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Two fixed point theorems on quasi-metric spaces via mw- distances

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Abstract

In this paper we prove a Banach-type fixed point theorem and a Kannan-type theorem in the setting of quasi-metric spaces using the notion of mw-distance. These theorems generalize some results that have recently appeared in the literature.

Keywords: *fixed point, generalized contraction, w-distance, mw -distance, complete quasi-metric space.*
MSC: *47H10, 54H25, 54E50.*

1. Introduction

In his celebrated fixed point theorem, Banach proved that if (X, d) is a complete metric space and the map $T : X \to X$ is a contraction, i.e., $d(Tx, Ty) \leq rd(x, y)$ for some $r \in [0, 1)$ and all $x, y \in X$, then T has a unique fixed point. Later, in

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[4], Kannan proved that if T is a self map on a complete metric space (X, d) such that $d(Tx, Ty) \leq r(d(x, Tx) + d(y, Ty))$ for some $r \in [0, 1/2)$ and all $x, y \in X$, then T has a unique fixed point. Since then, many successful attempts have been made to improve the Banach and Kannan theorems, mainly in two directions. On the one hand, by replacing the underlying metric space with a more general space, for example, a partial metric space, a generalized metric space, a quasi-metric space etc., and on the other, by finding better contractivity conditions on the map T. In [3] and [1] the authors extend these theorems by replacing the complete metric space by a kind of complete quasi-metric space. In this paper we improve these results using a mw-distances in the contractivity conditions instead of the quasi-metric.

In order to fix our terminology we recall the following notions.

A *quasi–metric* on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$ (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If the quasi-metric d satisfies the stronger condition (i") $d(x, y) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X.

A T_1 *quasi-metric space* is a pair (X, d) such that X is a non-empty set and d is a T_1 quasi-metric on X.

Each quasi-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X, called *conjugate quasi-metric*, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X

A quasi-metric space (X, d) is called d-sequentially complete if every Cauchy sequence in (X, d^s) converges with respect to the topology τ_d, i.e., there exists $z \in X$ such that $d(z, x_n) \to 0$.

A quasi-metric space (X, d) is called d^{-1}-sequentially complete if every Cauchy sequence in (X, d^s) converges with respect to the topology $\tau_{d^{-1}}$, i.e., there exists $z \in X$ such that $d(x_n, z) \to 0$.

According to [2], an mw-distance on a quasi-metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:
(W1) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$;
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;
(mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ then $d(y, z) \leq \varepsilon$.

Obviously, each quasi- metric d on a set X is a mw -distance for the quasi-metric space (X, d).

2. The results

Lemma 1. *Let (X, d) be a quasi-metric space, q an $mw-$distance on (X, d) and $(x_n)_{n \in \omega}$ a sequence in X. If for each $\varepsilon > 0$ there exists $n_0 \in \omega$ such that $q(x_n, x_m) \leq \varepsilon$ for all $n, m \geq n_0$, $n \neq m$, then $(x_n)_{n \in \omega}$ is a Cauchy sequence in (X, d^s).*

Proof. Let $\varepsilon > 0$. By $(mW3)$, there exists $\delta > 0$ such that if $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ then $d(y, z) \leq \varepsilon$. By hypothesis, there exists n_0 such that $q(x_n, x_m) \leq \delta/2$ whenever $n, m \geq n_0$, $n \neq m$. Then, $q(x_m, x_m) \leq q(x_m, x_n) + q(x_n, x_m) \leq \delta/2 + \delta/2 = \delta$ whenever $n, m \geq n_0$, $n \neq m$. Consequently, $d(x_n, x_m) \leq \varepsilon$ whenever $n, m \geq n_0$. Therefore, $d^s(x_n, x_m) \leq \varepsilon$ for all $n, m \geq n_0$. □

Theorem 2. *Let T be a self mapping of a d^{-1}-sequentially complete quasi-metric space (X, d) and let q be an $mw-$distance on (X, d). If there exists $r \in [0, 1)$ such that*

$$q(Tx, Ty) \leq rq(y, x)$$

for every $x, y \in X$ then there exists $z \in X$ such that $d(Tz, z) = 0$. Moreover, if $Tu = u$ then $q(u, u) = 0$.

Proof. Fix $x_0 \in X$. For each $n \in \omega$ let $x_n = T^n x_0$. Then

$$q(x_n, x_{n+1}) \leq r^n max\{q(x_0, x_1), q(x_1, x_0)\}$$

$$q(x_{n+1}, x_n) \leq r^n max\{q(x_0, x_1), q(x_1, x_0)\}$$

for all $n \in \omega$.

