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Topological characterization of Gelfand and zero dimensional semiring

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ABSTRACT

Let R be a conmutative semiring with 0 and 1, and let $Spec(R)$ be the set of all proper prime ideals of R. $Spec(R)$ can be endowed with two topologies, the Zariski topology and the D-topology. Let $MaxR$ denote the set of all maximal prime ideals of R. We prove that the two topologies coincide on $Spec(R)$ and on $MaxR$ if and only if R is zero dimensional and Gelfand semiring, respectively.

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KEYWORDS: *Zariski topology; D-topology; conmutative semiring; Gelfand semiring; zero dimensional semiring.*

1. BASIC FACTS

Recall that a semiring (conmutative with non zero identity) is an algebra $(R, +, \cdot, 0, 1)$, where R is a set with $0, 1 \in S$, and $+$ and $\cdot$ are binary operations on R called sum and multiplication, respectively, which satisfy the following:

(1) $(R, +, 0)$ and $(R, \cdot, 1)$ are conmutative monoid with $1 \neq 0$.
(2) $a \cdot (b + c) = a \cdot b + a \cdot c$ for every $a, b, c \in R$.
(3) $a \cdot 0 = 0$ for every $a \in R$.

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A subset I of R will be called an *ideal* of R if $a, b \in I$ and $r \in R$ implies $a+b \in I$ and $ra \in I$. A *prime ideal* of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. The *nilradical* of R, denoted by $N(R)$, is the intersection of all the prime ideals de R. $Max(R)$ and $Min(R)$ denote the set of all maximal and minimal prime ideals of R, respectively. R is said to be *Gelfand* if every prime ideal is contained in at most one maximal ideal. R is said to be *zero dimensional* if every prime ideal of R is maximal.

For $x \in R$, let $(0:x) = \{y \in R : \ xy = 0\}$. An ideal I of R is called a *ó-ideal* if for every $x \in R$, $I(0:x) = R$, that is to say there exist $x_1 \in I$ and $y \in (0:x)$ such that $1 = x_1 y$. A element $x \in R$ is called a *complemented element* in R if there is $y \in R$ such that $xy = 0$ and $x + y = 1$, y is called the *complement of* x.

For any semiring R, $Spec(R)$ denotes the set of all proper prime ideals of R. This set can be given the Zariski topology τ_z as follows: For every proper set I of R, let $(I)_0 = \{P \in Spec(R) : I \subseteq P\}$ and let $D(I) = Spec(R) \setminus (I)_0 = \{P \in Spec(R) : I \nsubseteq P\}$. If I is the ideal generated by $a \in S$, we write $I = (a)$. Note that $(a)_0 = \{P \in Spec(R) : a \in P\}$ and $D(a) = \{P \in Spec(R) : a \notin P\}$. The sets $D(a) \subseteq Spec(R)$ with $a \in R$, constitute a basis for τ_z, and the sets $(I)_0$ with I ideal of R are the closed sets for τ_z.

Let (X, τ) a topological space, τ^* denote the family of τ-closed subset of X. τ is said to be *Alexandroff* if it is closed under arbitrary intersections. By identifying a set with its characteristic function, we can view τ as a subset of 2^X with the product topology, then its closure $\overline{\tau}$ is also a topology, even more,

$$\overline{\tau} = \left\{ A \subseteq X : \ A = \bigcap_{\theta \in \mathcal{L}} \theta, \, \mathcal{L} \subseteq \tau \right\}$$

and it is the smallest Alexandroff topology containing τ_z (see [5]).

Note that $A \in \overline{\tau}^*$ if and only if A^c is $\overline{\tau}$-open, let say $A^c = \bigcap_{\theta \in \mathcal{L}} \theta$ for some $\mathcal{L} \subseteq \tau$, then

$$\begin{aligned} x \in A \Rightarrow x \notin A^c \quad &\Rightarrow \quad x \notin \theta, \text{ for some } \theta \in \mathcal{L} \\ &\Rightarrow \quad x \in \theta^c, \text{ for some } \theta \in \mathcal{L} \\ &\Rightarrow \quad \overline{\{x\}} \subseteq \overline{\theta^c} = \theta^c \\ &\Rightarrow \quad \overline{\{x\}} \subseteq \left(\bigcap_{\theta \in \tau} \theta \right)^c = A \end{aligned}$$

and A is just the union of the τ-closure of each of its points.

A subset A of a topological space (X, τ) is *τ-saturated* if $\overline{\{a\}} \subseteq A$ for all $a \in A$, that is to say, if $A \in \overline{\tau}^*$. In particular, $A \subseteq Spec(R)$ is τ_z-saturated if and only if for each $P \in A$, $(P)_0 \subseteq A$. The set of all τ_z-open and τ_z-saturated subsets of $Spec(R)$ defines a topology on $Spec(R)$ called the *D-topology*, this is to say that the D-topology is just $\tau_Z \cap \overline{\tau_Z}^*$.

