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A relationship between relaxed metrics and indistinguishability operators

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Abstract

In 1982, E. Trillas introduced the notion of indistinguishability operator with the main aim of fuzzifying the crisp notion of equivalence relation. In the study of such a class of operators, an outstanding property must be stressed. Concretely, there exists a relationship between indistinguishability operators and metrics. The aforesaid relationship was deeply studied by several authors that introduced a few techniques to generate metrics from indistinguishability operators. The main purpose of the present paper is to explore the possibility of making explicit a relationship between indistinguishability operators and relaxed metrics in such a way that the aforementioned classical techniques to generate the former concept from the other, can be extended to the new framework.

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1. Introduction

Throughout this paper we will assume that the reader is familiar with the basics of triangular norms (see [9] for a deeper treatment of the topic). In [12], E. Trillas introduced the notion of T-indistinguishability operator with the aim of fuzzifying the classical (crisp) notion of equivalence relation. Let us recall that, according to [12] (see also [9, 11]), given a t-norm T, a T-indistinguishability operator on a nonempty set X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying for all $x, y, z \in X$ the following conditions

(i) $E(x, x) = 1$ (Reflexivity),
(ii) $E(x, y) = E(y, x)$ (Symmetry),a
(iii) $T(E(x, y), E(y, z)) \leq E(x, z)$ (T-Transitivity).

A T-indistinguishability operator E is said to separate points when $E(x, y) = 1 \Rightarrow x = y$ for all $x, y \in X$.

In the literature the relationship between metrics and T-indistinguishability operators has been studied in depth for several authors [2, 6, 8, 9, 10, 11, 13]. Let us recall a few facts about metric spaces in order to explicitly state the aforesaid relationship. Following [4], a pseudo-metric on a nonempty set X is a function $d : X \times X \to [0, \infty]$ such that, for all $x, y, z \in X$, the following properties hold:

(i) $d(x, x) = 0$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A pseudo-metric d on X is called pseudo-ultrametric if it satisfies, in addition, for all $x, y, z \in X$ the following inequality: (iv) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Of course, a pseudo-metric (pseudo-ultrametric) d on X is called a metric (ultrametric) provided that it satisfies $d(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$:

Regarding the relationship between (pseudo-)metrics and indistinguishability operators, the next result makes it explicit. In fact, it introduces a technique that allows to construct (pseudo-)metrics from indistinguishability operators.

Theorem 1. *Let X be a nonempty set and let T^* be a t-norm with additive generator $f_{T^*} : [0,1] \to [0,\infty]$. Let $d_E : X \times X \to [0,\infty]$ be the function defined by $d_E(x,y) = f_{T^*}(E(x,y))$ for all $x,y \in X$. If T is a t-norm, then the following assertions are equivalent:*

1) $T^* \leq T$ (i.e., $T^*(x,y) \leq T(x,y)$ for all $x,y \in [0,1]$).
2) For any T-indistinguishability operator E on X the function d_E is a pseudo-metric on X.
3) For any T-indistinguishability operator E on X that separates points the function d_E is a metric on X.

In the last years a few generalizations of the metric notion have been introduced in the literature with the purpose of developing suitable mathematical tools for quantitative models in Computer Science and Artificial Intelligence. Concretely, the notion of dislocated metric, dislocated ultrametric, weak partial (pseudo-)metric and partial (pseudo-)metric have been studied and applied to Logic Programming, Domain Theory, Denotational Semantics and Asymptotic Complexity of Programs, respectively. Each of the preceding generalized metric notions can be retrieved as a particular case of a new notion, called relaxed metric, which has been introduced recently in [4].