Let $\varepsilon > 0$ and let $m > n$. Then

$$q(x_n, x_m) \leq q(x_n, x_{n+1}) + \cdots + q(x_{m-1}, x_m) \leq$$

$$(r^n + \cdots + r^{m-1}) max\{q(x_0, x_1), q(x_1, x_0\} \leq$$

$$\frac{r^n}{1-r} max\{q(x_0, x_1), q(x_1, x_0)\}.$$

Similarly, if $m < n$, then

$$q(x_n, x_m) \leq \frac{r^m}{1-r} max\{q(x_0, x_1), q(x_1, x_0)\}.$$

Hence, there exists $n_0 \in \omega$ such that $q(x_n, x_m) \leq \varepsilon$ whenever $n, m \geq n_0$, $n \neq m$. From Lemma 1, we have that $(x_n)_{n \in \omega}$ is a Cauchy sequence in (X, d^s).

Since (X, d) is d^{-1}-sequantially complete, there exists $z \in X$ such that $d(x_n, z) \to 0$.

Next we prove that $q(x_n, z) \to 0$.

Let $n \in \omega$ be fixed. Since, $q(x_n, \cdot)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$, we have that given $\varepsilon > 0$ there exists $m_0 > n$ such that

$$q(x_n, z) - q(x_n, x_m) < \varepsilon$$

for all $m \geq m_0$.

Then

$$q(x_n, z) \leq q(x_n, x_m) + \varepsilon \leq \frac{r^n}{1-r} max\{q(x_0, x_1), q(x_1, x_0)\} + \varepsilon.$$

Consequently, $q(x_n, z) \to 0$.

Now, since $q(Tz, x_n) = q(Tz, Tx_{n-1}) \leq rq(x_{n-1}, z)$, we have taht $q(Tz, x_n) \to 0$.

Let $\varepsilon > 0$. By $(mW3)$ there exists $\delta > 0$ such that if $q(x, y) < \delta$ and $q(y, z) < \delta$ then $d(x, z) < \varepsilon$.

Since $q(Tz, x_n) \to 0$, there is $n_1 \in \mathbb{N}$ such that $q(Tz, x_n) < \delta$ for every $n \geq n_1$.

Since $q(x_n, z) \to 0$, there is $n_2 \geq n_1$ such that $q(x_n, z) < \delta$ for every $n \geq n_2$.

Thus, if $n \geq n_2$ we have that $q(Tz, x_n) < \delta$ and $q(x_{nn}, z) < \delta$. Therefore $d(Tz, z) = 0$.

Finally, if $Tu = u$ then

$$q(u, u) = q(Tu, T^2u) \leq rq(Tu, u) = rq(u, u)$$

and this implies that $q(u, u) = 0$. □

The following example shows that previous theorem can be applied for an appropriate mw-distance on a quasi-metric space (X, d) but not for d.

Example 3. Let $X = [0, 1]$ and let d be the the quasi-metric on X given by $d(x, y) = max\{y - x, 0\}$, for all $x, y \in X$. (X, d) is d^{-1}-sequentially complete. Define $T : X \to X$ as $Tx = x^2/2$ and let q be the $mw-$distance given by $q(x, y) = x + y$, for all $x, y \in X$. Then,

$$q(Tx, Ty) = \frac{x^2}{2} + \frac{y^2}{2} \leq \frac{x}{2} + \frac{y}{2} = \frac{1}{2}(y + x) = \frac{1}{2}q(y, x).$$

Thus, all conditions of Theorem 1 are satisfied. Nevertheless, the contraction condition of Theorem 1 is not satisfied for d. Indeed, suppose that there exists $r \in (0, 1)$ such that $d(Tx, Ty) \leq rd(y, x)$, for all $x, y \in X$. Then

$$d(T\frac{r}{2}, Tr) = \frac{r^2}{4} \leq rd(r, \frac{r}{2}) = 0,$$

and this is a contradiction.

Corollary 4. *Let T be a self mapping of a d^{-1}-sequentially complete T_1 quasi-metric space (X, d) and let q be an $mw-$distance on (X, d). If there exists $r \in [0, 1)$ such that*

$$q(Tx, Ty) \leq rq(y, x)$$

for every $x, y \in X$ then T has a unique fixed point. Moreover, if $Tu = u$ then $q(u, u) = 0$.

Proof. By Theorem 1, there exists $z \in X$ such that $d(Tz, z) = 0$, and this implies that $Tz = z$ because X is a T_1 space.

