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On the essentiality and primeness of λ-super socle of $C(X)$

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Dedicated to professor O.A.S. Karamzadeh on the occasion of his retirement and to appreciate his peerless activities in mathematics (especially, popularization of mathematics) for nearly half a century in Iran

Abstract

Spaces X for which the annihilator of $S_\lambda(X)$, the λ-super socle of $C(X)$ (i.e., the set of elements of $C(X)$ that cardinality of their cozerosets are less than λ, where λ is a regular cardinal number such that $\lambda \le |X|$) is generated by an idempotent are characterized. This enables us to find a topological property equivalent to essentiality of $S_\lambda(X)$. It is proved that every prime ideal in $C(X)$ containing $S_\lambda(X)$ is essential and it is an intersection of free prime ideals. Primeness of $S_\lambda(X)$ is characterized via a fixed maximal ideal of $C(X)$.

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1. Introduction

Unless otherwise mentioned all topological spaces are infinite Tychonoff and we will employ the definitions and notations used in [11] and [7]. $C(X)$ is the ring of all continuous real valued functions on X. The socle of $C(X)$, denoted by $C_F(X)$, is the sum of all minimal ideals of $C(X)$ which plays an important role in the structure theory of noncommutative Noetherian rings, see [12], but

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O.A.S. Karamzadeh initiated the research regarding the socle of $C(X)$ (see [16]), which is the intersection of all essential ideals in $C(X)$ (recall that, an ideal is essential if it intersects every nonzero ideal nontrivially), see[12] and [16]. Also the minimal ideals and the socle of $C(X)$ are characterized via their corresponding z-filters; see [16]. In [10] and [15], the socle of $C_c(X)$ (the functionally countable subalgebra of $C(X)$), and $L_c(X)$ (the locally functionally countable subalgebra of $C(X)$), are investigated. The concept of the super socle is introduced in [8], denoted by $SC_F(X)$, which is the set of all elements f in $C(X)$ such that $coz(f)$ is countable. Clearly, $SC_F(X)$ is a z-ideal containing $C_F(X)$. Recently, the concept of $SC_F(X)$ has been generalized to the λ-super socle of $C(X)$, $S_\lambda(X)$, where $S_\lambda(X) = \{f \in C(X) : |X \setminus Z(f)| < \lambda\}$, in which λ is a regular cardinal number with $\lambda \leq |X|$, is introduced and studied in [17]. It is manifest that $C_F(X) = S_{\aleph_0}(X)$ and $SC_F(X) = S_{\aleph_1}(X)$. It turns out, in this regard, the ideal $C_F(X)$ plays an important role in both concepts. As we know the prime ideals are very important in the context of $C(X)$. It turns out that every prime ideal in $C(X)$ is either an essential ideal or a maximal one, therefore the study of essential ideals in $C(X)$ is worthwhile. It is easy to see that for any ideal I in any commutative ring R, the ideal $I + Ann(I)$, where $Ann(I) = \{x \in X : xI = (0)\}$ is the annihilator of I, is an essential ideal in R. Hence an ideal I in a reduced ring is an essential ideal if and only if $Ann(I) = (0)$ (note: it suffices to recall that R is reduced if and only if $Z(R) = \{x \in R : Ann(x) \text{ is essential in R}\} = (0)$). In [16, Proposition 2.1], it is proved that $C_F(X)$ is an essential ideal in $C(X)$ if and only if the set of all isolated points of X is dense in X. We note that in this case the socle is the smallest essential ideal in $C(X)$. Also the ideal $SC_F(X)$ (the super socle of $C(X)$) is an essential ideal in C(X) if and only if the set of countably isolated points of X is dense in X, see [8, Corollary 3.2]. Similarly, in what follows, we aim to relate the density of the set of λ-isolated points to an algebraic property of $C(X)$. In [3, Proposition 2.5], it is shown that the socle of $C(X)$, i.e., $C_F(X)$ is never a prime ideal in $C(X)$, but in [8], it is seen that $SC_F(X)$ can be a prime ideal (or even a maximal ideal) which this may be considered as an advantage of $SC_F(X)$ over $C_F(X)$. In this article we will see that $S_\lambda(X)$ can be a prime ideal, as well.