Definition 2. A relaxed pseudo-metric on a nonempty set X is a function $d : X \times X \to [0,\infty]$ which satisfies for all x,y,z the following:

(i) $d(x,y) = d(y,x)$,
(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

We will say that a relaxed pseudo-metric d on a nonempty set satisfies the small self-distances (SSD for short) property in the spirit of [7] whenever $d(x,x) \leq d(x,y)$ for all $x,y \in X$. Moreover, a relaxed pseudo-metric d is a relaxed metric provided that it satisfies the following separation property for all $x,y \in X$: (iii) $d(x,x) = d(x,y) = d(y,y) \Rightarrow x = y$. Furthermore, a relaxed (pseudo-)metric d on X will be called a relaxed (pseudo-)ultrametric if satisfies in addition, for all x,y,z, the following inequality: (iv) $d(x,y) \leq \max\{d(x,z), d(z,y)\}$.

Recently, it has been discussed that the notion of indistinguishability operator and relaxed metric are closely related. Indeed, in [4, 5] it has been stated that the logical counterpart for relaxed metrics is, in some sense, a generalized indistinguishability operator.

Definition 3. Let X be a non-empty set and let T be a t-norm. A relaxed T-indistinguishability operator E on X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying the following properties for any $x, y, z \in X$:

(i) $E(x, y) = E(y, x)$,
(ii) $T(E(x, z), E(z, y)) \leq E(x, y)$.

Moreover, a relaxed T-indistinguishability operator E satisfies the small-self indistinguishability (SSI for short) property provided that (i) $E(x, y) \leq E(x, x)$ for all $x, y \in X$. Furthermore, a relaxed T-indistinguishability operator E is said to separate points provided that $E(x, y) = E(x, x) = E(y, y) \Rightarrow x = y$ for all $x, y \in X$.

Notice that the notion of T-indistinguishability operator is retrieved as a particular case of relaxed T-indistinguishability operator whenever the relaxed T-indistinguishability operator satisfies also the reflexivity. In fact, a relaxed indistinguishability operator is an indistinguishability operator if and only if it is reflexive. The same occurs when we consider T-indistinguishability operators that separate points.

Motivated, on the one hand, by the exposed facts and, on the other hand, by the utility of generalized metrics in Computer Science and Artificial Intelligence, the target of this paper is to study deeply the relationship between both concepts, relaxed indistinguishability operators and relaxed metrics, and try to extend the method given in Theorem 1 to this new context.

2. From relaxed indistinguishability operators to relaxed metrics

In this section we focus our work on the possibility of extending Theorem 1 to the relaxed framework. First, we will make clear the relationship between relaxed metrics and relaxed T_{Min}-indistinguishability operators, where T_{Min} stands for the minimum t-norm, and then we will specify the correspondence between realaxed T-indistinguishability operators and relaxed metrics whenever one considers t-norms T with additive generator.

According to [14] (see also [11]), the relationship between T_{Min}-indistinguishability operators and metrics is given by the next result.

Proposition 4. *Let X be a nonempty set and let $E : X \times X \to [0,1]$ be a fuzzy relation. Then the following assertions are equivalent:*

1) *E is a T_{Min}-indistinguishability operator.*
2) *The function d_E is a pseudo-ultrametric on X, where $d_E(x,y) = 1 - E(x,y)$ for all $x, y \in X$.*

Moreover, E separates points if and only if d_E is a ultrametric on X.

Next we show that the preceding result can be easily extended to our new context.

Proposition 5. *Let X be a nonempty set and let E be a fuzzy relation on X. Then the following assertions are equivalent:*

1) *E is a relaxed T_{Min}-indistinguishability operator.*
2) *The function d_E is a relaxed pseudo-ultrametric on X, where $d_E(x,y) = 1 - E(x,y)$ for all $x, y \in X$.*

Moreover, E separates points if and only if d_E is a relaxed ultrametric on X.

Corollary 6. *Let X be a nonempty set and let E be a T_{Min}-indistinguishability operator on X. Then the following assertions hold:*

1) *E fulfills the SSI property and, thus, d_E fulfills the SSD property.*
2) *E separates points if and only if d_E is a relaxed ultrametric.*

Next we focus our attention on the relationship that there exists between relaxed metrics and the relaxed indistinguishability operators when the t-norm under consideration admits an additive generator. Notice that the study developed before considers relaxed T_{Min}-indistinguishability operators and that the t-norm T_{Min}

does not admit additive generator. The next result provides an affirmative answer to the question about whether Theorem 1 can be stated when the t-norm admits additive generator.

