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A quantitative version of the Arzelà-Ascoli theorem based on the degree of nondensifiability and applications

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Abstract

We present a novel result that, in a certain sense, generalizes the Arzelà-Ascoli theorem. Our main tool will be the so called degree of nondensifiability, which is not a measure of noncompactness but can be used as an alternative tool in certain fixed problems where such measures do not work out. To justify our results, we analyze the existence of continuous solutions of certain Volterra integral equations defined by vector valued functions.

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1. Introduction

To set the notation, let $I := [0, 1]$, $(X, \|\cdot\|)$ a Banach space and $\mathcal{C}(I, X)$ the Banach space of continuous maps $x : I \longrightarrow X$, endowed the norm $\|x\|_\infty := \max\{\|x(t)\| : t \in I\}$, for each $x \in \mathcal{C}(I, X)$. Also, $\mathfrak{B}_X$ denotes the class of non-empty and bounded subsets of X, $\mathrm{Conv}(B)$ and $\bar{B}$, as usual, denote the convex hull and closure of a subset B of X, respectively.

It is well known, that one of the most useful tools to analyse the compactness of a subset of $\mathcal{C}(I, X)$ is the celebrated Arzelà-Ascoli theorem (see, for instance,

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[29]). So, it is not surprising that this result has been extended and generalized in many ways; see, for instance, [5, 7, 25] and references therein. Our main goal will be to state a novel generalization of such a result.

In our context, the Arzelà-Ascoli theorem can be stated as follows: a set $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$ is precompact (i.e., its closure is compact) if, and only if, the following conditions hold:

(1) For each $t \in I$, the set $B(t) := \{x(t) : x \in B\}$ is precompact.
(2) B is equicontinuous, that is, given $\varepsilon > 0$ there is $\delta > 0$ such that $\|x(t) - x(t')\| \leq \varepsilon$ for every $x \in B$, whenever $|t - t'| \leq \delta$.

For a better comprehension of the manuscript, we recall the concept of *measure of noncompactness*. As the definition of such measures may vary according to the author (see, for instance, [1, 3]), here we consider the following one given in [13]:

Definition 1.1. A map $\mu : \mathfrak{B}_X \longrightarrow \mathbb{R}_+ := [0, \infty)$ is said to be a measure of noncompactness, MNC, if it satisfies the following properties:

(i) Regularity: $\mu(B) = 0$ if and only if $B \in \mathfrak{B}_X$ is a precompact set.
(ii) Invariant under closure: $\mu(B) = \mu(\bar{B})$, for all $B \in \mathfrak{B}_X$.
(iii) Monotony: $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$, for all B_1, $B_2 \in \mathfrak{B}_X$.
(iv) Semi-homogeneity: $\mu(\lambda B) = |\lambda|\mu(B)$, for all $\lambda \in \mathbb{R}$ and $S \in \mathfrak{B}_X$.
(v) Invariance under translations: $\mu(x + B) = \mu(B)$, for all $x \in X$ and $B \in \mathfrak{B}_X$.

Two widely studied MNCs are these of Hausdorff and Kuratowski (again, [1, 3]), denoted by χ and κ respectively and defined as

$$\begin{aligned}\chi(B) :=& \inf\big\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls} \\ & \quad \text{of radius} \leq \varepsilon\big\} \quad \text{and} \\ \kappa(B) :=& \inf\big\{\varepsilon > 0 : B \text{ can be covered by a finite number of sets} \\ & \quad \text{of diameter} \leq \varepsilon\big\},\end{aligned}$$

for every $B \in \mathfrak{B}_X$. For instance, if U_X is the closed unit ball of X, we have that $\chi(U_X) = 1$, $\kappa(U_X) = 2$ if X has infinite dimension and $\chi(U_X) = \kappa(U_X) = 0$ if X has finite dimension.

There are some Arzelà-Ascoli results type for vector-valued functions based on the MNCs (see, for instance, [6, 14, 23] or [28, Theorem 12.5, p. 255]), the first of them, due to Ambrosetti [2], was proved in 1967:

Theorem 1.2. *Let $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$ be equicontinuous. Then,*

$$\kappa(B) = \sup\big\{\kappa\big(B(t)\big) : t \in I\big\},$$

where $B(t) := \{x(t) : x \in B\}$, for each $t \in I$.

Roughly speaking, the above result "quantifies" the compactness of an equicontinuous $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$. Of course, if $X := \mathbb{R}$ then the above result holds trivially from the equicontinuity of B and the Arzelà-Ascoli theorem. Therefore, Theorem 1.2 can be useful in infinite dimensional Banach spaces where, unlike the finite dimensional case, a bounded set is not necessarily precompact.

On the other hand, our main goal is to prove a more general result, not based on MNCs, than Theorem 1.2 using the so called *degree of nondensifiability* ϕ_d, explained in detail in Section 2. We prove this result in Theorem 3.2, and as a consequence we derive some bounds of the MNCs of arc-connected subsets of $\mathfrak{B}_{\mathcal{C}(I,X)}$.

As it is shown in [10, 11, 12], we can use ϕ_d as an alternative to the MNCs in certain fixed point problems, because in some cases it seems to work out under more general conditions than the MNCs. So, with a suitable combination of a Sadovskiĭ fixed point theorem type for ϕ_d (see Theorem 2.11) and Theorem 3.2, we will prove in Section 4 a result regarding the existence of solutions of certain Volterra integral equations.

