_M$.

By Theorem 2.4 in [4], X is zero-dimensional if and only if for each $x \in X$, the ideal O_x is generated by a set of idempotents. Hence for each $y \in X_z$ ($y = Q_x$ for some $x \in X$), the ideal O_y is generated by a set of idempotents. Now in view of Theorem 2.4 in [4] and using the isomorphism φ defined in the proof of Theorem 3.1, the following corollary is evident.

Corollary 4.6. *For each $x \in X$, the ideal O_{Q_x} in $C_s(X)$ is generated by a set of idempotents.*

Remark 4.7. By Theorem 4.9 in [6], two compact spaces X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic. It is clear that if $X \cong_s Y$, then $C_s(X) \cong C_s(Y)$. In fact, if $\sigma : Y \to X$ is a s-homeomorphism, then $f \to f \circ \sigma$ is a s-isomorphism between $C_s(X)$ and $C_s(Y)$. But in contrast to Theorem 4.9 in [6], we observe that $C_s(X) \cong C_s(Y)$ does not necessarily imply $X \cong_s Y$ even if X and Y are mildly compact. To see this, consider spaces $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = (\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})) \cup \{0\}$ as subspaces of $\mathbb{R}$. Clearly, X and Y are mildly compact. Also $C_s(X) \cong C_s(Y)$, in fact, every $g \in C_s(Y)$ is constant on each interval $(\frac{1}{n+1}, \frac{1}{n})$, by Proposition 2.1, say $g((\frac{1}{n+1}, \frac{1}{n})) = \{a_n\}$. Now if we define $f_g : X \to \mathbb{R}$ by $f_g(\frac{1}{n}) = a_n$ and $f_g(0) = g(0)$, then $f_g \in C_s(X)$ and $\theta : C_s(Y) \to C_s(X)$, $\theta(g) = f_g$ for each $g \in C_s(Y)$ is an isomorphism. Since there is no bijection function between X and Y, these two spaces are not s-homeomorphic.

Proposition 4.8. *If $X_z \cong Y_z$, then $C_s(X) \cong C_s(Y)$ and whenever X and Y are mildly compact, then the converse is also true.*

Proof. $X_z \cong Y_z$ implies that $C(X_z) \cong C(Y_z)$ and hence by Theorem 3.1, $C_s(X) \cong C_s(Y)$. For the converse, $C_s(X) \cong C_s(Y)$ implies $C(X_z) \cong C(Y_z)$

by Theorem 3.1. Now using Proposition 4.3, X_z and Y_z are compact, hence $X_z \cong Y_z$, by Theorem 4.9 in [6]. $\square$

5. Some relations between algebraic properties of $C_s(X)$ and topological properties of X

We call a space X a P_s-space if every zero-set in $Z_s(X)$ is open. Clearly, every P-space is a P_s-space but not conversely. For instance, $X = (0,1) \cup (1,2)$ as a subspace of $\mathbb{R}$ is not a P-space whereas it is a P_s-space, for $Z_s(X) = \{\varnothing, X, (0,1), (1,2)\}$, by Corollary 2.2. Every P_s-space is not necessarily a completely regular space. it is enough to consider a non-completely regular space with two components.

Whenever $f \in C_s(X)$, then $Z(f_z) = \{Q_x : f(x) = 0\}$ and $Z(f) = \bigcup_{Q_x \in Z(f_z)} Q_x$. These imply, by definition of open sets in X_z, that $Z(f)$ is s-open in X if and only if $Z(f_z)$ is open in X_z. On the oder hand, since $C_s(X) \cong C(X_z)$, by Theorem 3.1, the ring $C(X_z)$ is regular if and only if $C_s(X)$ is a regular ring. In view of these points, the following result is an immediate consequence of Problem 4J in [6].

