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Fuzzy contractive sequences

Valentín Gregori[1] *and Almanzor Sapena*

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Spain

(vgregori@mat.upv.es, alsapie@mat.upv.es)

Abstract

In this paper we survey some results on contractive sequences in fuzzy metric spaces in the sense of George and Veeramani.

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1. Introduction and preliminaries

George and Veeramani [1] gave the following definition of fuzzy metric which is a slight modification of the one given by Kramosil and Michalek [11].

Definition 1. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

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(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

It is also said that M is a fuzzy metric on X.

If we define $M(x, y, 0) = 0$ and (GV2) and (GV5) are replaced by

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous,

then $(X, M, *)$ is a KM-fuzzy metric space.

From both fuzzy metric spaces on can deduce on X a topology τ_M which has as a base the family of open sets of the form $B(x, \varepsilon, t) = \{x \in X, 0 < \varepsilon < 1, t > 0\}$ where $B(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $\varepsilon \in]0, 1[$ and $t > 0$.

If (X, d) is a metric space then $(X, M_d, *)$ is a fuzzy metric space which is called standard fuzzy metric space deduced from (X, d) where

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Definition 2. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ is called Cauchy if given $\varepsilon \in]0, 1[$ and $t > 0$ there exists $n_0 \in \mathbb{N}$, which depends on ε and t, such that $M(x_m, x_n, t) > 1 - \varepsilon$ for all $m, n \geq n_0$ or, equivalently, $\lim\limits_{m,n} M(x_m, x_n, t) = 1$ for all $t > 0$.

$(X, M, *)$, or simply M, is called complete if every Cauchy sequence in X is convergent in (X, τ_M).

2. Contractive sequences

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is called contractive if there exists $k \in]0, 1[$ such that $d(x_{n+2}, x_{n+1}) \leq k \cdot d(x_{n+1}, x_n),\ n \in \mathbb{N}$.

It is well known that every contractive sequence is Cauchy and hence in a complete metric space every contractive sequence is convergent. Cauchy sequences, and so contractive sequences, are interesting because one can assert their convergence (in complete metric spaces) ignoring the point of convergence. Notice that in some cases to verify the contractive condition on a sequence it can be easier than Cauchyness's condition.

The concept of contractive sequence is strongly related with the theory of fixed point theorems initiated by the Banach Contraction Principle. Indeed, suppose that f is a self-contractive mapping (a contraction) of a complete metric space (X, d), that is there exists $k \in]0, 1[$ such that $d(f(x), f(y)) \leq k \cdot d(x, y)$. Then, for each $x_0 \in X$, the sequence of iterates $\{x_n\}$, where $x_n = f(x_{n-1}),\ n = 1, 2, \ldots$ is contractive and then $\{x_n\}$ is convergent to a point $y \in X$, which is the fixed point for f.

The aim of this paper is to revise the results obtained on contractive sequences in our fuzzy setting.

3. On fuzzy contractive sequences

In order to obtain a fixed point theorem in fuzzy setting the authors gave the following definition.

Definition 3 ([5]). Let $(X, M, *)$ be a GV-fuzzy metric space. A mapping $f : X \to X$ is called fuzzy contractive if there exists $k \in]0, 1[$ such that $\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right)$ for all $x, y \in X$ and $t > 0$.

Accordingly to this definition the authors also gave the following concept.

Definition 4. A sequence $\{x_n\}$ in X is called fuzzy contractive if there exists $k \in]0, 1[$ such that.

$$\text{(1)} \qquad \frac{1}{M(f(x_{n+2}), f(n_{n+1}), t)} - 1 \leq k \left(\frac{1}{M(x_{n+1}, x_n, t)} - 1 \right)$$

for all $n \in \mathbb{N}$ and $t > 0$.

Radu [15] rewrote the notion of fuzzy contractive mapping in the equivalent form

$$M(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))} \tag{2}$$

which is more convenient that (1) because it remains valid in the context of $KM-$ fuzzy metric spaces in which $M(x, y, t)$ can take the value 0.