2. The degree of nondensifiability

In 1997 the concepts of α-dense curve and densifiable set were introduced by Cherruault and Mora [18] in a metric space (Y, d):

Definition 2.1. Let $\alpha \geq 0$ and $B \in \mathfrak{B}_Y$, the class of non-empty and bounded subsets of Y. A continuous map $\gamma : I \longrightarrow Y$ is said to be an α-dense curve in B if the following conditions hold:

(i) $\gamma(I) \subset B$.
(ii) For any $x \in B$, there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.

B is said to be densifiable if for every $\alpha > 0$ there is an α-dense curve in B.

Let us note that, given $B \in \mathfrak{B}_Y$, the class of α-dense curves in B is non-empty for any $\alpha \geq \mathrm{Diam}(B)$, the diameter of B. Indeed, fixed $x_0 \in B$, the map given by $\gamma(t) := x_0$, for each $t \in I$, is an α-dense curve in B, for any $\alpha \geq \mathrm{Diam}(B)$. Also, note that for $B := I^n$, $n > 1$, a 0-dense curve in B is, precisely, a *space-filling curve* in B (see [26]). So, the α-dense curves are a generalization of these curves.

It can be proved that the class of densifiable sets is strictly between the class of Peano Continua (i.e. those sets that are the continuous image of I) and the class of connected and precompact sets. For a detailed exposition of the above concepts, see [8, 17, 18, 19, 20] and references therein.

On the other hand, from the α-dense curves we can define the following (see [13, 19]):

Definition 2.2. The degree of nondensifiability $\phi_d : \mathfrak{B}_Y \longrightarrow \mathbb{R}_+$ is defined as

$$\phi_d(B) := \inf \left\{ \alpha \geq 0 : \Gamma_{\alpha,B} \neq \varnothing \right\},$$

for each $B \in \mathfrak{B}_Y$, $\Gamma_{\alpha,B}$ being the class of α-dense curves in B.

Note that ϕ_d is well defined, as we have pointed out above, $\Gamma_{\alpha,B} \neq \varnothing$ for every $\alpha \geq \mathrm{Diam}(B)$ for each $B \in \mathfrak{B}_Y$. So, $\phi_d(B) \leq \mathrm{Diam}(B)$ for each $B \in \mathfrak{B}_Y$. For instance, if Y is a Banach space, we have that $\phi_d(U_Y) = 0$ if Y is finite dimensional (see Proposition 2.4 below) and $\phi_d(U_Y) = 1$ if Y is infinite dimensional (see [19]), U_Y being the closed unit ball of Y.

Example 2.3. Let L^1 be the Banach space of absolute value Lebesgue integrable functions defined on I, endowed its usual norm, and define the set

$$D := \big\{ f \in L^1 : f \geq 0 \text{ and } \int_0^1 f(x)dx = 1 \big\}.$$

Then, one can check that $\phi_d(D) = 2$ (see [13]). So, the inequalities

$$1 = \phi_d(U_{L^1}) = \phi_d(U_{L^1} \cup D) < \max\{\phi_d(D), \phi_d(U_{L^1})\} = 2,$$

hold, where U_{L^1} is the closed unit ball of L^1.

By the above example follows that the ϕ_d is not a MNC, as the monotony condition (iii) of Definition 1.1 is not satisfied. However, the degree of nondensifiability shares some properties with the MNCs, as we show in the next result (see [13]).

Proposition 2.4. *In a complete metric space (Y, d), the degree of nondensifiability ϕ_d satisfies:*

(i) It is regular on the subclass $\mathfrak{B}_{a,Y}$ of bounded and arc-connected sets of the class $\mathfrak{B}_Y$ of bounded sets of Y: for a given $B \in \mathfrak{B}_{a,Y}$, $\phi_d(B) = 0$ if, and only if, B is precompact.

(ii) It is invariant under closure: $\phi_d(B) = \phi_d(\overline{B})$, for any $B \in \mathfrak{B}_Y$.

(iii) It is semi-additive on disjoint sets of $\mathfrak{B}_{a,Y}$, that is to say

$$\phi_d(B_1 \cup B_2) = \max\{\phi_d(B_1), \phi_d(B_2)\}.$$

whenever $B_1 \cap B_2 \neq \varnothing$.

Furthermore, if Y is a Banach space, then ϕ_d also satisfies:

(I) $\phi_d\big(\mathrm{Conv}(B_1)\big) \leq \phi_d(B_1)$, $\phi_d\big(\mathrm{Conv}(B_1 \cup B_2)\big) \leq \max\{\phi_d(B_1), \phi_d(B_2)\}$, for every $B_1, B_2 \in \mathfrak{B}_Y$.

(II) It is semi-homogeneous, that is, $\phi_d(\lambda B) = |\lambda|\phi_d(B)$, for $\lambda \in \mathbb{R}$ and $B \in \mathfrak{B}_Y$.

(III) It is invariant under translations, that is, $\phi_d(y + B) = \phi_d(B)$, for any $y \in Y$ and $B \in \mathfrak{B}_Y$.

The degree of nondensifiability ϕ_d and the Hausdorff MNC χ are related by the following result (see [13]):

Proposition 2.5. *For each arc-connected $B \in \mathfrak{B}_X$ we have*

$$\chi(B) \leq \phi_d(B) \leq 2\chi(B),$$

and these inequalities are the best possible if X is infinite dimensional.