Remember that a topology τ on X is said T_0 if for each pair of distinct elements x and y in X, exist a open set containing either x or y, and τ is T_1 if for each pair of distinct elements x and y in X, exist a open set containing x and not y and an open set containing y and not x.

The following results, given in [2] and [5], characterize the topologies T_0 and T_1 of the following manner:

Theorem 1.1. *Let τ be a topology on X then,*

(i) τ *is* T_0 *if and only if* $\overline{\tau}$ *is* T_0.
(ii) τ *is* T_0 *if and only if* $\overline{\tau} \vee \overline{\tau}^* = \wp(X)$
(iii) τ *is* T_1 *if and only if* $\overline{\tau} = \wp(X)$.

In [1], Al-Ezeh endowed $Spec(L)$, where L is a distributive lattice with 0 and 1, with two topologies, the τ_z-topology and the D-topology, and he proved that this two topologies coincide on $Spec(L)$ and $Max(L)$ iff L is a boolean and normal lattice, respectively. In [3], Rafi and Rao introduced the concept of D-topology on $Spec(R)$, where R is a almost distributive lattice (ADL), and characterized those ADLs for which topologies coincide on $Spec(R)$ and $Min(R)$. In this paper, the concept of D-topology is introduced on $Spec(R)$, where R is a semiring, we do a similar study, as a consequence, we obtain a result given in [1] for distributive lattices.

2. Main results

We begin by establishing some relationships between the τ_z-open sets and D-open and between the τ_z-clopen and the D-clopen.

Remark 2.1. If I is a ó-ideal of a semiring R, then $D(I)$ is D-open. In effect, let $P \in D(I)$, we will show that $(P)_0 \subseteq D(I)$. Let $Q \in (P)_0$, since $P \in D(I)$, $I \nsubseteq P$, hence exists $x \in I$ such that $x \notin P$. Since I is a ó-ideal, there exist $x_1 \in R$ and $y \in (0 : x)$ such that $x_1 + y = 1$, note that $y \in P \subseteq Q$ (because $xy = 0 \in P$ and $x \notin P$) so $x_1 \notin Q$ (otherwise $1 = x_1 + y \in Q$) implying $I \nsubseteq Q$, in consequence, $Q \in D(I)$.

The reciprocal of the previous remarks it is not true, as shown in the following example.

Example 2.2. Let A a non-empty subset of a set X, and let $L = \{\varnothing, A, A^c, X\}$, $(L, \cup, \cap)$ is a semiring where the sum and multiplication are the union and intersection, respectively, and the identities of the sum and multiplication are the empty set and the whole set X, even more $(L, \cup, \cap)$ is a distributive lattice. The ideals of L are $\{\varnothing\}$, $\langle A\rangle = \{\varnothing, A\}$, $\langle A^c\rangle = \{\varnothing, A^c\}$, $\langle X\rangle = L$. $Spec(L) = \{\langle A\rangle, \langle A^c\rangle\}$ and $D(\langle A\rangle) = \{\langle A^c\rangle\}$, clearly $D(\langle A\rangle)$ is τ-saturated, but $\langle A\rangle$ it is not a ó-ideal, since $(\varnothing : A^c) = \{\varnothing, A\}$ and $\langle A\rangle \cup (\varnothing : A^c) = \{\varnothing, A\} \neq L$.

Proposition 2.3. *Let R be a semiring with trivial nilradical and let I an ideal of R. Then $D(I)$ is D-clopen if and only if $D(I) = D(x)$ for some complemented element x in R.*

Proof. Assume that $D(I)$ is clopen. Then $Spec(R) \setminus D(I)$ is also an open set, so there exists an ideal J of R such that $D(J) = Spec(R) \setminus D(I)$. Now $D(I) \cap D(J) = D(IJ) = \varnothing$, this implies $IJ \subseteq P$ for all $P \in Spec(R)$, this is, $IJ \subseteq N(R) = \{0\}$. Also now, $Spec(R) = D(I) \cup D(J) = D(I + J)$, this implies $I + J = R$, thus exist $x \in I$ and $y \in J$ such that $x + y = 1$. Since $IJ = \{0\}$, we have $xy = 0$, then x is complemented. We see that $I = (x)$, let $z \in I$, $z = z1 = z(x + y) = zx + zy = zx \in (x)$ since $zy \in IJ = \{0\}$. Thus I

is a principal ideal generated by x, in consequence $D(I) = D(x)$. Conversely, assume that x is a complemented element in R, then there exists an element $y \in R$ such that $xy = 0$ and $x+y = 1$. Now $D(x) \cap D(y) = D(xy) = D(0) = \varnothing$ and $D(x) \cup D(y) = D(x+y) = D(1) = Spec(R)$. Therefore $D(x)$ is clopen. $\square$

Now we characterize those semiring for which the Zariski topology and the D-topology coincide on $Spec(R)$.

