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The distribution function of a probability measure on a Polish ultrametric space

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Abstract

In this work we elaborate a theory of a cumulative distribution function on a Polish ultrametric space from a probability measure defined in this space. With that purpose, the idea is to define an order in the space from the collection of balls and show that the function defined plays a similar role to that played by a cumulative distribution function in the classical case.

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1. Introduction

This work collects some results on a theory of a cumulative distribution function in a separable complete ultrametric space. It is a preview of [3].

With that purpose, the idea is to define an order in a space from the collection of balls and show that the function defined from its order plays a similar role to that played by a cumulative distribution function in the classical case.

Moreover, we define its pseudo-inverse and study its properties. Those properties will allow us to generate samples of a distribution and give us the chance to calculate integrals with respect to the related probability measure.

2. Ultrametric spaces

First of all, we recall that an ultrametric space (X, d) is a metric space for which the metric d satisfies that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, for each $x, y, z \in X$.

Now, following [2, Def 18.1.1], we recall that

Definition 1. A Polish metric space is a complete metric space which has a countable dense subset.

Given $x \in X$ and $n \in \mathbb{N}$, we will denote by $U_{xn} = \{y \in X : d(x, y) \leq \frac{1}{2^n}\}$ the closed ball, with respect to the ultrametric d, centered at x with radius $\frac{1}{2^n}$. The collection of these balls will be denoted by $\mathcal{G} = \bigcup_{n \in \mathbb{N}} G_n$ where $G_n = \{U_{xn} : x \in X\}$, for each $n \in \mathbb{N}$.

Next we collect some properties of an ultrametric space according to the notation we have just introduced and [1, Ex. 2.1.15]:

Proposition 2. *Let (X, d) be an ultrametric space. Then:*

(1) A ball, U_{xn}, has diameter at most $\frac{1}{2^n}$.
(2) Every point of a ball is a center: that is, if $y \in U_{xn}$, then $U_{xn} = U_{yn}$, for each $x \in X$ and $n \in \mathbb{N}$. Consequently, G_n is a partition of X, that is, it covers X and given $x, y \in X$ it follows that $U_{xn} = U_{yn}$ or $U_{xn} \cap U_{yn} = \emptyset$.
(3) U_{xn} is open and closed in $\tau(d)$ for each $x \in X$ and $n \in \mathbb{N}$.

Note that, according to the previous properties, G_{n+1} is a refinement of G_n (that is, each element of G_{n+1} is contained in some element of G_n) for each $n \in \mathbb{N}$.

In this work, we will assume that (X, d) is a Polish ultrametric space (that is, d is a separable and complete ultrametric). Note that this implies that G_n is countable. Moreover, we will denote by τ the topology induced by d.

3. Defining an order in X

We first define an order in X from the collection of balls $G_n = \{U_{xn} : x \in X\}$ as follows:

Definition 3. We can enumerate $G_1 = \{g_1, g_2, \ldots\}$. Since each element of G_1 can be decomposed into a countable number of elements of G_2 we can write $g_i = g_{i1} \cup g_{i2} \cup \cdots$ for each $g_i \in G_1$, and define the lexicographical order in G_2. Hence, we can enumerate G_2 by considering, first, the elements which are contained in g_1, then those which are contained in g_2, ... Recursively, we define an order in G_n for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, this order induces an order in X given by $x \leq_n y$ if, and only if $U_{xn} \leq U_{yn}$. From that orders, we define an order in X given by $x \leq y$ if, and only if $x \leq_n y$ for each $n \in \mathbb{N}$.

It can be proven that $(G_n, \leq_n)$ is a well ordered set (that is, $\leq_n$ is a total order and each subset has a minimum). Indeed, $(X, \leq)$ is a totally ordered set with a bottom. If G_n is finite for each $n \in \mathbb{N}$ (that is, d is totally bounded), then it also has a top.

From the previous order we define the set $]a, b] = \{x \in X : a < x \leq b\}$. Analogously, we define $]a, b[, [a, b]$ and $[a, b[$. Moreover, $(\leq a)$ is given by $(\leq a) = \{x \in X : x \leq a\}$. $(< a), (\geq a)$ and $(> a)$ are defined similarly.

The previous order also suggests the definition of a new topology in X, τ_o, which is the topology in X given by the order $\leq$, that is, the topology given by the subbase $\{(< a) : a \in X\} \cup \{(> a) : a \in X\}$.

τ_o is related to the topology induced by the ultrametric, τ, in the next sense

Proposition 4. $\tau_o \subseteq \tau$.