For the Kuratowski MNC κ, we have the following result (again, [13]):

Proposition 2.6. *For each arc-connected $B \in \mathfrak{B}_X$ we have*

$$\frac{1}{2}\kappa(B) \leq \phi_d(B) \leq \kappa(B).$$

Another interesting property of the degree of nondensifiability ϕ_d is that it is an "universal" upper bound for any MNC. Formally (see [13]):

Proposition 2.7. *$\mu : \mathfrak{B}_X \longrightarrow \mathbb{R}_+$ a MNC. Then,*

$$\mu(B) \leq \mu(U_X)\phi_d(B),$$

for each $B \in \mathfrak{B}_X$, U_X being the closed unit ball of X.

Next, we recall some well known definitions regarding with the MNCs (see, for instance, [1, 3]):

Definition 2.8. Let $T : C \subseteq X \longrightarrow X$ continuous and μ a MNC. Given $k \geq 0$, we say that T is k-μ-contractive if

$$\mu\big(T(B)\big) \leq k\mu(B),$$

for each $B \in \mathfrak{B}_X \cap C$. If for every non-precompact $B \in \mathfrak{B}_X \cap C$, $\mu\big(T(B)\big) < \mu(B)$ we say that T is μ-condensing.

Now, we can state the following fixed point result (again, [1, 3]):

Theorem 2.9 (Sadovskiĭ fixed point theorem)**.** *Let $C \in \mathfrak{B}_X$ be closed and convex, and $T : C \longrightarrow C$ μ-condensing, for certain MNC μ invariant under the convex hull. Then, T has some fixed point.*

Before to state a Sadovskiĭ fixed point theorem for the degree of nondensifiability ϕ_d, is convenient to formalize the concepts of Definition 2.8 for ϕ_d.

Definition 2.10. Let $T : C \subseteq X \longrightarrow X$ continuous and $k \geq 0$. We will say that T is k-ϕ_d-contractive if

$$\phi_d\big(T(B)\big) \leq k\phi_d(B),$$

for each convex $B \in \mathfrak{B}_X \cap C$. If for every convex and non-precompact $B \in \mathfrak{B}_X \cap C$, $\phi_d\big(T(B)\big) < \phi_d(B)$, then T is said to be ϕ_d-condensing.

Now, we have the following result (we skip the proof because it follows in the same way than the given in [3], noticing Proposition 2.4, or in [10, Theorem 3.2]):

Theorem 2.11. *Let $C \in \mathfrak{B}_X$ be closed and convex, and $T : C \longrightarrow C$ ϕ_d-condensing. Then, T has some fixed point.*

As we have pointed out above, the degree of nondensifiability ϕ_d and the MNCs (and, in particular, the Hausdorff and Kuratowski MNCs) are essentially different. Then, as expected, Definitions 2.8 and 2.10 are essentially different. This fact is evidenced in the following examples.

Example 2.12. Let ℓ_2 be the Banach space of the real sequences of summable square, endowed with the usual norm $\|\cdot\|$. Fixed $2^{-1/2} < \beta < 1$, let $(\xi_n)_{n\geq 1}$ be a dense sequence in βU_{ℓ_2}, U_{ℓ_2} being the closed unit ball of ℓ_2, and define $T : \ell_2 \longrightarrow \ell_2$ as

$$T(x) := \sum_{n\geq 1} \max\Big\{0, 1 - \frac{\|x - e_n\|}{1-\beta}\Big\}\xi_n, \quad \forall x := (x_n)_{n\geq 1} \in \ell_2,$$

where $\{e_n : n \geq 1\}$ is the standard basis of ℓ_2. Clearly, T is continuous and $T(U_{\ell_2}) \subset \beta U_{\ell_2}$.

Then, in [9, Remark 3.10] it is proved that

(2.1)
$$\kappa(\{e_n : n \geq 1\}) = \sqrt{2} > 2\beta = \kappa(T(\{e_n : n \geq 1\})) \quad \chi(T(B)) \leq \beta \leq \chi(B),$$

for every non-precompact $B \in \mathfrak{B}_{\ell_2}$. That is, T is 1-χ-contractive but not 1-κ-contractive.

Now, let B be a non-empty, convex and non-precompact subset of U_{ℓ_2}. By Proposition 2.4, we can assume without loss of generality that $\theta \in B$, θ being the null vector of ℓ_2. Then, from the definition of T, $T(\theta) = \theta$ and $\|T(x)\| \leq \beta$ for every $x \in U_{\ell_2}$. Consequently, the identically null map $\gamma : I \longrightarrow \ell_2$ is a β-dense curve in $T(U_{\ell_2})$. Therefore,

$$\phi_d(T(B)) \leq \beta. \tag{2.2}$$

Finally, in view of (2.1), (2.2) and Proposition 2.5 we conclude that

$$\phi_d(T(B)) \leq \beta \leq \chi(B) \leq \phi_d(B),$$

that is, T is a 1-ϕ_d-contractive map.

