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Some observations on a fuzzy metric space

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Abstract

Let (X, d) be a metric space. In this paper we provide some observations about the fuzzy metric space in the sense of Kramosil and Michalek $(Y, N, \wedge)$, where Y is the set of non-negative real numbers $[0, \infty[$ and $N(x, y, t) = 1$ if $d(x, y) \leq t$ and $N(x, y, t) = 0$ if $d(x, y) \geq t$.

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1. Introduction and preliminaries

In 1975, Kramosil and Michalek extended the concept of Menger space to the fuzzy setting [11], providing a concept of fuzzy metric space which, in modern terminology, is the following.

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Definition 1 ([2, 3]). A *KM-fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left-continuous (Also written as $M_{x,y}(t) = M(x, y, t)$).

If $(X, M, *)$ is a KM-fuzzy metric space, it is also said that M is a KM-fuzzy metric on X.

Further, if the fuzzy set M, in the above definition, takes values in $]0, 1]$, and so $(KM1)$ is removed, and $(KM2)$ is replaced by $M(x, y, t) = 1$ if and only if $x = y$, and $(KM5)$ is strengthened demanding continuity to the function $M_{x,y}$ then, we obtain the concept of GV-fuzzy metric space due to George and Veeramani [2]. Both concepts will be referred as fuzzy metric space whenever distinction is not necessary. In fact, a GV-fuzzy metric can be considered a KM-fuzzy metric defining $M(x, y, 0) = 0$ for each $x, y \in X$.

A fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

A significant difference between KM-fuzzy metrics and GV-fuzzy metrics is that the first ones admit completion (see [1, 15]) and the second ones can not be completable (see [9]).

An interesting example of KM-fuzzy metric space [14] used by D. Mihet in [12] for proving the existence of non-Cauchy sequences which are fuzzy contractive, in the sense of Gregori and Sapena [10] is the following.

Example 2. Let (Y, d) be the usual metric on the real interval $Y = [0, \infty[$. Then $(Y, N, *)$ is a KM-fuzzy metric space, for every continuous t-norm, where

$$N(x, y, t) = \begin{cases} 0, & \text{if } d(x, y) \geq t; \\ 1, & \text{if } d(x, y) < t. \end{cases}$$

The aim of this paper is to provide some observations about this last example on some concepts defined in fuzzy metric spaces. These observations will point out significant differences between KM-fuzzy metrics and GV-fuzzy metrics, in some aspects. The mentioned example will be denoted by $(Y, N, \wedge)$, where $\wedge$ is considered the minimum t-norm, throughout the paper.

2. Observations to $(Y, N, \wedge)$

2.1. **Degree of nearness in** $(Y, N, \wedge)$**.** If M is a fuzzy metric space on X then George and Veeramani [2] interpreted $M(x, y, t)$ as the degree of nearness between x and y, with respect to t. Under this interpretation we observe in the case of $(Y, N, \wedge)$ that any two distinct points x and y are *infinitely* separated with respect to t whenever $0 \leq t < d(x, y)$, since in this case $N(x, y, t) = 0$, and, suddenly, they are *infinitely* close if $t > d(x, y)$, since in this case $N(x, y, t) = 1$.

Definition 3. A fuzzy metric space $(X, M, *)$, or simply M, is called *strong* [8] if for each $x, y, z \in X$ and $t > 0$ it satisfies

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t)$$

Proposition 4. *The fuzzy metric space* $(Y, N, *)$ *is not strong, for each continuous* t*-norm.*

Proof. We have that $N(1, 3, 3) = N(3, 5, 3) = 1$ and $N(1, 5, 3) = 0$. Thus,

$$0 = N(1, 5, 3) < N(1, 3, 3) * N(3, 5, 3) = 1 * 1 = 1,$$

for each $*$ continuous t-norm. □

2.2. **Topology of** $(Y, N, \wedge)$**.** Recall that an open ball centered at $x \in X$ of radius $r \in]0,1[$ and parameter $t > 0$, denoted by $B(x, r, t)$, is formed by those points $y \in Y$ satisfying $N(x, y, t) > 1 - r$. So, let $x \in X$, $r \in]0,1[$ and $t > 0$, then the open ball $B(x, r, t)$ is the set $\{y : N(x, y, t) > 1-r\}$, that is those points $y \in Y$ such that $N(x, y, t) = 1$, or equivalently, $\{y \in Y : d(x, y) < t\}$. Therefore, $B(x, r, t)$ coincides with the open d-ball centered at x and radius $t > 0$, denoted usually by $B_t(x)$. Consequently, τ_M coincides with $\tau(d)$ (the topology on X deduced from d).