Proposition 5.1. *A space X is a P_s-space if and only if $C_s(X)$ is a regular ring.*

The counterparts of the other conditions of Problem 4J in [6] can be obtained more or less for regularity of $C_s(X)$. For example, $C_s(X)$ is regular if and only if $M_{Q_x} = O_{Q_x}$, for each $x \in X$ if and only if every ideal in $C_s(X)$ is a z-ideal and so on. Note that for each $f, g \in C_s(X)$, it is easy to see that $Z(f) \subseteq Z(g)$ if and only if $Z(f_z) \subseteq Z(g_z)$ and this implies that an ideal I in $C_s(X)$ is a z-ideal if and only if $\varphi(I)$ is a z-ideal in $C(X_z)$.

We already observed that every P_s-space is not necessarily a P-space. By the following result, this happens if and only if X is zero-dimensional.

Proposition 5.2. *A space X is a P-space if and only if X is a zero-dimensional P_s-space.*

Proof. Clearly, every P-space is basically disconnected, hence using Problem 16O in [6], every P-space is zero-dimensional. Every P-space is a P_s-space as well. Conversely, since X is zero-dimensional, $C(X) = C_s(X)$ by Lemma 1.1 and since X is a P_s-space, $C_s(X)$ is a regular ring, by Proposition 5.1. This implies that $C(X)$ is also a regular ring. Now using Problem 4J in [6], X is a P-space. $\square$

We call a space X an almost P_s-space if every non-empty zero-set in $Z_s(X)$ has a non-empty s-interior. However the notion of almost P_s-space is the counterpart of that of almost P-space but the class of almost P-spaces and the class of almost P_s-spaces are dissimilar. The following example shows that these classes are not comparable and non of them is larger than the other.

Example 5.3. Whenever every quasi-component in a space X is open, in particular, if X is locally connected, then X is a P_s-space. In fact if $Z(f) \neq \varnothing$,

$f \in C_s(X)$, then $Z(f)$ is a union of quasi-components in X, by Corollary 2.2. Since each quasi-component in X is clopen, $Z(f)$ is s-open. This implies that $Y = (0,1) \cup (1,2)$ as a subspace of $\mathbb{R}$ is a P_s-space. Hence Y is an almost P_s-space but clearly, it is not an almost P-space. Also every almost P-space need not be an almost P_s-space. To see this let X be a (completely regular Hausdorff) connected almost P-space, see Proposition 2.3 in [8] for existence of such a space. Take a point $\sigma \in X$ and let $Y = X \cup \mathbb{N}$ with the topology as follows: elements of $\mathbb{N}$ are considered to be isolated points, neighborhoods of all points of X, except σ will be the same as in the space X while each neighborhood of σ in Y will be of the form $G \cup A$, where G is an open set in X containing σ and A is a subset of $\mathbb{N}$ such that $\mathbb{N} \setminus A$ is finite. If we define $f : Y \to \mathbb{R}$ with $f(n) = \frac{1}{n}$ and $f(X) = \{0\}$, then $f \in C_s(Y)$. Since X is connected and $X = \bigcap_{n=1}^{\infty}(Y \setminus \{n\})$, it is a quasi-component in Y. Also it does not contain any clopen subset of Y (note that X itself is not open in Y, for σ is the cluster point of the subset $\mathbb{N}$ of Y and hence $\mathbb{N}$ is not closed in Y). Hence $\text{int}_s Z(f) = \varnothing$, i.e., Y is not an almost P_s-space.

It remains to show that Y is an almost P-space. Let $f \in C(Y)$. If $Z(f) \cap \mathbb{N} \neq \varnothing$, then clearly $\text{int}_Y Z(f) \neq \varnothing$. Now suppose that $Z(f) \cap \mathbb{N} = \varnothing$. Whenever $\sigma \notin Z(f)$, then $\text{int}_Y Z(f) = \text{int}_X Z(f|_X) \neq \varnothing$, for X is an almost P-space. Finally, suppose that $\sigma \in Z(f)$, then $Z(f) \neq \{\sigma\}$, for otherwise $\text{int}_X Z(f|_X) = \text{int}_X\{\sigma\} \neq \varnothing$ implies that σ is an isolated point of X which contradicts the connectedness of X. Therefore there exists $x \neq \sigma$ such that $x \in Z(f)$. Since Y is completely regular Hausdorff, define $h \in C(Y)$ so that $h(x) = 0$ and $h(\sigma) = 1$. Now take $g = f^2 + h^2$, then $\sigma \notin Z(g) \subseteq X$ implies that $\varnothing \neq \text{int}_X Z(g) = \text{int}_Y Z(g) \subseteq \text{int}_Y Z(f)$. Hence Y is an almost P-space.