Example 2.13. Let the closed and convex set $C := \{x \in \mathcal{C}(I, \mathbb{R}) : 0 = x(0) \leq x(t) \leq 1 = x(1), t \in I\}$ and $T : C \longrightarrow C$ given by

$$T(x)(t) := \begin{cases} \dfrac{1}{2}x(2t), & \text{for} \quad 0 \leq t \leq \frac{1}{2} \\[2ex] \dfrac{1}{2}x(2t-1) + \dfrac{1}{2}, & \text{for} \quad \frac{1}{2} < t \leq 1 \end{cases}$$

for each $x \in C$. Then, T is continuous (in fact, affine) and $T(C) \subset C$. One can check (see [3, Example 2, p. 169]) that T is $\frac{1}{2}$-κ-contractive and 1-χ-contractive, because of $\chi(T(C)) = \chi(C) = 1/2$, whereas in [11] we show that T is $\frac{1}{2}$-ϕ_d-contractive.

To end this section, note that the above examples show that ϕ_d and the MNCs χ and κ are essentially different. Moreover, if we consider the "inner" Hausdorff MNC (see, for instnace, [1]) defined as

$$\chi_i(B) := \inf\big\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls}$$
$$\text{centered at points of } B \text{ and radius} \leq \varepsilon\big\},$$

for each $B \in \mathfrak{B}_X$, in general, $\chi_i \neq \phi_d$. Indeed, for the set

$$B := \big\{(x, \sin(1/x)) : x \in [-1, 0] \cup (0, 1]\big\} \cup \big\{[0, y] : y \in [-1, 1]\big\} \subset \mathbb{R}^2,$$

we find $\chi_i(B) = 0 < 1 = \phi_d(B)$. Another example that shows that ϕ_d and χ_i are essentially different can be found in [12, Example 3.4]. Despite its name, χ_i is not a MNC (see, for instance, [1, p. 9]).

3. The main result

Before to prove our main result, we need to recall the following definition (see, for instance, [28, p. 253]):

Definition 3.1. Given $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$, the uniform modulus of equicontinuity of B is defined as

$$\omega(B) := \inf_{\delta>0} \sup_{|t-s|<\delta} \sup_{x\in B} \|x(t) - x(s)\|.$$

Is clear that $\omega(B)$ is well defined, as $\omega(B) < \infty$ for every $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$. Also, let us note that B is equicontinuous if, and only if, $\omega(B) = 0$.

Now, we are ready to prove our main result:

Theorem 3.2. *Let $C \in \mathfrak{B}_{\mathcal{C}(I,X)}$ be arc-connected. Then,*

$$\max\big\{\Phi_d(C), \frac{1}{2}\omega(C)\big\} \le \phi_d(C) \le \Phi_d(C) + 2\omega(C). \tag{3.1}$$

where $C(t) := \{x(t) : x \in C\}$ and $\Phi_d(C) := \sup\{\phi_d\big(C(t)\big) : t \in I\}$, for each $t \in I$. If $C(t)$ is precompact for every $t \in I$, then

$$\frac{1}{2}\omega(C) \le \phi_d(C) \le \omega(C). \tag{3.2}$$

Proof. The inequality

$$\Phi_d(C) \le \phi_d(C). \tag{3.3}$$

has been proved in [10, Lemma 3.2]. Let $\alpha := \phi_d(C)$, $\varepsilon > 0$ and γ an $(\alpha+\varepsilon)$-dense curve in C. Then, given $x \in C$ there is $y \in \gamma(I)$ such that

$$\|x - y\|_\infty \le \alpha + \varepsilon. \tag{3.4}$$

Therefore, fixed $\delta > 0$, in view of (3.4), for each $t, s \in I$ with $|t - s| < \delta$ we have

$$\|x(t) - x(s)\| \le \|x(t) - y(t)\| + \|y(t) - y(s)\| + \|y(s) - x(s)\|$$

$$\le 2(\alpha + \varepsilon) + \|y(t) - y(s)\|,$$

and, as $\delta > 0$ can be arbitrarily small, we infer

$$\omega(C) \le 2(\alpha + \varepsilon) + \omega\big(\gamma(I)\big),$$

but, as $\omega\big(\gamma(I)\big) = 0$ (by the Arzelà-Ascoli Theorem, $\gamma(I)$ is equicontinuous), letting $\varepsilon \to 0$,

$$\omega(C) \le 2\phi_d(C). \tag{3.5}$$

Then, from (3.3) and (3.5), we conclude that

$$\max\big\{\Phi_d(C), \frac{1}{2}\omega(C)\big\} \le \phi_d(C). \tag{3.6}$$

On the other hand, fixed $\varepsilon > 0$ there is some $\delta > 0$ such that for each $t, s \in I$ with $|t - s| < \delta$ we have $\|x(t) - x(s)\| < \omega(C) + \varepsilon$ for each $x \in C$. By the compactness of I there is $\{t_1, \ldots, t_n\} \subset I$ such that $I \subset \cup_{i=1}^{n} B(t_i, \delta)$ (the open balls centered at t_i and radius δ). So,

$$C(t) \subset \bigcup_{i=1}^{n} C(t_i) + (\omega(C) + \varepsilon) U_X, \tag{3.7}$$