4. Defining the cumulative distribution function

The definition of the cumulative distribution function related to a probability measure defined on X is the next one:

Definition 5. The cumulative distribution function (in short, cdf) of a probability measure μ on a Polish ultrametric space X is a function $F : X \to [0, 1]$ defined by $F(x) = \mu(\leq x)$.

Its properties are collected in the next

Proposition 6. *Let F be a cdf. Then:*

1. *F is monotically non-decreasing.*
2. *F is right τ_o-continuous and, consequently, it is right τ-continuous.*
3. $\lim_{x\to\infty} F(x) = 1$ *(this means that for each $\varepsilon > 0$ there exists $y \in X$ with $x \leq y$ such that $1 - F(y) < \varepsilon$).*

The previous proposition makes us wonder the next question which will be answered in [4] by using a fractal structure.

Question 7. *Let $F : X \to [0, 1]$ be a function satisfying the properties collected in the previous proposition, does there exist a probability measure μ on X such that its cdf, F_μ, is F?*

Moreover, given a probability measure on a Polish ultrametric space, we can define $F_- : X \to [0, 1]$, by $F_-(x) = \mu(< x)$, for each $x \in X$.

Its properties are collected in the next proposition.

Proposition 8. *Let μ be a probability measure on X and F its cdf, then:*

1. *F_- is monotically non-decreasing.*
2. *F_- is left τ_o-continuous. Consequently, F_- is also left τ-continuous.*
3. $F_-(\min X) = 0$.

From F and F_- we can get the measure of some sets, as next results show:

Lemma 9. *Let μ be a probability measure on X and F its cdf. Given $x \in X$, it holds that $F(x) = F_-(x) + \mu(\{x\})$.*

Proposition 10. *Let μ be a probability measure on X and F its cdf, then $\mu(]a, b]) = F(b) - F(a)$ for each $a, b \in X$ with $a < b$.*

Corollary 11. *Let μ be a probability measure on X and F its cdf, then:*

(1) $\mu([a, b]) = F(b) - F_-(a)$.
(2) $\mu(]a, b[) = F_-(b) - F(a)$.
(3) $\mu([a, b[) = F_-(b) - F_-(a)$.

5. The pseudo-inverse of a cdf

Finally we see how to define the pseudo-inverse of a cdf F defined on X and we gather some properties which relate this function to both F and F_-. Moreover, we prove that it is measurable.

Let F be a cdf. We define its pseudo-inverse (also called quantile function), $G : [0, 1] \to X$, by $G(x) = \inf\{y \in X : F(y) \geq x\}$ for each $x \in [0, 1]$.

Its properties are collected in the next result.

Proposition 12. *Let F be a cdf and let $x \in X$ and $r \in [0, 1]$. Then:*

(1) *G is monotically non-decreasing.*
(2) $G(F(x)) \leq x$.
(3) $F(G(r)) \geq r$.
(4) *$G(r) \leq x$ if, and only if $r \leq F(x)$.*
(5) *$F(x) < r$ if, and only if $G(r) > x$.*
(6) *If $F_-(x) < r$, then $x \leq G(r)$.*
(7) *If $F_-(x) < r \leq F(x)$, then $G(r) = x$.*
(8) *If $r < F_-(x)$, then $G(r) < x$.*
(9) *If $r = F_-(x)$, then $G(r) \leq x$.*
(10) $F_-(G(r)) \leq r \leq F(G(r))$.

(11) *If* $F(G(r)) > r$*, then* $\mu(\{G(r)\}) > 0$.

(12) *If* $\mu(\{G(r)\}) = 0$*, then* $F(G(r)) = r$.

(13) $G^{-1}(U_{xn}) \in \sigma([0,1])$*, where* $\sigma([0,1])$ *denotes de Borel* σ*-algebra with respect to the Euclidean topology.*

(14) G *is measurable with respect to the Borel* σ*-algebras.*

6. Generating samples

Proposition 13. *Let* μ *be a probability measure, then* $\mu(A) = l(G^{-1}(A))$ *for each* $A \in \sigma([0,1])$*, where* l *is the Lebesgue measure.*

Results in sections 5 and 6 allow us to generate samples with respect to the probability measure μ by following the classical procedure: generate a random uniform sample on $[0,1]$ and then apply G to obtain a sample in (X,μ).

Remark 14. We can also calculate integrals with respect to μ, so, for $g : X \to \mathbb{R}$, it holds

$$\int g(x)d\mu(x) = \int g(G(t))dt$$

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