Accordingly, a sequence $\{x_n\}$ is fuzzy contractive in a fuzzy metric spaces $(X, M, *)$ if

$$M(x_{n+2}, x_{n+1}, t) \geq \frac{M(x_{n+1}, x_n, t)}{M(x_{n+1}, x_n, t) + k(1 - M(x_{n+1}, x_n, t))} \tag{3}$$

The given concept of fuzzy contractive sequence is appropriate in the sense that if (X, d) is a metric space then $\{x_n\}$ is contractive in (X, d) if and only if $\{x_n\}$ is fuzzy contractive in $(X, M_d, \cdot)$. Consequently, in $(X, M_d, *)$ a fuzzy contractive sequence is Cauchy. In order to prove that this assertion is true for any fuzzy metric space, the authors posed in [5] the following question, which is the subject of this paper, stated in GV-fuzzy metric spaces.

Question 5. *Is a fuzzy contractive sequence a Cauchy sequence (in George and Veeramani's sense)?*

A negative response to this question, but in the context of KM-fuzzy metric spaces was given by D.Mihet, which we reproduce, in a slight different way, in the following example.

Example 6 ([12])**.** Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define

$$M(x, y, t) = \begin{cases} 0 & \text{if} \quad t \leq d(x, y) \\ 1 & \text{if} \quad t > d(x, y) \end{cases}$$

Then $(X, M, *)$ is a KM-fuzzy metric space for the t-norm minimum and consequently for every continuous t-norm, but clearly M is not a GV-fuzzy metric. Further, τ_M agrees with the topology on X deduced from d, and $(X, M, *)$ is complete.

Now, in this case it is easy to verify that a sequence $\{x_n\}$ is fuzzy contractive if and only if $M(x_{n+2}, x_{n+1}, t) \geq M(x_{n+1}, x_n, t)$ for all $n \in \mathbb{N}$ and $t > 0$, or equivalently if and only if $d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n,)$ for all $n \in \mathbb{N}$.

Consider the sequence $\{x_n\}$ where $x_n = n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is fuzzy contractive since it fulfills the last condition, and obviously $\{x_n\}$ is not Cauchy since $\lim_{m,n} M(x_n, x_m, \frac{1}{2}) = 0$.

4. Affirmative partial responses to Question 5

In the following we will see two recent partial affirmative responses to Question 5.

Definition 7 ([13])**.** Let Ψ be the class of all mappings $\psi :]0,1] \to]0,1]$ such that ψ is continuous, non-decreasing and $\psi(s) > s$ for all $s \in]0,1]$. Let $(X, M, *)$ be a fuzzy metric space and $\psi \in \Psi$. A mapping $f : X \to X$ is called ψ-contractive if $M(f(x), f(y), t) \geq \psi(M(x, y, t))$ for all $x, y \in X$ and $t > 0$.

A sequence $\{x_n\}$ in X is called fuzzy ψ-contractive if $M(x_{n+2}, x_{n+1}, t) \geq \psi(M(x_{n+1}, x_n, t))$ for all $x, y \in X$ and $t > 0$.

In order to obtain fixed point theorems in GV-fuzzy metric spaces, in [7] the authors gave the following results.

Proposition 8 ([7], Corollary 3.8)**.** *Let $(X, M, *)$ be a GV-fuzzy metric space such that $\bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$. Then every fuzzy ψ-contractive sequence is Cauchy.*

Proposition 9 ([7], Lemma 3.12)**.** *Let $(C, M, *)$ be a strong GV-fuzzy metric space. Then, every fuzzy ψ-contractive sequence is Cauchy.*

It is easy to observe, after seeing equation (3), that every fuzzy contractive sequence is fuzzy ψ-contractive for $\psi(s) = \dfrac{s}{s + k(1-s)}$. Consequently, the last two propositions are affirmative partial responses to Question 5.