U_X being the closed unit ball of X. Given $\alpha > \Phi_d(C)$, let γ_i be an α-dense curve in $C(t_i)$. Then, as each $\gamma_i(I)$ is compact there is a finite set $\{y_1^{(i)}(t_i), \ldots, y_{r_i}^{(i)}(t_i)\} \subset \gamma_i(I)$, with $y_{r_i}^{(i)} \in C$, for each $i = 1, \ldots, n$, such that

$$C(t_i) \subset \bigcup_{i=1}^{n} \gamma_i(I) + \alpha U_X \subset \bigcup_{i=1}^{n} \{y_1^{(i)}(t_i), \ldots, y_{r_i}^{(i)}(t_i)\} + (\alpha + \varepsilon) U_X, \tag{3.8}$$

and joining (3.7) and (3.8), we find

$$C(t) \subset \bigcup_{i=1}^{n} \{y_1^{(i)}(t_i), \ldots, y_{r_i}^{(i)}(t_i)\} + (\omega(C) + \alpha + 2\varepsilon) U_X. \tag{3.9}$$

for each $t \in I$. Next, we claim:

$$C \subset \bigcup_{i=1}^{n} \{y_1^{(i)}, \ldots, y_{r_i}^{(i)}\} + (2\omega(C) + \alpha + 3\varepsilon) U_{\mathcal{C}(I,X)}, \tag{3.10}$$

where $U_{\mathcal{C}(I,X)}$ is the closed unit ball of $\mathcal{C}(I, X)$. In other case, there is $x_0 \in C$ such that

$$\|x_0 - y_{r_i}^{(i)}\|_\infty > 2\omega(C) + \alpha + 3\varepsilon, \tag{3.11}$$

for each $i = 1, \ldots, n$. Take, for each $i = 1, \ldots, n$, $\tilde{t}_i \in I$ such that $\|x_0(\tilde{t}_i) - y_{r_i}^{(i)}(\tilde{t}_i)\| = \|x_0 - y_{r_i}^{(i)}\|_\infty$, and let $t_i \in I$ be such that $\tilde{t}_i \in B(t_i, \delta)$. Then, in view of (3.9)

$$\|x_0 - y_{r_i}^{(i)}\|_\infty = \|x_0(\tilde{t}_i) - y_{r_i}^{(i)}(\tilde{t}_i)\| \leq \|x_0(\tilde{t}_i) - y_{r_i}^{(i)}(t_i)\|$$

$$+ \|y_{r_i}^{(i)}(t_i) - y_{r_i}^{(i)}(\tilde{t}_i)\| \leq \omega(C) + \alpha + 2\varepsilon + \omega(C) + \varepsilon = 2\omega(C) + \alpha + 3\varepsilon,$$

which, is contradictory with (3.11). Therefore, (3.10) holds as claimed. Now, if γ is a continuous map joining the functions $y_1^{(i)}, \ldots, y_{r_i}^{(i)}$ for $i = 1, \ldots, n$ (this is possible because C is arc-connected), from (3.10), we find that γ is an $(2\omega(C) + \alpha + 3\varepsilon)$-dense curve in C. Letting $\varepsilon \to 0$ and $\alpha \to \Phi_d(C)$, we conclude that

$$\phi_d(C) \leq 2\omega(C) + \Phi_d(C), \tag{3.12}$$

and consequently, the inequality (3.1) follows from (3.6) and (3.12).

To prove the inequality (3.2) of the statement, if $C(t)$ is precompact for every $t \in I$, then $\phi_d(C(t)) = 0$ by (i) of Proposition 2.4. So, from (3.1), $\phi_d(C) \geq \omega(C)/2$. By [23, Theorem 1], $\kappa(C) \leq \omega(C)$ and therefore, taking into account Proposition 2.6, the inequality (3.2) holds. The proof is now complete. □

Some comments are necessary before continuing.

(a) Replacing the Banach space X by an arbitrary complete metric space, the above result remains true.
(b) The above result has been proved in [23, Theorem 1] for the Kuratowski MNC κ. However, as we have pointed out in Section 2 (and, in particular, in Example 2.12), the degree of nondensifiability is essentially different from κ.
(c) As Theorem 1.2, our result "quantifies" the compactness of the bounded and arc-connected subsets of $\mathfrak{B}_{\mathcal{C}(I,X)}$. Indeed, if an arc-connected $C \in \mathfrak{B}_{\mathcal{C}(I,X)}$ is precompact if, and only if, is equicontinuous (that is, $\omega(C)=0$) and for each $t \in I$, $C(t)$ is precompact (that is, $\phi_d(C(t))=0$ by virtue of Proposition 2.4).
(d) If C is equicontinuous then Theorem 1.2 remains true for the degree of nondensifiability ϕ_d, as in such case $\omega(C)=0$.

In the following examples we show that the estimates given in the inequality (3.2) of Theorem 3.2 are the best possible.

Example 3.3 (see [23, Example 2])**.** Define the sequence of functions $x_n : I \longrightarrow \mathbb{R}$ as

$$x_n(t) := \begin{cases} 0, \quad \text{for } 0 \le t \le 1/2 - 1/n \\ \\ (t-1/2+1/n)n/2, \quad \text{for } 1/2-1/n \le t \le 1/2+1/n \\ \\ 1, \quad \text{for } 1/2+1/n \le t \le 1 \end{cases}$$

for every integer $n \ge 2$. Then for $B := \{x_n : n \ge 2\}$ we find $\kappa(B)=1/2$ and $\omega(B)=1$.