In Section 2, some concepts and preliminary results which are used in the subsequent sections are given. In Section 3, we deal with the essentiality of $S_\lambda(X)$ and also with the essential ideals containing $S_\lambda(X)$. In this section, we characterize spaces X for which the annihilator of $S_\lambda(X)$ is generated by an idempotent. Consequently, this enables us to find an algebraic property equivalent to the density of the set of λ-isolated points in a space X. In contrast to the fact that $C_F(X)$ is never a prime ideal in $C(X)$, in Section 4, we characterize spaces X for which $S_\lambda(X)$ is a prime ideal (even maximal ideal).

In the final section, for a class of topological spaces, including maximal λ-compact ones, we prove that the λ-super socle of $C(X)$ is the intersection of the essential ideals O_x containing $S_\lambda(X)$, where x runs through the set of non-λ-isolated points in X. Also we show that the z-filter corresponding to the λ-super socle of $C(X)$ is the intersection of all essential z-filters containing $S_\lambda(X)$.

2. Preliminaries

First we cite the following results and definitions which are in [14] and [17].

Definition 2.1. An element $x \in X$ is called a λ-isolated point if x has a neighborhood with cardinality less than λ. The set of all λ-isolated points of X is denoted by $I_\lambda(X)$. If every point of X is λ-isolated, then X is called a λ-discrete space, i.e., $I_\lambda(X) = X$.

Definition 2.2. A topological space X is said to be λ-compact whenever each open cover of X can be reduced to an open cover of X whose cardinality is less than λ, where λ is the least infinite cardinal number with this property.

Definition 2.3. X is a P_λ-space if every intersection of a family of cardinality less than λ of open sets (i.e., G_λ-set) is open.

We begin with the following well-known result for $S_\lambda(X)$, see [17, Lemma 2.6].

Theorem 2.4. *$\bigcap Z[S_\lambda(X)]$ is equal to the set of non-λ-isolated points, i.e., $\bigcap Z[S_\lambda(X)] = X \setminus I_\lambda(X)$. In particular, if $x \in X$ is a λ-isolated point, then there exists $f \in S_\lambda(X)$, such that $f(x) = 1$.*

Corollary 2.5. *For any space X the following statements hold.*

(1) *An element $x \in X$ is a λ-isolated point if and only if $M_x + S_\lambda(X) = C(X)$.*
(2) *X is a λ-discrete space if and only if for all $x \in X$, $M_x + S_\lambda(X) = C(X)$.*
(3) *The ideal $S_\lambda(X)$ is a free ideal in $C(X)$ if and only if for all $x \in X$,*
$$M_x + S_\lambda(X) = C(X).$$
(4) *An element $x \in X$ is non-λ-isolated point if and only if $S_\lambda(X) \subseteq M_x$.*
(5) *If $|X| \geq \lambda$ and $|I_\lambda(X)| < \lambda$, then $S_\lambda(X) = \bigcap_{x \in X \setminus I_\lambda(X)} M_x$.*

3. On the essentiality of $S_\lambda(X)$ in $C(X)$

We begin with the following theorem, which is, in fact, our main result in this section.

Theorem 3.1. *$Ann(S_\lambda(X)) = (e)$, where e is an idempotent in $C(X)$ if and only if $X = A \cup B$, where A and B are two disjoint open subsets of X such that the set of λ-isolated points of X is a dense subset of A and B has no λ-isolated points of X .*

