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A characterization of quasi-metric completeness

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Abstract

Hu proved in [4] that a metric space (X,d) is complete if and only if for any closed subspace C of (X,d), every Banach contraction on C has fixed point. Since then several authors have investigated the problem of characterizing the metric completeness by means of fixed point theorems. Recently this problem has been studied in the more general context of quasi-metric spaces for different notions of completeness. Here we present a characterization of a kind of completeness for quasi-metric spaces by means of a quasi-metric versions of Hu's theorem.

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1. Introduction

It is an obvious consequence of the Banach contraction principle that every Banach contraction on any closed subspace of a complete metric space has fixed point.

Hu proved in [4] that if every Banach contraction on any closed subset of a metric space (X, d) has fixed point then (X, d) is complete. Indeed, suppose that (X, d) is not complete, so X contains a nonconvergent Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct terms. For each x_n define $l_n = \inf\{d(x_n, x_m) : m > n\}$. By the Cauchyness, given $r \in (0, 1)$ and l_n there exists $k(n) > n$ such that $d(x_i, x_j) < rl_n$ for all $i, j \geqslant k(n)$. If not $d(x_i, x_j) < rl_n < l_n$, with $j > i$, a contradiction. Then, the mapping T defined as $Tx_n = x_{k(n)}$ for all $n \in \mathbb{N}$ is a Banach contraction on the closed set $\{x_n : n \in \mathbb{N}\}$ with no fixed point.

Therefore, a metric space (X, d) is complete if and only if for any closed subspace C of (X, d), every Banach contraction on C has fixed point.

Since Hu obtained this result, several authors have investigated the problem of characterizing the metric completeness by means of fixed point theorems. Recently this problem has been studied in the more general context of quasi-metric spaces for different notions of quasi-metric completeness ([1, 5, 6]). Here we present a characterization of a kind of completeness for quasi-metric spaces by means of a quasi-metric version of Hu's theorem.

2. Basic notions and preliminary results

Our basic reference for quasi-metric spaces is [2].

By a quasi-metric on a set X we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ is also a quasi-metric on X, called the conjugate of d, and the function d^s defined by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ is a metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open ball $\{B_d(x,r) : x \in X, \varepsilon > 0\}$, where $B_d(x,r) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A subset C of a quasi-metric space (X,d) is called doubly closed if C is closed with respect to τ_d and with respect to $\tau_{d^{-1}}$.

If τ_d is a T_1 (resp. a Hausdorff) topology on X, we say that (X,d) is a T_1 (resp. a Hausdorff) quasi-metric space. Note that a quasi-metric space (X,d) is T_1 if and only if for each $x, y \in X$, condition $d(x,y) = 0$ implies $x = y$.

A quasi-metric space (X,d) is called d-sequentially complete if every Cauchy sequence in the metric space (X,d^s) converges with respect to the topology τ_d. Similarly, a quasi-metric space (X,d) is called d^{-1}-sequentially complete if every Cauchy sequence in the metric space (X,d^s) converges with respect to the topology $\tau_{d^{-1}}$.

Definition 1 [3]. Let (X,d) be a quasi-metric space.

A d-contraction on (X,d) is a mapping $T : X \to X$ such that there is a constant $r \in [0,1)$ satisfying $d(Tx,Ty) \le rd(x,y)$, for all $x, y \in X$.

A d^{-1}-contraction on (X,d) is a mapping $T : X \to X$ such that there is a constant $r \in [0,1)$ satisfying $d(Tx,Ty) \le rd(y,x)$, for all $x, y \in X$.

A d^{-1}-contraction on a subset C of (X,d) is a mapping $T : C \to C$ such that there is a constant $r \in [0,1)$ satisfying $d(Tx,Ty) \le rd(y,x)$, for all $x, y \in C$.

If (X,d) is a metric space, the notions of d-contraction and d^{-1}-contraction coincide, and they coincide with the classical notion of (Banach) contraction for metric spaces.

It is easy to see ([3, Proposition 3]) that if T is a d-contraction or a d^{-1}-contraction on (X,d), then T is a contraction on the metric space (X,d^s), so for any $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (X,d^s).

In order to obtain a suitable quasi-metric extension of Hu's theorem we shall consider d^{-1}-contractions but no d-contractions since there exist examples of T_1 d-sequentially complete quasi-metric spaces for which there are d-contractions without fixed point. In any case, the following example shows that such an extension is very difficult in the realm of d-sequentially complete quasi-metric spaces (and hence, in the realm of stronger forms of quasi-metric completeness, as left K-sequential completeness, right K-sequential completeness, Smyth completeness, etc.) and motivates the notion of completeness introduced in Definition 2 below.

Example 1. Let d be the quasi-metric on $\mathbb{N}$ given as $d(n,n) = 0$ for all $n \in \mathbb{N}$ and $d(n,m) = \frac{1}{n}$ if $n \neq m$. Then $(\mathbb{N}, d)$ is a Hausdorff non d-sequentially complete quasi-metric space. Let C be any (nonempty) subset of $\mathbb{N}$ and $T : C \to C$ a d^{-1}contraction on C. It is easy to check that for each $x \in C$, Tx is a fixed point of T.

Definition 2. A quasi-metric space (X,d) is called half sequentially complete if every Cauchy sequence in the metric space (X, d^s) converges with respect to the topology τ_d or $\tau_{d^{-1}}$.

Observe that the space of Example 1 is d^{-1}-sequentially complete and hence half sequentially complete.