Putting $C := \operatorname{Conv}(B)$, by Proposition 2.6, we have $\phi_d(C) \le \kappa(C) = \kappa(B) = 1/2$. Then, noticing (3.2) of Theorem 3.2, we have

$$\phi_d(C) = \frac{1}{2} = \frac{1}{2}\omega(C).$$

Example 3.4. Consider the following (closed and convex) set

$$C := \big\{x \in \mathcal{C}(I,\mathbb{R}) : 0 = x(0) \le x(t) \le x(1) = 1, \text{ for each } t \in I\big\}.$$

Then, $\omega(C)=1$ (see also the above example). We will prove in the following lines that $\phi_d(C)=1$.

As $\operatorname{Diam}(C)=1$ we have $\phi_d(C) \le 1$. If $\phi_d(C)<1$, we can take an α-dense curve in C, put γ, with $0<\alpha<1$. Noticing that $\gamma(I)$ is a compact subset of C, by the Arzelà-Ascoli theorem, it is equicontinuous. So, for $0<3\varepsilon<1-\alpha$ there is $\delta_1>0$ such that

$$|y(t)-y(t')| \le \varepsilon \quad \text{for all } y \in \gamma(I), \tag{3.13}$$

whenever $|t-t'| \le \delta_1$. Taking into account that $y(1)=1$ for each $y \in \gamma(I)$, again by the equicontinuity of the set $\gamma(I)$, there is $\delta_2 > 0$ such that

$$y(t') \ge 1-\varepsilon \quad \text{for all } y \in \gamma(I), \tag{3.14}$$

whenever $1-t' \le \delta_2$. Therefore, taking $0 < t' < t < 1$ satisfying $1-t' \le \min\{\delta_1, \delta_2\}$ and an integer $n \ge 1$ such that $f(t) := t^n \le \varepsilon$, as γ is an α-dense curve in C we can pick $y \in \gamma(I)$ with

$$\|f-y\|_\infty \le \alpha. \tag{3.15}$$

Finally, from (3.13), (3.14) and (3.15) we infer

$$\alpha \ge \|f-y\|_\infty \ge |f(t)-y(t')| - |y(t')-y(t)| \ge 1-2\varepsilon - |y(t')-y(t)| \ge 1-3\varepsilon,$$

which is contradictory with the choice of α. So, we conclude that $\phi_d(C)=1$.

On the other hand, we can derive from Theorem 3.2 some bounds for the MNCs of arc-connected subsets of $\mathfrak{B}_{\mathcal{C}(I,X)}$. In view of Proposition 2.7, our first corollary is the following:

Corollary 3.5. *Let $\mu : \mathfrak{B}_{\mathcal{C}(I,X)} \longrightarrow \mathbb{R}_+$ be a MNC, and $C \in \mathfrak{B}_{\mathcal{C}(I,X)}$ arc-connected. Then, we have*

$$\mu(C) \le \mu(U_{\mathcal{C}(I,X)})\big[\Phi_d(C) + 2\omega(C)\big],$$

$U_{\mathcal{C}(I,X)}$ being the closed unit ball of $\mathcal{C}(I,X)$, and $\Phi_d(C)$ defined in Theorem 3.2. In particular, if $C(t)$ is precompact for every $t \in I$, then

$$\mu(C) \le \mu(U_{\mathcal{C}(I,X)})\omega(C).$$

Example 3.6. Let X be an infinite dimensional Banach space, $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$ convex and $k : I^2 \longrightarrow \mathbb{R}$, $f : I \times X \longrightarrow X$ continuous. Assume that there is $M > 0$ with $\|f(s,x(s))\| \le M$ for every $s \in I$ and $x \in B$. Consider the set

$$C := \Big\{ \int_0^t k(t,s) f(s,x(s))ds : x \in C \Big\},$$

meaning the above integral in the Bochner sense (see, for instance, [27]). From the convexity of B and the continuity of f, it follows that C is arc-connected. Also, given $0 \le t' < t \le 1$ we have

$$\Big\| \int_0^t k(t,s)f(s,x(s))ds - \int_0^{t'} k(t',s)f(s,x(s))ds \Big\| \le$$

$$\Big\| \int_0^t (k(t,s)-k(t',s))f(s,x(s))ds \Big\| + \Big\| \int_{t'}^t k(t',s)f(s,x(s))ds \Big\| \le$$

$$M \int_0^t |k(t,s)-k(t',s)|ds + MM_k(t-t'),$$

and therefore, the set C is equicontinuous, M_k denoting the maximum of $|k|$ over I^2. Then, from Corollary 3.5, we find the inequalities

$$\chi(C) \leq \Phi_d(C) \quad \text{and} \quad \kappa(C) \leq 2\Phi_d(C),$$

with $\Phi_d(C)$ as in Theorem 3.2.