For the proof of the following proposition, we need the following lemma.

Lemma 5.4. *Let $f, g \in C_s(X)$.*

(1) *If $Z(g) \subseteq int_s Z(f)$, then f is a multiple of g.*
(2) *$Z_s(X)$ is closed under countable intersection.*

Proof. (1) Let X_z, φ and f_z for each $f \in C_s(X)$ be as in the proof of Theorem 3.1. Let $Q_x \in Z(g_z)$, where $x \in X$. Hence $x \in Z(g)$ and so $x \in \text{int}_s Z(f)$, by our hypothesis. Since $\text{int}_s Z(f)$ is s-open, there exists a clopen set U such that $x \in U \subseteq Z(f)$. Now $H = \{Q_y : y \in U\}$ is clopen in X_z and $Q_x \in H \subseteq Z(f_z)$. This implies that $Z(g_z) \subseteq \text{int}_{X_z} Z(f_z)$ and using Problem 1D in [6], there is $k_z \in C(X_z)$, where $k \in C_s(X)$ such that $f_z = k_z g_z$. Now it is clear that $f = kg$.

(2) It is easy to see that whenever $\{S_n\}$ is a sequence in $C_s(X)$ converges uniformly to a function f, then $f \in C_s(X)$. Now, as in 1.14(a) in [6], if for each $n \in \mathbb{N}$, we consider $Z_n = Z(f_n)$, where $f_n \in C_s(X)$ and $|f_n| \leq 1$ (note that if $f \in C_s(X)$, then $\frac{f}{1+|f|} \in C_s(X)$, $Z(\frac{f}{1+|f|}) = Z(f)$ and $|\frac{f}{1+|f|}| \leq 1$), then the sequence $S_n = \sum_{i=1}^{n} f_i/2^i$ converges uniformly to a function $f \in C_s(X)$. Clearly $\bigcap_{n=1}^{\infty} Z(f_n) = Z(f)$. □

Proposition 5.5. *For a topological space X, the following statements are equivalent.*

(1) *X is an almost P_s-space.*
(2) *For each non-unit $f \in C_s(X)$, $f = ef$ for some idempotent $e \neq 1$ in $C_s(X)$.*
(3) *For each non-unit $f \in C_s(X)$, $fe = 0$ for some idempotent $e \neq 0$ in $C_s(X)$.*
(4) *Every non-empty countable intersection of s-open sets has a non-empty s-interior.*

Proof. (1)⇒(2) If $f \in C_s(X)$ is not unit, then $\text{int}_s Z(f) \neq \varnothing$. Since $\text{int}_s Z(f)$ is s-open, there exists a non-empty clopen set U contained in $Z(f)$. Take the idempotent e with $Z(e) = U$. Clearly $e \neq 1$, for $U \neq \varnothing$. Since $Z(e) \subseteq \text{int}_s Z(f)$, f is a multiple of e, by Lemma 5.4. Hence $f = eg$ for some $g \in C_s(X)$. But $f = g$ on $X \setminus Z(e)$, so $f = ef$.

(2)⇒(3) $f = ef$ implies $f(1-e) = 0$, where $1-e$ is a non-zero idempotent.

(3)⇒(4) Let $A = \bigcap_{n=1}^{\infty} A_n \neq \varnothing$, where each A_n is s-open. Let $x \in A$. Hence there is an idempotent $e_n \in C_s(X)$ such that $x \in Z(e_n) \subseteq A_n$, for each $n \in \mathbb{N}$. Now by Lemma 5.4, $\bigcap_{n=1}^{\infty} Z(e_n)$ is a zero-set, say $Z(g)$, where $g \in C_s(X)$. Since g is non-unit ($x \in Z(g)$), there exists an idempotent $0 \neq e \in C_s(X)$ such that $ge = 0$, by our hypothesis. Therefore $\varnothing \neq Z(1-e) \subseteq Z(g) \subseteq A$ which means that A has a non-empty s-interior.