Proof. Let us first get rid of the case that $Ann(S_\lambda(X)) = (1)$. Clearly, this case holds if and only if $S_\lambda(X) = (0)$, or equivalently if and if X has no λ-isolated point, since $1.g = 0$, for each $g \in S_\lambda(X)$, i.e., $S_\lambda(X) = (0)$. Conversely, if $S_\lambda(X) = (0)$, then $Ann(S_\lambda(X)) = C(X) = (1)$. So put $X = A \cup B$, where $A = \phi$ and $B = X$, see Theorem 2.4. Now let $Ann(S_\lambda(X)) = (e)$, where e is an idempotent in $C(X)$ and $H = I_\lambda(X)$ be the set λ-isolated points of X. We claim $cl(H) = Z(e)$. In view to Theorem 2.4, for each $x \in H$, there exists $f \in S_\lambda(X)$ such that $f(x) = 1$. But by assumption, $ef = 0$, implies $e(x) = 0$, i.e., $H \subseteq Z(e)$ and consequently $cl(H) \subseteq Z(e)$. Now let $x \in Z(e) \setminus cl(H)$ and seek a contradiction. By complete regularity of X, there exists $g \in C(X)$, such that $g(x) = 1$ and $g(cl(H)) = (0)$. On the other hand for each $y \in X \setminus H$ and every $f \in S_\lambda(X)$, we have $f(y) = 0$, see Theorem 2.4, this implies that $gf = 0$, for every $f \in S_\lambda(X)$, which in turn implies $g \in Ann(S_\lambda(X)) = (e)$. Since $x \in Z(e)$ and $g = he$, $g(x) = h(x).e(x) = 0$, which is a contradiction. Consequently, $cl(H) = Z(e)$ and so $cl(H)$ is clopen. Now put $A = cl(H)$ and $X \setminus cl(H) = B$, thus we are done. Conversely, let $X = A \cup B$ such that A and B are two disjoint open subsets of X, where A and B have the assumed properties. We may define

$$e(x) = \begin{cases} 0 & , x \in A \\ 1 & , x \in B \end{cases}$$

It is clear $e \in C(X)$ and $e^2 = e$. We claim $Ann(S_\lambda(X)) = (e)$. If $f \in S_\lambda(X)$ then $|X \setminus Z(f)| < \lambda$ and this implies $X \setminus Z(f) \subseteq A = Z(e)$, i.e., $fe = 0$ or $e \in Ann(S_\lambda(X))$. It reminds to be shown that if $f \in Ann(S_\lambda(X))$, then $f \in (e)$. First, we prove that if $f \in Ann(S_\lambda(X))$, then $Z(e) \subseteq Z(f)$. To see this, put $H = I_\lambda(X)$, since for each $x \in H$, we infer that there exists $g \in S_\lambda(X)$ such that $g(x) = 1$. Hence $(fg)(x) = 0$ implies that $f(x) = 0$, for every $x \in H$. So $f(cl(H)) = 0$ (note, $f(cl(H)) \subseteq clf(H)$). So $cl(H) = A = Z(e) \subseteq Z(f)$, and since $Z(e)$ is clopen, $Z(e) \subseteq int\, Z(f)$ and by [11, Problem 1D], f is a multiple of e, thus $f \in (e)$ and we are done. □

As previously mentioned, the set of isolated points in a space X is dense if and only if the socle of $C(X)$ is essential. Similarly, in [8, Corollary 3.2], it has shown that the ideal $SC_F(X)$ is an essential ideal if and only if the set of countably isolated points of X is dense in X. But in the following corollary, we generalize this result for λ-super socle.

Corollary 3.2. *The ideal $S_\lambda(X)$ is an essential ideal in $C(X)$ if and only if the set of λ-isolated points of X is dense in X.*

Proof. Let $S_\lambda(X)$ be essential ideal, as the previous result $Ann(S_\lambda(X)) = (0)$, see[1, Proposition 3.1]. Therefore by the comment preceding Theorem 3.1, $e = 0$ and $A = Z(e) = X$, i.e., $I_\lambda(X)$ is dense in X. Conversely, let $cl(I_\lambda(X)) = X$, since $int(\bigcap Z[S_\lambda(X)]) = int((I_\lambda(X))^c) = (cl(I_\lambda(X))^c = \phi$, we infer that $S_\lambda(X)$ is essential in $C(X)$, see[1, Proposition 3.1]. □

Clearly, every essential ideal in any commutative ring R contains the socle of R. Now the following definition is in order.

Definition 3.3. An essential ideal in $C(X)$ containing $S_\lambda(X)$ is called a λ-essential ideal where λ is a cardinal number greater than or equal $\aleph_0$.