Theorem 2.4. *Let R be a semiring. Then the Zariski topology and the D-topology coincide on $Spec(R)$ if and only if R is zero dimensional.*

Proof. Note that W is D-open if and only if $W \in \tau_z \cap \overline{\tau_z}^*$, thus the Zariski topology and the D-topology coincide on $Spec(R)$ if and only if $\tau_z = \tau_z \cap \overline{\tau_z}^*$, but

$$\begin{aligned}
\tau_z = \tau_z \cap \overline{\tau_z}^* \quad &\Leftrightarrow \quad \tau_z \subseteq \overline{\tau_z}^* \\
&\Leftrightarrow \quad \overline{\tau_z} \subseteq \overline{\tau_z}^* \\
&\Leftrightarrow \quad \overline{\tau_z} = \overline{\tau_z}^* \\
&\Leftrightarrow \quad \overline{\tau_z} \vee \overline{\tau_z}^* = \overline{\tau_z} \\
&\Leftrightarrow \quad \overline{\tau_z} = \wp(Spec(S)) \quad (\overline{\tau_z} \text{ is } T_0 \text{ and by Theorem 1.1 part } (i)) \\
&\Leftrightarrow \quad \tau_z \text{ is } T_1 \quad (\text{ by Theorem 1.1 part } (iii))
\end{aligned}$$

Now τ_z is T_1 if and only if every $P \in Spec(R)$ is closed impliying $\{P\} = (P)_0$, or equivalently, every prime ideal of R is maximal, this is, R is zero dimensional. $\square$

Theorem 2.5 ([4])**.** *Let L be a distributive lattice. Then L is a boolean algebra if and only if every prime ideal of L is a maximal ideal.*

As a consequence of the Theorem 2.4 we obtain the following result given in [1] for distributive lattices.

Corollary 2.6. *Let L be a distributive lattice with 0 and 1. Then the Zariski topology and the D-topology coincide on $Spec(R)$ if and only if L is a boolean lattice (a lattice every element of which has a complement).*

Proof. immediately of Theorem 2.4 and Theorem 2.5, since every lattice is a semiring. $\square$

Theorem 2.7. *R is a Gelfand semiring if and only if τ_z and $\tau_z \cap \overline{\tau_z}^*$ agree on $Max(R)$.*

Proof. Suppose R is a Gelfand semiring. We want to prove that τ_z and $\tau_z \cap \overline{\tau_z}^*$ agree on $Max(R)$. Since $\tau_z \cap \overline{\tau_z}^* \subseteq \tau_z$, it remains to show that $\tau_z \subseteq \tau_z \cap \overline{\tau_z}^*$ on $Max(R)$. So we want to show that each $D(x)$ in τ_z restricted to $Max(R)$ is an open set in $\tau_z \cap \overline{\tau_z}^*$ restricted to $Max(R)$. For each $x \in R$, let $D_x^* = D(x) \cap Max(R)$ and let

$$W = \{P \in Spec(R) : x \notin M_P\},$$

where M_P is the unique maximal ideal of R containing P. Let us prove then that D^*_x is a $\tau_z \cap \overline{\tau_z}^*$-open subset in $Max(R)$. Now, for each $P \in W$ we have that $(P)_0 \subseteq W$, then $W \in \overline{\tau}^*_z$. Since $D^*_x = W \cap Max(R)$ then, D^*_x is a $\tau_z \cap \overline{\tau_z}^*$-open set in $Max(R)$. Therefore the conclusion follows.

Conversely, suppose τ_z and $\tau_z \cap \overline{\tau_z}^*$ agree on $Max(R)$ and take $P \in Spec(R)$ which is contained in two different maximal ideals M and N. Take, without lost of generality, $x_0 \in M$ such that $x_0 \notin N$. So $N \in D(x_0) \cap Max(R) = W \cap Max(R)$ for some $\tau_z \cap \overline{\tau_z}^*$ open set W. Now since $P \subseteq N$ and $N \in W$ it follows that $(P)_0 \subseteq W$. Therefore $(P)_0 \cap Max(R) \subseteq W \cap Max(R)$ impliying that $x_0 \notin M$, which is a contradiction. $\square$

Question 2.8. *Under what conditions on R, the τ_z-topology and the D-topology coincide on $Min(R)$?*

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