If we suppose that $Tv = v$, then $q(v, z) = q(Tv, Tz) \leq rq(z, v) \leq r^2 q(v, z)$, so that $q(v, z) = 0$. Since, $q(z, z) = 0$, by (mW3) we have that $d(v, z) = 0$, i.e., $v = z$. □

Definition 5 (Definition 2 of [3]). A d-contraction on a quasi-metric space (X, d) is a mapping $T : X \to X$ such that there is $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$.

A d^{-1}-contraction on a quasi-metric space (X, d) is a mapping $T : X \to X$ such that there is $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(y, x)$ for all $x, y \in X$.

Corollary 6 (Corollary 8 of [3]). *Let (X, d) a T_1 quasi-metric space d^{-1}-sequentially complete. Every d^{-1}-contraction on (X, d) has a unique fixed point.*

Corollary 7 (Theorem 7 of [3]). *Let (X, d) a T_1 quasi-metric space d-sequentially complete. Every d^{-1}-contraction on (X, d) has a unique fixed point.*

Proof. Let $d_0 = d^{-1}$, then (X, d_0) is a T_1 d_0^{-1}-sequentially complete quasi-metric space. If T is a d^{-1}-contraction on (X, d), then

$$d_0(Tx, Ty) = d(Ty, Tx) \leq rd(x, y) = rd_0(y, x),$$

i.e., T is a d_0^{-1}-contraction on (X, d_0). Applying Corollary 2, we have that T has a unique fixed point. □

Theorem 8. *Let T be a self mapping of a d^{-1}-sequentially complete quasi-metric space (X, d) and let q be an $mw-$distance on (X, d). If there exists $k \in [0, 1/2)$ such that*

$$q(Tx, Ty) \leq k(q(Tx, x) + q(Ty, y))$$

for every $x, y \in X$ then there exists $z \in X$ such that $d(Tz, z) = 0$. Moreover, if $Tu = u$ then $q(u, u) = 0$.

Proof. Fix $x_0 \in X$. For each $n \in \omega$ let $x_n = T^n x_0$. Then

$$q(x_{n+1}, x_n) \leq k(q(x_{n+1}, x_n) + q(x_n, x_{n-1})).$$

Put $r = \frac{k}{1-k} < 1$. We have

$$q(x_{n+1}, x_n) \leq r q(x_n, x_{n-1}).$$

Hence, by $(W1)$,

$$q(x_{n+1}, x_n) \leq r^n q(x_1, x_0),$$

for all $n \in \omega$.

Let $\varepsilon > 0$ and let $n, m \in \mathbb{N}$. Then

$$q(x_n, x_m) \leq k(q(x_n, x_{n-1}) + q(x_m, x_{m-1}))$$

$$\leq k(r^{n-1} + r^{m-1}) q(x_1, x_0)$$

Therefore there exists $n_0 \in \omega$ such that $q(x_n, x_m) \leq \varepsilon$ whenever $n, m \geq n_0$. From Lemma 1 it follows that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $z \in X$ such that (x_n) converges to z with respect to the topology $\tau_{d^{-1}}$, i.e., $d(x_n, z) \to 0$.

Next we show that $q(x_n, z) \to 0$. Let $n \in \omega$ be fixed and let $\varepsilon > 0$. Since $q(x_n, \cdot)$ is lower semicontinuous, there exists $m_0 > n$ such that

$$q(x_n, z) - q(x_n, x_m) < \varepsilon$$

for all $m \geq m_0$.

Therefore

$$q(x_n, z) \leq q(x_n, x_m) + \varepsilon \leq 2k q(x_1, x_0) r^{n-1} + \varepsilon.$$

This implies that $q(x_n, z) \to 0$.

Now we prove that $q(Tz, z) = 0$: Indeed,

$$q(Tz, z) \leq q(Tz, Tx_n) + q(Tx_n, z) \leq k(q(Tz, z) + q(Tx_n, x_n)) + q(x_{n+1}, z) \leq$$

$$kq(Tz, z) + kq(x_{n+1}, x_n) + q(x_{n+1}, x_n) + q(x_n, z) \leq$$

$$kq(Tz, z) + (k+1) r^n q(x_1, x_0) + q(x_n, z),$$

for every $n \in \omega$. Then,

$$q(Tz, z) \leq kq(Tz, z),$$

and so $q(Tz, z) = 0$.

Since $q(Tz,Tz) \leq 2kq(Tz,z)$, it follows that $q(Tz,Tz) = 0$. Finally, from condition $(mW3)$ we obtain that $d(Tz,z) = 0$.