Recalling that two MNCs μ and ν are said to be comparable (see, for instance, [3, Definition 1.1, p. 168]) if there are $a, b > 0$ such that

$$a\nu(B) \leq \mu(C) \leq b\nu(B),$$

for every $B \in \mathfrak{B}_X$, the next result follows directly from Proposition 2.5 and Theorem 3.2:

Corollary 3.7. *Assume $\mu : \mathfrak{B}_{\mathcal{C}(I,X)} \longrightarrow \mathbb{R}_+$ is a comparable MNC with the Hausdorff MNC χ, that is to say, there are $a, b > 0$ such that*

$$a\chi(B) \leq \mu(B) \leq b\chi(B),$$

for every $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$. Let $C \in \mathfrak{B}_{\mathcal{C}(I,X)}$ be arc-connected. Then, we have

$$\frac{a}{2} \max\left\{\Phi_d(C), \frac{1}{2}\omega(C)\right\} \leq \mu(C) \leq b\big[\Phi_d(C) + 2\omega(C)\big],$$

with $\Phi_d(C)$ as in Theorem 3.2. In particular, if $C(t)$ is precompact for every $t \in I$, then

$$\frac{a}{4}\omega(C) \leq \mu(C) \leq b\omega(C).$$

Roughly speaking, the above corollaries provide us bounds for $\mu(C)$ (of an arc-connected $C \in \mathfrak{B}_{\mathcal{C}(I,X)}$) from the degree of nondensifiability of $C(t)$ or from the uniform modulus of equicontinuity of C, depending if $C(t)$ is, or not, precompact for each $t \in I$.

4. Application to Volterra integral equations

For $X := \mathbb{R}^n$ one of the most important example of compact operators in $\mathcal{C}(I, X)$ (i.e. continuous maps that transform bounded sets into precompact sets) are those defined by integral operators with sufficient regular kernels (see, for intance, [16]). However, as it is shown in [15] by several examples, if X is an infinite dimensional Banach space, the situation is very different. It is due, essentially, to the fact that in infinite dimensional Banach spaces bounded subsets are not necessarily precompact. So, the MNCs are a useful tool to analyze the existence of solutions of differential and integral equations posed in infinite dimensional Banach spaces, see, for instance, [4, 21, 22, 24] and references therein. In this section, we will use the results exposed in previous sections to prove the existence of solutions of certain Volterra integral equation, which can not be solved by the MNCs techniques.

Specifically, we consider the following Volterra integral equation

$$x(t) = x_0(t) + \int_0^t k(t,s) f(s, x(s)) ds, \tag{4.1}$$

where $x_0 \in \mathcal{C}(I, X)$, $k : I^2 \longrightarrow \mathbb{R}$ and $f : I \times X \longrightarrow X$ are known, and the integral stands in the Bochner sense (see, for instance, [27]).

When X is a sequence Banach space (such as c_0, ℓ_p, etc.), the above integral equation is equivalent to certain infinite systems of second-order differential equations (see [21] and references therein).

Let the following conditions:

(C1) The functions $k : I^2 \longrightarrow \mathbb{R}$ and $f : I \times X \longrightarrow X$ are continuous.

(C2) There is $R > 0$ such that

$$\frac{\|x_0(t)\| + \| \int_0^t k(t,s) f(s, x(s)) ds\|}{R} \leq 1 \quad \text{for all } t \in I,$$

whenever $\|x\|_\infty \leq R$.

(C3) For each $B \in \mathfrak{B}_{\mathcal{C}(I,X)}$, there is $c : I \longrightarrow \mathbb{R}_+$, bounded and Lebesgue integrable, and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ continuous and increasing with $\psi(r) < r$ for every $r > 0$ and $\psi(0) = 0$, such that

$$\phi_d\big(\{f(s, x(s)) : x \in B\}\big) \leq c(s)\psi\big(\phi_d\big(\{x(s) : x \in B\}\big)\big) \quad \text{for all } s \in I.$$

(C4) The functions c and k obey

$$\int_0^t |k(t,s)| c(s) ds \leq 1, \quad \text{for all } t \in I.$$

Regarding the above conditions, we can do the following observations.

- Unlike many of the results based in MNCs for integral equations of type (4.1), we do not require the uniform continuity of the family of functions $(f(s, x(s))_{s \in I}$ (see the above cited references).
- Similar conditions to (C1)-(C4) are assumed in [12] to prove the existence of solutions of certain integral equations. In fact, condition (C3) is required in [12] by taking $c(s)$ equal to a positive constant.
- From Example 2.13 and [12, Examples 3.2 and 3.4], not detailed here for lack of space, we can define some maps $f(t, x)$ such that condition (C3) is satisfied by the ϕ_d but not by the Hausdorff MNC χ. This fact will be also evidenced in Example 4.2.
- In particular, conditions (C3) and (C4) are assumed to prove that the operator which defined the integral equation (4.1) is ϕ_d-condensing and then, we can apply Theorem 2.11. Let us note that, as the concepts of ϕ_d-condensing and μ-condensing are essentially different (see Example 2.13), Sadovskiĭ fixed point theorem and Theorem 2.11 too.

Now, we can state and prove the main result of this section.

Theorem 4.1. *Assume conditions (C1)-(C4). Then, equation (4.1) has some solution $x^* \in \mathcal{C}(I, X)$.*

Proof. Clearly, the existence of solutions of (4.1) is equivalent to the existence of fixed points of the map

$$T(x)(t) := x_0(t) + \int_0^t k(t,s) f(s, x(s)) ds \quad \text{for all } x \in \mathcal{C}(I, X),$$

which, from condition (C1), satisfies $T(x) \in \mathcal{C}(I,X)$ for every $x \in \mathcal{C}(I,X)$. Also, a routine check shows us that T is continuous. We will apply Theorem 2.11 to prove the existence of fixed points of T.