Now, the authors in [2] proved that closed balls, in a GV-fuzzy metric space, are closed set. Nevertheless, this assertion is not true, in general, in a KM-fuzzy metric space. In fact, in the fuzzy metric space $(N, Y, \wedge)$ the situation is different as we will see in the following.

Recall that a closed ball centered at $x \in X$ of radius $r \in]0,1[$ and parameter $t > 0$, $B[x, r, t]$ is the set $\{y \in Y : N(x, y, t) \geq 1 - r\}$. Then,

$$B[x, r, t] = \{y \in Y : N(x, y, t) = 1\} = B(x, r, t) = B_t(x).$$

That is, closed balls in $(N, y, \wedge)$ are open sets. Further, for each $r \in]0,1[$ we have that $B[x, r, t] = B_t(x)$.

We continue studying if the fuzzy metric space $(Y, N, \wedge)$ satisfies two topological properties defined in the context of fuzzy metric spaces, which have no sense in classical metrics.

We will see that $(Y, N, \wedge)$ is not principal.

Recall that a GV-fuzzy metric space is called *principal* [4] if the family $\{B(x, r, t) : r \in]0,1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$. Extending this concept to KM-fuzzy metric spaces we can observe that $(Y, N, \wedge)$ is not principal. Indeed, given $x \in Y$, for a fixed $t > 0$ we have that $\{B(x, r, t) : r \in]0,1[\} = \{B_t(x)\}$, as we have observed, and obviously $\{B_t(x)\}$ is not a local base at x, for the usual topology of $\mathbb{R}$ restricted to Y.

We will see that $(Y, N, \wedge)$ is co-principal.

Recall that a GV-fuzzy metric space is called *co-principal* [5] if the family $\{B(x,r,t) : t > 0\}$ is a local base at x, for each $x \in X$ and $r \in]0,1[$. Now, if we extend this concept to the context of KM-fuzzy metric spaces, we can observe that $(Y, N, \wedge)$ is co-principal. Indeed, let $x \in X$ and fix $r \in]0,1[$, then $\{B(x,r,t) : t > 0\} = \{B_t(x) : t > 0\}$, which is a local base at x.

2.3. **Completeness of $(Y, N, \wedge)$.** In this subsection, we will study if the fuzzy metric space $(Y, N, \wedge)$ is complete, attending to different notions of fuzzy metric completeness appeared in the literature.

First, we recall the concept of Cauchy sequence given formerly by H. Sherwood in Probabilistic Metric spaces [15] and later by George and Veeramani [2] in the fuzzy metric context.

Definition 5. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *Cauchy* if for each $\epsilon \in]0,1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is M-Cauchy if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$, where $\lim_{n,m}$ denotes the double limit as $n \to \infty$, and $m \to \infty$.

X is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Proposition 6. *The fuzzy metric space $(Y, N, \wedge)$ is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(Y, N, \wedge)$. We will see that it is a convergent sequence in Y for τ_N.

By definition, given $\epsilon \in]0,1[$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $N(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$, and so $N(x_n, x_m, t) = 1$ for all $n, m \geq n_0$. Consequently, $d(x_n, x_m) < t$ for all $n, m \geq n_0$ (notice that this assertion is valid for every $\epsilon \in]0,1[$). Therefore, for each $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < t$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} d(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a d-Cauchy sequence, i.e. is a Cauchy sequence for the metric space (Y, d). Now, taking into account that (Y, d) is a complete metric space, we can find $x_0 \in Y$ such that $\{x_n\}$ converges for the topology $\tau(d)$. Finally, since, as we have observed, τ_N coincides with $\tau(d)$, we have that $\{x_n\}$ is convergent as we claimed. □

The next notion of Cauchy sequence was formerly given by M. Grabiec [3], although we present it here attending to a reformulation given by D. Mihet in [13].