Moreover, if $Tu = u$, then

$$q(u,u) = q(Tu,Tu) \leq 2kq(u,u)$$

and hence $q(u,u) = 0$. □

Corollary 9. *Let T be a self mapping of a d^{-1}-sequentially complete quasi-metric space (X,d). If there exists $k \in [0,1/2)$ such that*

$$d(Tx,Ty) \leq k(d(Tx,x) + d(Ty,y))$$

for every $x,y \in X$ then T has a unique fixed point.

Proof. From Theorem 2, taking $q = d$ we obtain that there exists $z \in X$ such that $d(Tz,z) = 0$. Now we show that Tz is a fixed point of T.

Since $d^s(Tx,Ty) \leq k(d(Tx,x) + d(Ty,y))$, for all $x,y \in X$, we have

$$d^s(T^2z,Tz) \leq k(d(T^2z,Tz) + d(Tz,z)) = kd(T^2z,Tz) \leq kd^s(T^2z,Tz).$$

Therefore $d^s(T^2z,Tz) = 0$, i.e, $T^2z = Tz$.

Suppose that u,v are fixed points of T. Then $d^s(u,v) = d^s(Tu,Tv) \leq k(d(Tu,u) + d(Tv,v)) = 0$, and thus $u = v$. □

Corollary 10 (Theorem 2.5 of [1]). *Let T be a self mapping of a d-sequentially complete quasi-metric space (X,d). If there exists $k \in [0,1/2)$ such that*

$$d(Tx,Ty) \leq k(d(x,Tx) + d(yTy))$$

for every $x,y \in X$ then T has a unique fixed point.

Proof. Let $d_0 = d^{-1}$. Then (X,d_0) is a d_0^{-1}-sequentially complete quasi-metric space. Since

$$d_0(Tx,Ty) = d(Ty,tx) \leq k(d(x,Tx) + d(y,Ty)) =$$

$$= k(d_0(Tx,x) + d_0(Ty,y)),$$

from Corollary 4, it follows that T has a unique fixed point. □

It is well known that the Banach and Kannan theorems are independent, therefore Theorem 2 and Theorem 8 are also. However, for the sake of completeness we include here two examples that illustrate this fact.

Example 11. Let $X = [-1, 1]$ and let d be the the quasi-metric on X given by $d(x, y) = max\{y - x, 0\}$, for all $x, y \in X$. (X, d) is d^{-1}-sequentially complete. Define $T : X \to X$ as $Tx = -x/2$ and let $q = d$. We can apply Theorem 1 to T because if $x > y$, then $d(Tx, Ty) = (-y/2 + x/2) \vee 0 = \frac{1}{2}(-y + x) = \frac{1}{2}d(y, x)$, and if $x \leq y$, then $d(Tx, Ty) = 0$. Nevertheless, T does not satisfy the condition of Theorem 2. Indeed, if $x = -1/2$ and $y = -1$ then $d(Tx, Ty) = 1/4$ and $(d(Tx, x) + d(Ty, y)) = 0$.

Example 12. Let $X = [0, 1]$ and let d be the quasi-metric on X given by $d(x, y) = max\{y - x, 0\}$, for all $x, y \in X$. (X, d) is d^{-1}-sequentially complete. Define $T : X \to X$ as $Tx = 1/3$ if $x \neq 1$ and $T1 = 0$ and let $q = d$. We can apply Theorem 2 to T. Indeed, if $x < 1/3$, $d(T1, 1) + d(Tx, x) = 1 = 3d(T1, Tx)$, and if $x \geq 1/3$, $d(T1, 1) + d(Tx, x) = 2/3 + x \geq 1 = 3d(T1, Tx)$. Consequently, $d(T1, Tx) \leq \frac{1}{3}(d(T1, 1) + d(Tx, x))$. Note that $d(Tx, T1) = 0$ for every $x \in X$. T does not satisfy the contraction condition of Theorem 1 because $d(T1, T\frac{2}{3}) = 1/3 = d(\frac{2}{3}, 1) > rd(\frac{2}{3}, 1)$ for all $r \in (0, 1)$.

References

[1] C. Alegre, H. Dag, S. Romaguera, P. Tirado, Characterizations of quasi-metric completeness in terms of Kannan-type fixed point theorems, Hacettepe Journal of Mathematics and Statistic 46 (1) (2017), 67-76.

[2] C. Alegre, J. Marín, Modified w-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces, Topology and its Applications 203 (2016), 32-41.

[3] H. Dag, S. Romaguera, P. Tirado, The Banach contraction principle in quasi-metric spaces revisited, Proceedings of the Workshop on Applied Topological Structures WATS'15 (2015) 25-31.

[4] R. Kannan, Some results on fixed points theorems, Bull. Calcutta Math. Soc. 60 (1968), 71-76.