Theorem 7. *Let X be a nonempty set and let T^* be a t-norm with additive generator f_{T^*}. Let d_E be a function defined by $d_E^{f_{T^*}}(x,y) = f_{T^*}(E(x,y))$ for all $x,y \in X$. If T is a t-norm, then the following assertions are equivalent:*

1) *$T^* \leq T$.*
2) *For any relaxed T-indistinguishability operator E on X the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X.*
3) *For any relaxed T-indistinguishability operator E on X that separates points the function $d_E^{f_{T^*}}$ is a relaxed metric on X.*

It is worth pointing out that Theorems 1 and 7 disclose a surprising connection (equivalence) between indistinguishability operators and the relaxed ones.

In [3, 6, 13] (see also [1, 2]), the subsequent characterization was given establish the relationship between indistinguishability operators and (pseudo-)metrics. Concretely, the aforesaid characterization states the following.

Theorem 8. *Let X be a nonempty set and let E be a fuzzy binary relation on X. Let d_E be the function defined by $d_E^{f_{T_L}}(x,y) = 1 - E(x,y)$ for all $x,y \in X$. If T is a t-norm, then the following assertions are equivalent:*

1) *$T_L \leq T$.*
2) *For any T-indistinguishability operator the function $d_E^{f_{T_L}}$ is a pseudo-metric on X.*
3) *For any T-indistinguishability operator that separates points the function $d_E^{f_{T_L}}$ is a metric on X.*

Taking in Theorem 7, T^* as the Lukasiewicz t-norm T_L and the function f_{T^*} as the function $f_{T_L} : [0,1] \to [0,\infty]$ given by $f_{T_L}(x) = 1-x$ for all $x \in [0,1]$ we obtain as a particular case the following results, one of them, Corollary 9, providing an extension of Theorem 8.

Corollary 9. *Let X be a nonempty set and let $E : X \times X \to [0,1]$ be a fuzzy relation. Let $d_E : X \times X \to \mathbb{R}^+$ be the function defined by $d_E(x,y) = 1 - E(x,y)$ for all $x, y \in X$. If T is a t-norm, then the following assertions are equivalent:*

1) *$T_L \leq T$.*
2) *For any relaxed T-indistinguishability operator the function d_E is a relaxed pseudo-metric on X.*
3) *For any relaxed T-indistinguishability operator that separates points the function d_E is a relaxed metric on X.*

When we consider in Theorem 7 the t-norm T as the minimum t-norm T_M and the function f_{T^*} as the additive generator of any t-norm T^* we retrieve as a particular case the following result.

Corollary 10. *Let E be a relaxed T_{Min}-indistinguishability operator on a nonempty set X. Then the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X for any additive generator f_{T^*} of a t-norm T^*.*

Of course the preceding results agree with Theorem 4 because every relaxed pseudo-ultrametric is a relaxed pseudo-metric.

If we consider in Theorem 7 the t-norm T^* as the Drastic product T_D and the function f_{T^*} as an additive generator of T_D, i.e., $f_{T_D}(x) = 2 - x$ if $x \in [0,1[$ and $f(1) = 0$, then we get as a consequence the following result.

Corollary 11. *If E is a relaxed T-indistinguishability operator on a nonempty set X, then the function $d_E^{f_{T_D}}$ is a relaxed pseudo-metric on X.*

If we consider in Corollaries 10 and 11 indistinguishability operators that separate points then the obtained relaxed pseudo-metrics become relaxed metrics.

Clearly Theorem 7 provides a technique to generate relaxed pseudo-metrics from relaxed indistinguishability operators. Observe that in spite of the aforementioned equivalence between Theorems 1 and 7, the new technique gives instances of relaxed pseudo-metric which are not pseudo-metrics.

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