Next we present an example of a half sequentially complete quasi-metric space that is not d-sequentially complete and not d^{-1}-sequentially complete.

Example 2. Let $X = \{0, \infty\} \cup \mathbb{N} \cup \{\frac{1}{n+1} : n \in \mathbb{N}\}$. Define a function d on $X \times X$ by $d(0,0) = d(\infty,\infty) = 0$, $d(\frac{1}{n+1}, m) = d(m, \frac{1}{n+1}) = 1$, $d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$, $d(\frac{1}{n+1}, \frac{1}{m+1}) = \left|\frac{1}{n+1} - \frac{1}{m+1}\right|$ if $n, m \in \mathbb{N}$, $d(n,\infty) = 1/n$, $d(0, \frac{1}{n+1}) = \frac{1}{n+1}$, and $d(\infty, n) = d(\frac{1}{n+1}, \infty) = d(\infty, \frac{1}{n+1}) = d(\frac{1}{n+1}, 0) = d(0,n) = d(n,0) = 1$, for all $n \in \mathbb{N}$. Then (X,d) is a Hausdorff quasi-metric space. Moreover, every non eventually constant Cauchy sequence in (X, d^s) is a subsequence of $\{n\}_{n\in\mathbb{N}}$ or of $\{\frac{1}{n+1}\}_{n\in\mathbb{N}}$. Since $\{n\}_{n\in\mathbb{N}}$ converges with respect to $\tau_{d^{-1}}$ (but not with respect to τ_d) and $\{\frac{1}{n+1}\}_{n\in\mathbb{N}}$ converges with respect to τ_d (but not with respect to $\tau_{d^{-1}}$), we deduce that (X,d) is half sequentially complete but not d-sequentially complete and not d^{-1}-sequentially complete.

3. The main result

Theorem 1. *A T_1 quasi-metric space (X,d) is half sequentially complete if and only if every d^{-1}-contraction on any doubly closed subset of (X,d) has a fixed point.*

Proof. Let (X,d) be a T_1 half sequentially complete quasi-metric space, C a doubly closed subset of (X,d) and T a d^{-1}-contraction on C. Fix $x_0 \in C$, then $\{T^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s) such that $\{T^n x_0 : n \in \mathbb{N}\} \subset C$. Since (X,d) is half sequentially complete then $\{T^n x_0\}_{n\in\mathbb{N}}$ converges with respect to τ_d or with respect to $\tau_{d^{-1}}$. If $\{T^n x_0\}_{n\in\mathbb{N}}$ converges with respect to τ_d there exists $y \in X$ such that $d(y, T^n x_0) \to 0$ as $n \to \infty$. Since C is doubly closed then $y \in C$. Since T is a d^{-1}-contraction, there exists $r \in [0,1)$ such that $d(T^{n+1}x_0, Ty) \leq rd(y, T^n x_0)$ for all $n \in \mathbb{N}$. Consequently $d(T^{n+1}x_0, Ty) \to 0$ as $n \to \infty$. From the triangle inequality we deduce $d(y,Ty) = 0$. Therefore $y = Ty$ because (X,d) is a T_1 quasi-metric space. If $\{T^n x_0\}_{n\in\mathbb{N}}$ converges with respect to $\tau_{d^{-1}}$ there exists $y \in X$ such that $d(T^n x_0, y) \to 0$ as $n \to \infty$. Since C is doubly closed then $y \in C$. Since T is a d^{-1} contraction, there exists $r \in [0,1)$ such that $d(Ty, T^{n+1}x_0) \leq rd(T^n x_0, y)$ for all $n \in \mathbb{N}$. Consequently $d(Ty, T^{n+1}x_0) \to 0$ as $n \to \infty$. From the triangle inequality we deduce $d(Ty, y) = 0$. Therefore $y = Ty$ because (X,d) is a T_1 quasi-metric space.

For the converse suppose that there exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,d^s) of distinct terms that is nonconvergent with respect to τ_d and nonconvergent with respect to $\tau_{d^{-1}}$. Then, the set $C := \{x_n : n \in \mathbb{N}\}$ is a doubly closed subset of (X,d). For each x_n we define $l_n = d(x_n, \{x_m : m > n\}) \wedge d(\{x_m : m > n\}, x_n)$. Thus $l_n > 0$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s), given $r \in (0,1)$, for each $n \in \mathbb{N}$ there exists $k(n) > n$ such that $d^s(x_{n'}, x_{m'}) < rl_n$ for all $m', n' \geq k(n)$. (Obviously we can take $k(m) > k(n)$ when $m > n$.)

Now we construct a d^{-1}-contraction on C without fixed point. Indeed, define $T : C \to C$ as $Tx_n = x_{k(n)}$ for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$, and suppose, without loss of generality, that $m > n$. Then $d^s(Tx_n, Tx_m) = d^s(x_{k(n)}, x_{k(m)}) < rl_n \leq r(d(x_n, x_m) \wedge d(x_m, x_n))$. Hence $d(Tx_n, Tx_m) \leq rd(x_m, x_n)$ and $d(Tx_m, Tx_n) \leq$

$rd(x_n, x_m)$. We deduce that T is a d^{-1}-contraction on the doubly closed subset C. This concludes the proof.

Finally, we observe that the above theorem cannot be generalized to non T_1 quasi-metric spaces since there are examples of half sequentially complete non T_1 quasi-metric spaces for which there exist d^{-1}-contractions without fixed point.

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