Definition 7. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. We will say that $\{x_n\}$ is a *G-Cauchy* sequence if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

We will say that $(X, M, *)$ is *G-complete* if each G-Cauchy sequence is convergent.

We will say that $(X, M, *)$ is *weak G-complete* [6, 7] if each G-Cauchy sequence has, at least, a cluster point.

Attending to the last concepts about completeness, it is obvious that ever G-complete fuzzy metric spaces is weak G-complete.

Proposition 8. *The fuzzy metric space* $(Y, N, \wedge)$ *is not (weak)* G*-complete.*

Proof. Consider the sequence $\{s_n\}$ (harmonic series), where $s_n = \sum_{i=1}^{n} \frac{1}{i}$, for each $n \in \mathbb{N}$. We claim that $\{s_n\}$ is G-Cauchy in $(Y, N, \wedge)$. Indeed, if we take take $t > 0$, then we can find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t$. Thus $d(s_m, s_{m+1}) = \frac{1}{m+1} < t$ for all $m \geq n_0$, and consequently $N(s_m, s_{m+1}, t) = 1$ for all $m \geq n_0$, i.e. $\lim_m N(s_m, s_{m+1}, t) = 1$ and so it is G-Cauchy.

It is obvious that $\{s_n\}$ has not any cluster point in Y and hence $(Y, N, \wedge)$ is not weak G-complete. □

To finish, we will study the completeness of $(Y, N, \wedge)$ related to the concept of p-convergence introduced by D. Mihet in [13].

Definition 9. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. We will say that $\{x_n\}$ is *p-convergent* to x_0 if there exists $t > 0$ such that $\lim_n M(x_n, x_0, t) = 1$.

$\{x_n\}$ is called *p-Cauchy* [4] if there exists $t > 0$ such that $\lim_{n,m} M(x_n, x_m, t) = 1$.

$(X, M, *)$ is called *(weak) p-complete* if every p-Cauchy sequence in X is (p-) convergent.

Proposition 10. *The fuzzy metric space* $(Y, N, \wedge)$ *is weak* p*-complete.*

Proof. Let $\{x_n\}$ be a sequence in $(Y, N, \wedge)$. First, we claim that $\{x_n\}$ is p-Cauchy if and only if $\{x_n\}$ is d-bounded. Indeed, suppose that $\{x_n\}$ is p-Cauchy. Then, $\lim_{n,m} M(x_n, x_m, t) = 1$ for some $t > 0$. Hence, given $\epsilon \in]0, 1[$ we can find $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\epsilon$ for all $n, m \geq n_0$, that is $d(x_n, x_m) < t$ for all $n, m \geq n_0$. Let $K = \max\{d(x_n, x_m) : n, m \leq n_0\}$, then obviously $K + t$ is a d-bound of $\{x_n\}$. Conversely, suppose that $\{x_n\}$ is d-bounded. Let $K > 0$ be an upper bound of $\{x_n\}$. Then, $d(x_n, x_m) \leq K < K + 1$ and so $\lim_{n,m} N(x_n, x_m, K + 1) = 1$. Thus, $\{x_n\}$ is p-Cauchy.

Let $\{x_n\}$ be a Cauchy sequence. By the last observation, we can find $K > 0$ such that $d(x_n, x_m) < K$. Then, for each $x \in]0, K[$ we have that $d(x_n, x) < K$ and so $\lim_n N(x_n, x, K) = 1$. Thus, $\{x_n\}$ is p-convergent to x. (Moreover, one can show that $\{x_n\}$ is p-convergent to x for each $x \in Y$.) □

Proposition 11. *The fuzzy metric space* $(Y, N, \wedge)$ *is not p-complete.*

Proof. By the observation in the proof of the last proposition, the bounded sequence $\{1, 2, 1, 2, 1, \ldots\}$ is p-Cauchy, but, obviously, it is not convergent. □

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