(4)⇒(1) Since every zero-set in $Z_s(X)$ is a countable intersection of s-open sets, the proof is evident. We note that whenever $f \in C_s(X)$, then $Z(f) = \bigcap_{n=1}^{\infty} f^{-1}((-\frac{1}{n}, \frac{1}{n}))$ and each $f^{-1}((-\frac{1}{n}, \frac{1}{n}))$ is s-open, by Theorem 2.2 in [10]. □

We call a space X s-basically (s-extremally) disconnected if for every zero-set $Z(f) \in Z_s(X)$ (s-closed subset H of X), $\text{int}_s Z(f)$ ($\text{int}_s H$) is s-closed. Equivalently, X is a s-basically (s-extremally) disconnected space if and only if for each $X \setminus Z(f)$, $f \in C_s(X)$ (s-open subset G of X), $\text{cl}_s(X \setminus Z(f))$ ($\text{cl}_s G$) is s-open. We show the counterparts of Theorems 3.3 and 3.5 in [4] that the s-basically (s-extremally) disconnectedness of X is equivalent to saying that $C_s(X)$ is a *p.p.* ring (Baer ring). Recall that a ring R is said to be *p.p.* ring (Baer ring) if for each $a \in R$ ($S \subseteq R$), $\text{Ann}(a)$ ($\text{Ann}S$) is generated by an idempotent, where $\text{Ann}(a) := \{r \in R : ar = 0\}$ ($\text{Ann}S := \{r \in R : rs = 0, \forall s \in S\}$). First we need the following lemma.

Lemma 5.6. *Let X be a topological space and X_z be the space mentioned in the proof of Theorem 3.1.*

(1) *If $f \in C_s(X)$, then $\bigcap_{g \in Ann(f)} Z(g) = cl_s(X \setminus Z(f))$.*
(2) *X is s-extremally (s-basically) disconnected if and only if X_z is extremally (basically) disconnected.*

Proof. (1) Since $\bigcap_{g\in \mathrm{Ann}(f)} Z(g) = \bigcap_{X\setminus Z(f)\subseteq Z(g)} Z(g)$, we have

$$X \setminus Z(f) \subseteq \bigcap_{g\in \mathrm{Ann}(f)} Z(g).$$

But $\bigcap_{g\in \mathrm{Ann}(f)} Z(g)$ is s-closed, hence $\mathrm{cl}_s(X\setminus Z(f)) \subseteq \bigcap_{g\in \mathrm{Ann}(f)} Z(g)$. Conversely, let $x \in \bigcap_{g\in \mathrm{Ann}(f)} Z(g)$ but $x \notin \mathrm{cl}_s(X \setminus Z(f))$. Hence there exists an idempotent $e \in C_s(X)$ such that $e(x) = 1$ and $e(\mathrm{cl}_s(X \setminus Z(f))) = 0$, by Proposition 2.3. This implies that $e \in \mathrm{Ann}(f)$, but $e(x) = 1$ yields that $x \notin \bigcap_{g\in \mathrm{Ann}(f)} Z(g)$, a contradiction.

(2) Let X be s-extremally disconnected and H be an open set in X_z. Then $G = \bigcup_{Q_x\in H} Q_x = \{x \in X : Q_x \in H\}$ is a s-open set in X. Hence cl_sG is s-open, by our hypothesis. Now let $Q_x \in \mathrm{cl}_{X_z}H$. Then every clopen set in X_z containing Q_x intersects H and this implies that every clopen set in X containing x intersects G as well. Therefore $x \in \mathrm{cl}_sG$. But cl_sG is s-open, then there exists a clopen set U in X containing x contained in cl_sG. Now $V = \{Q_y : y \in U\}$ is a clopen set in X_z containing Q_x contained in $\mathrm{cl}_{X_z}H$, i.e., $\mathrm{cl}_{X_z}H$ is open in X_z and hence X_z is extremally disconnected. The proof of the converse is similar. In case of s-basically disconnectedness, the proof goes along the lines of the above arguments, so it is left to the reader. □