It is well known that the intersection of the essential ideals in a commutative ring R is equal to the socle of R. More generally, any ideal containing the socle of R is also an intersection of essential ideals, see [13, 3N]. It is obvious that $S_\lambda(X)$ is the intersection of the λ-essential ideals of $C(X)$.

Proposition 3.4. *Let X be a λ-discrete space, then the set of λ-essential ideals and the set of free ideals containing $S_\lambda(X)$ coincide. In particular, $S_\lambda(X)$ is the intersection of free ideals containing it.*

Proof. Let X be a λ-discrete space and E be a free ideal containing $S_\lambda(X)$, it is well known that every free ideal in $C(X)$ is an essential ideal, see [2, Proposition 2.1] and the comment preceding it, hence E is a λ-essential ideal which implies that the set of λ-essential ideals and the set of free ideals containing $S_\lambda(X)$ coincide. □

It is clear that every maximal ideal containing the socle of any commutative ring is essential, see [16]. So each maximal ideal M containing $S_\lambda(X)$ is λ-essential, since $C_F(X) \subseteq S_\lambda(X)$. We also recall that every prime ideal in $C(X)$ is either essential or it is a maximal ideal which is generated by idempotent and it is a minimal prime too, see [4]. In view of these facts and using the above proposition and the fact that $S_\lambda(X)$ is a z-ideal (hence it is an intersection of prime ideals), we immediately have the following proposition.

Proposition 3.5. *Every prime ideal P in $C(X)$ containing $S_\lambda(X)$ (or even $C_F(X)$) is an essential ideal. In particular if X is a λ-discrete space, then $S_\lambda(X)$ is an intersection of free prime ideals.*

4. On the primeness of $S_\lambda(X)$ in C(X)

Our main aim in this section is to investigate the primeness of the λ-super socle. First, we give an example to show that $S_\lambda(X)$ can be a prime ideal (even a maximal ideal), which is the difference between $S_\lambda(X)$ and $C_F(X)$.

Example 4.1. Let $X = Y \cup \{x\}$ be one point λ-compactification of a discrete space Y, see [17, Definition 2.11]. We claim that $C(X) = \mathbb{R} + S_\lambda(X)$, i.e., $S_\lambda(X)$ is a real maximal ideal. Let $f \in C(X)$, then we consider two cases. Let us first take $x \in Z(f)$, since X is a P_λ-space, $Z(f)$ is open and so $|X \setminus Z(f)| < \lambda$ implies $f \in S_\lambda(X) \subseteq \mathbb{R} + S_\lambda(X)$. Now, we suppose $x \notin Z(f)$, so there exists $0 \neq r \in \mathbb{R}$ such that $f(x) = r$. Put $g = f - r$, hence $x \in Z(g)$ and therefor $g \in S_\lambda(X)$. We are done.

Using Corollary 2.5, it is evident that if $x \in X$ is the only non-λ-isolated point of X, then M_x is the unique fixed maximal ideal in $C(X)$ such that $S_\lambda(X) \subseteq M_x$. It is well-known that every prime ideal in $C(X)$ is contained in a unique maximal ideal, see [11, Theorem 2.11]. Now let $S_\lambda(X)$ be a prime ideal in $C(X)$, then $S_\lambda(X)$ is contained in the unique maximal ideal M_x, such that x is the only non-λ-isolated point. So the space X has only one non-λ-isolated point. Consequently, if X has more than one non-λ-isolated point then $S_\lambda(X)$ can not be a prime ideal in $C(X)$, see 2.5. Now we have the following results.