From condition (C2), there is $R>0$ such that $T(C) \subset C$, $C := RU_{\mathcal{C}(I,X)}$. Also, from condition (C3) and Theorem 3.2, for every $s \in I$ the degree of nondensifiablity of the set $\{f(s,x(s)) : x \in C\}$ remains bounded by $\sup\{c(t) : t \in I\}\psi(\phi_d(C)) + 2\omega(C)$ and, therefore, the set $\{f(s,x(s)) : x \in C\}$ must be bounded. So, in view of Example 3.6, we find that $T(C)$ is equicontinuous.

Next, let $B \subset C$ be non-empty, non-precompact and convex. By Theorem 3.2 and the above considerations, we have

$$\phi_d\big(T(B)\big) \le \Phi_d\big(T(B)\big) + 2\omega\big(T(B)\big) = \sup\{\phi_d\big(T(B)(t) : t \in I\big\}, \tag{4.2}$$

meaning $T(B)(t) := \{T(x)(t) : x \in B\}$.

On the other hand, by condition (C3), for each $s \in I$, if $\alpha_s := \phi_d(B(s)) > 0$ then for an arbitrarily small $\varepsilon > 0$, there is an $(\psi(\alpha_s)+\varepsilon)$-dense curve in $\{f(s,x(s)) : x \in B\}$, put $\tau \in I \longmapsto \gamma_\tau(s)$. So, given $x \in B$ there is $\tau \in I$ such that

$$\|f(s,x(s)) - \gamma_\tau(s)\| \le c(s)\psi(\alpha_s), \tag{4.3}$$

for each $s \in I$. Now, consider the map

$$\tau \in I \longmapsto \tilde{\gamma}_\tau(t) := x_0(t) + \int_0^t k(t,s)\gamma_\tau(s)ds, \quad \text{for all } t \in I,$$

which is continuous and $\gamma(I) \subset T(B)$.

Then, noticing (4.3) and condition (C4), for every $t \in I$, we have

$$\|T(x)(t) - \tilde{\gamma}_\tau(t)\| \le \int_0^t |k(t,s)| \|f(s,x(s)) - \gamma_\tau(s)\| ds \le$$

$$\int_0^t |k(t,s)| c(s)(\psi(\alpha_s)+\varepsilon) ds \le \sup\{\psi(\alpha_s) : s \in I\} + \varepsilon,$$

and letting $\varepsilon \to 0$

$$\|T(x)(t) - \tilde{\gamma}_\tau(t)\| \le \sup\{\psi(\alpha_s) : t \in I\} = \sup\{\psi(\alpha_t) : t \in I\},$$

or, in other words, $\tilde{\gamma}$ is a $\sup\{\psi\big(\phi_d(B(t))\big) : t \in I\}$-sense curve in $T(B)(t)$.

Then, noticing (4.2), the properties of ψ and Theorem 3.2

$$\phi_d\big(T(B)\big) \le \sup\big\{\psi\big(\phi_d(B(t))\big) : t \in I\big\} \le \psi\big(\sup\{\phi_d(B(t)) : t \in I\}\big) <$$

$$\sup\big\{\phi_d(B(t)) : t \in I\big\} \le \max\big\{\sup\{\phi_d(B(t)) : t \in I\}, \tfrac{1}{2}\omega(B)\big\} \le \phi_d(B).$$

So, T is a ϕ_d-condensing map and the proof is complete. □

We conclude our exposition with an example.

Example 4.2. Consider the following integral equation in the Banach space c_0 of the null sequences:

$$x_n(t) = \frac{4n}{3n+1} \int_0^t 3ts^3 \arctan\big(|x_n(s)|\big) ds. \quad \text{for all } n \geq 1. \tag{4.4}$$

In this case, $x_0 \equiv 0$, $f(s,x) := (\frac{4n}{3n+1} \arctan(|x_n|))_{n\geq 1}$ and $k(t,s) := 3ts^3$. So, condition (C1) is trivially satisfied and (C2) taking, for instance, $R := \pi$.

From the elementary properties of the arctan function, it is immediate to check that, if γ is an α-dense curve in B, then $f(s,\gamma(s))$ is an $\frac{4}{3}\arctan(\alpha)$-dense curve in the set $\{f(s,x(s)) : s \in B\}$, for every $B \in \mathfrak{B}_{\mathcal{C}(I,c_0)}$. So, condition (C3) is fulfilled for $c(s) := \frac{4}{3}$ and $\psi(r) := \arctan(r)$. Also,

$$\frac{4}{3} \int_0^t 3ts^3 ds = t^5 \leq 1 \quad \text{for all } t \in I,$$

and therefore condition (C4) holds. Then, by Theorem 4.1, the above integral equation has some continuous solution.

On the other hand, from the following formula (see, for instance, [1, p. 5])

$$\chi(B) = \lim_n \sup_{x\in B} \big\{ \sup\{|x_i| : i \geq n\}\big\},$$

for every $B \in \mathfrak{B}_{\mathcal{C}(I,c_0)}$, we deduce that

$$\chi(\{e_n : n \geq 1\}) = 1 < \frac{\pi}{3} = \chi(\{f(s,e_n) : n \geq 1\}),$$

$e_n \in c_0$ being the vector with 1 in the n-th position and zeros otherwise. In other words, condition (C3) is not satisfied for the Hausdorff MNC χ and the above functions $c(s)$ and $\psi(r)$. Moreover, Sadovskiĭ fixed point theorem (see Theorem 2.11) can not be used in this example for χ as, by the above inequality, T is not χ-condensing.

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