Proposition 5.7. *Let X be a topological space.*

1. *$C_s(X)$ is a p.p. ring if and only if X is a s-basically disconnected space.*
2. *$C_s(X)$ is a Baer ring if and only if X is a s-extremally disconnected space.*

Proof. (1) We may apply each part of Lemma 5.6, we prefer to use part (1). If $C_s(X)$ is a *p.p.* ring, then for each $f \in C_s(X)$, $\mathrm{Ann}(f) = (e)$ for some idempotent e. Now by Lemma 5.6, $Z(e) = \bigcap_{g\in \mathrm{Ann}(f)} Z(g) = \mathrm{cl}_s(X \setminus Z(f))$ which implies that $\mathrm{cl}_s(X \setminus Z(f))$ is clopen and hence it is s-open. Therefore X is a s-basically disconnected space.

Conversely, suppose that X is s-basically disconnected. Hence $\bigcap_{g\in \mathrm{Ann}(f)} Z(g) = \mathrm{cl}_s(X \setminus Z(f))$ is s-open and hence it is clopen. Now take an idempotent e with $Z(e) = \mathrm{cl}_s(X \setminus Z(f))$. Since $X \setminus Z(f) \subseteq \mathrm{cl}_s(X \setminus Z(f)) = Z(e)$, we have $ef = 0$, i.e., $e \in \mathrm{Ann}(f)$. On the other hand if $g \in \mathrm{Ann}(f)$, then $Z(e) = \mathrm{cl}_s(X \setminus Z(f)) \subseteq Z(g)$ implies that $Z(e) \subseteq \mathrm{int}_sZ(g)$ and by Lemma 5.4, $g \in (e)$, i.e., $\mathrm{Ann}(f) \subseteq (e)$.

(2) If $C_s(X)$ is a Baer ring, then $C(X_z)$ is also a Baer ring, for $C_s(X) \cong C(X_z)$, by Theorem 3.1. Now by Theorem 2.5 in [4], X_z is extremally disconnected. Thus using Lemma 5.6, X is s-extremally disconnected. The proof of the converse is similar. □

It is manifest that every basically (extremally) disconnected space is a s-basically (s-extremally) disconnected space. The converse is not true in general. For example let $X = (0, 1) \cup (1, 2)$ be as a subspace of $\mathbb{R}$. In fact X is a P_s-space which is not basically disconnected and it is not extremally disconnected as well. It is not hard to see that every s-basically disconnected almost P_s-space is a P_s-space. The following result states that the zero-dimensionality and s-basically (s-extremally) disconnectedness is equivalent to basically (extremally) disconnectedness.

Proposition 5.8. *A space is basically (extremally) disconnected if and only if it is s-basically (s-extremally) disconnected zero-dimensional.*

Proof. By Problem 16O in [6], every basically (extremally) disconnected space is zero-dimensional and since every basically (extremally) disconnected space is also s-basically (s-extremally) disconnected, the left-to-right implication is immediate. For the converse, whenever X is zero-dimensional, then by Lemma 1.1, $C(X) = C_s(X)$. Now if X is s-basically (s-extremally) disconnected, then by Proposition 5.7, $C(X) = C_s(X)$ is a *p.p.* (Baer) ring. Now by Theorems 3.3 and 3.5 in [4], X is basically (extremally) disconnected. □

The socle $C_F(X)$ of $C(X)$ which is the intersection of all essential ideals in $C(X)$ is the set of all functions which vanish everywhere except on a finite number of isolated points of X, see Proposition 3.3 in [7]. Corollary 2.3 in [2] and Proposition 2.1 in [7] show that the socle of $C(X)$ is essential if and only if the set of isolated points of X is dense in X.