Proposition 4.2. *If X is a topological space with more than one non-λ-isolated point in X, i.e., $|X \setminus I_\lambda(X)| > 1$, then $S_\lambda(X)$ is not a prime ideal in $C(X)$.*

Theorem 4.3. *Let X be a P_λ-space, then the following statements are equivalent.*

(1) *$S_\lambda(X) = M_x$, for som $x \in X$.*
(2) *X is a λ-compact space containing only one non-λ-isolated point.*

Proof. ((1) $\Rightarrow$ (2)) Evidently, $x \in X$ is the only non-λ-isolated point in X, see Corollary 2.5 and Proposition 4.2. Now we show that X is a λ-compact space. Put $X = \bigcup_{i\in I} G_i$, such that G_i is an open set in X, for each $i \in I$ and $|I| \geq \lambda$. Since $x \in \bigcup_{i\in I} G_i$, there exists $k \in I$, such that $x \in G_k$. But by complete regularity of X, there exists $f \in C(X)$ such that $x \in int(Z(f)) \subseteq G_k$. Since X is a P_λ-space, $x \in Z(f)$ and therefore $f \in M_x = S_\lambda(X)$. Thus $|X \setminus G_k| \leq |X \setminus Z(f)| = |coz(f)| < \lambda$, i.e., $X = (\bigcup_{j\in J} G_j) \bigcup G_k$, where $J \subseteq I$ and $|J| < \lambda$. Now, it is sufficient to show that λ is the least infinite cardinal number with this property. To see this we show that there exists an open cover of X with cardinality $\beta < \lambda$ which is not reducible to a subcover with cardinality less than β. By [17, Lemma 2.13], there exists a closed subspace $F \subset X$, such that $|F| = \beta$ and $x \in F$. Now, by complete regularity of X, for each $s \in F$ and $y \in F \setminus \{s\}$, there exists $f_y \in C(X)$, such that $f_y(s) = 0$ and $f_y(y) = 1$. Therefore $s \in \bigcap_{y\in F\setminus\{s\}} Z(f_y) = G_s$ and since X is a p_λ-space, G_s is an open set of X. So $X = (X \setminus F) \cup \{G_s\}_{s\in F}$ is an open cover of X. It goes without saying that $G_s \cap F = \{s\}$ and therefore the above cover cannot reduce to an open cover of X with cardinality less than β. Consequently, X is a λ-compact space.
((2) $\Rightarrow$ (1)) It is sufficient to show that $M_x \subseteq S_\lambda(X)$, where x is the only non-λ-isolated point of X. Let $f \in M_x$, i.e., $x \in Z(f)$. Since each point of X except x is a λ-isolated point we infer that for every $y \in X \setminus Z(f)$, there exists a neighborhood of y in X, say G_y, with cardinality less than λ. Hence $(X \setminus Z(f)) \subseteq \bigcup_{i\in I} G_{y_i}$, where $|I| < \lambda$ and y_i is a λ-isolated point, for each $i \in I$. Thus $|\bigcup_{i\in I} G_{y_i}| < \lambda$ implies that $|X \setminus Z(f)| < \lambda$ and we are done. □

We note that if X has at most one non-λ-isolated point, then by criterion for recognizing the essential ideals in $C(X)$, see [1, theorem 3.1], $S_\lambda(X)$ is essential in $C(X)$ and by Proposition 4.2, it is an essential prime ideal of $C(X)$. If X is the one point λ-compactification of a discrete space, then $S_\lambda(X)$ is an essential maximal ideal, see Theorem 4.3. The above discussion refers to the following proposition which is proved in [1, Proposition 4.1].

Proposition 4.4. *If X is an infinite space, there is an essential ideal in $C(X)$ which is not a prime ideal.*

The following theorem is the counterpart of the above proposition.

Theorem 4.5. *Let X be a topological space with $|X| \geq \lambda$ such that $|X \setminus I_\lambda(X)| > 1$, then there exists a λ-essential ideal in $C(X)$ which is not a prime ideal.*

Proof. By assumption, there exist two distinct non-λ-isolated points, say x and y. Now, define $E = \{f \in C(X) : \{x, y\} \subseteq Z(f)\}$, then $\bigcap Z[E] = \{x, y\}$ and therefore by the criterion for recognizing the essential ideals, E is essential. Since $x, y \in \bigcap Z[S_\lambda(X)]$, by Theorem 2.4 we infer that $S_\lambda(X) \subseteq E$. It is evident that E is not a prime ideal, see [11, Theorem 2.11] and we are done. ☐

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