Proposition 5.9. *Let X be a topological space and X_z be the space defined in the proof of Theorem 3.1.*

(1) *The socle $S_s(X)$ of $C_s(X)$ is free if and only if every quasi-component in X is open, if and only if X_z is discrete.*
(2) *The socle of $C_s(X)$ is essential if and only if the union of open quasi-components in X is s-dense in X (A subset D of X is called s-dense in X if every non-empty s-open subset of X intersects D).*

Proof. We remind the reader that a subset H of X_z is open if and only if $\bigcup_{Q_x \in H} Q_x$ is s-open in X. This implies that for each $x \in X$, the quasi-component Q_x is an isolated point of X_z if and only if Q_x is s-open in X and hence it should be clopen. Since $C_F(X_z)$ is the set of all functions in $C(X_z)$ which vanish everywhere except on a finite set of isolated points of X_z, $S_s(X)$ will be the set of all functions in $C_s(X)$ which vanish everywhere except on a finite union of open quasi-components in X. Therefore, $\bigcap_{f \in S_s(X)} Z(f)$ is the union of all non-open quasi-components in X. Now it is clear that $S_s(X)$ is free if and only if every quasi-components in X is open and this is equivalent to saying that X_z is discrete. For the proof of part (2), it is easy to see that the density of isolated points of X_z is equivalent to the density of the union of open quasi-components in X. Since $C_s(X) \cong C(X_z)$, the socle of $C(X_z)$ is essential if and only if the socle of $C_s(X)$ is. Now using Corollary 2.3 in [2], we are done. □

It is known that every extremally disconnected P-space of non-measurable cardinal is discrete, see Problem 12H in [6]. We conclude the paper by the counterpart of this fact.

Proposition 5.10. *Every quasi-component in a s-extremally disconnected P_s-space of non-measurable cardinal is open.*

Proof. Let X be s-extremally disconnected P_s-space of non-measurable cardinal. Then $C_s(X)$ is a Baer ring by Proposition 5.7. But using Proposition 3.1, $C_s(X) \cong C(X_z)$, hence $C(X_z)$ is also a Baer ring. Therefore X_z is extremally disconnected, by Theorem 3.5 in [4]. On the other hand, since X is P_s-space, $C_s(X)$ will be regular by Proposition 5.1, whence $C(X_z)$ is also regular and hence X_z is a P-space by Problem 4J in [6]. Finally, $|X_z| \leq |X|$ implies that the cardinal of X_z is non-measurable, see part (i) in the proof of Theorem 12.5 in [6]. Now X_z is extremally disconnected P-space of non-measurable cardinal which means that X_z is discrete, by Problem 12H in [6]. Therefore each Q_x is an isolated point in X_z and hence each Q_x should be open in X. □

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References

[1] S. Afrooz, F. Azarpanah and O. A. S. Karamzadeh, Goldie dimension of rings of fractions of $C(X)$, Quaest. Math. 38, no. 1 (2015), 139–154.

[2] F. Azarpanah, Intersection of essential ideals in $C(X)$, Proc. Amer. Math. Soc. 125, no. 7 (1997), 2149–2154.

[3] F. Azarpanah, On almost P-spaces, Far East J. Math. Sci. Special volume (2000), 121-132.

[4] F. Azarpanah and O. A. S. Karamzadeh, Algebraic characterizations of some disconnected spaces, Italian J. Pure Appl. Math. 12 (2002), 155–168.

[5] R. Engelking, General Topology, Sigma Ser. Pure Math., Vol. 6, Heldermann Verlag, Berlin, 1989.

[6] L. Gillman and M. Jerison, Rings of Continuous Functions, Springer-Verlag, 1976.

[7] O. A. S. Karamzadeh and M. Rostami, On the intrinsic topology and some related ideals of $C(X)$, Proc. Amer. Math. Soc. 93 (1985), 179–184.

[8] R. Levy, Almost P-spaces, Can. J. Math. XXIX, no. 2 (1977), 284–288.

[9] I. L. Reilly and M. K. Vamanamurthy, On supercontinuous mappings, Indian J. Pure Appl. Math. 14, no. 6 (1983), 767-772.

[10] D. Singh, cl-supercontinuous functions, Applied Gen. Topol. 8, no. 2 (2007), 293–300.

[11] R. Staum, The algebra of bounded continuous functions into non-archimedean field, Pacific J. Math. 50, no. 1 (1974), 169–185.

[12] S. Willard, General Topology, Addison-Wesley Publishing Company, Inc., 1970.

