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The distribution function of a probability measure on the completion of a space with a fractal structure

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Abstract

In this work we show how to define a probability measure with the help of a fractal structure. One of the keys of this approach is to use the completion of the fractal structure. Then we use the theory of a cumulative distribution function on a Polish ultrametric space and describe it in this context. Finally, with the help of fractal structures, we prove that a function satisfying the properties of a cumulative distribution function on a Polish ultrametric space is a cumulative distribution function with respect to some probability measure on the space.

Keywords: *probability; fractal structure; non-archimedean quasi-metric; measure; cumulative distribution function; ultrametric; Polish space.*
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1. Introduction

This work collects and advances some results on a research line on the construction of a probability measure with the help of a fractal structure, which is in current development ([2], [3], [4], [5]).

First, we show how to define a probability measure on the completion of a fractal structure. Second, we show a theory of the cumulative distribution function on Polish ultrametric spaces. Finally, we use fractal structures to prove that a probability measure on a Polish ultrametric space can be fully described by a cumulative distribution function.

2. Fractal structures and non archimedean quasi metrics

Fractal structures were introduced in [1] to study non archimedean quasi metrization, but they have a wide range of applications (see for example [6]).

Let X be a set and Γ_1 and Γ_2 be coverings of X. Γ_2 is said to be a strong refinement of Γ_1 if it is a refinement (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $A \in \Gamma_1$ we have that $A = \cup\{B \in \Gamma_2 : B \subseteq A\}$.

Definition 1. A fractal structure $\mathbf{\Gamma}$ on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that each cover Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Cover Γ_n is called level n of the fractal structure.

A quasi pseudo metric on a set X is a function $d : X \times X \to [0, \infty[$ such that:

(1) $d(x, x) = 0$, for each $x \in X$.
(2) $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

d is called a pseudo metric if it also satisfies that $d(x, y) = d(y, x)$ for each $x, y \in X$. A quasi pseudo metric (resp. a pseudo metric) is said to be a T_0 quasi metric (resp. a metric) if $d(x, y) = d(y, x) = 0$ implies that $x = y$, for each $x, y \in X$.

If d is a quasi (pseudo) metric, the function defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi (pseudo) metric, called conjugate quasi (pseudo) metric of d. Furthermore, the function $d^*(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo) metric.

A quasi pseudo metric is said to be non archimedean if $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for each $x, y, z \in X$.

If d is a non archimedean quasi (pseudo) metric, then d^{-1} is also a non archimedean quasi (pseudo) metric and d^* is a non archimedean (pseudo) metric. A non-archimedean metric is also called an ultrametric.

A fractal structure $\mathbf{\Gamma}$ induces a non archimedean quasi pseudo metric $d_{\mathbf{\Gamma}}$ given by:

$$d_{\mathbf{\Gamma}}(x,y) = \begin{cases} \frac{1}{2^n} & \text{if} \quad y \in U_{xn} \backslash U_{x,n+1} \\ \\ 1 & \text{if} \quad y \notin U_{x1} \end{cases}$$

where $U_{xn} = X \setminus \bigcup\{A \in \Gamma_n : x \notin A\}$ for each $x \in X$ and $n \in \mathbb{N}$.

In this work, we will assume that the induced topology is T_0, and hence $d_{\mathbf{\Gamma}}$ is a non archimedean T_0-quasi metric. It follows that $d^*_{\mathbf{\Gamma}}$ is an ultrametric.

Given $x \in X$ and $n \in \mathbb{N}$, we will denote by $U^*_{xn} = \{y \in X : d^*(x,y) \leq \frac{1}{2^n}\}$ the closed ball, with respect to the ultrametric d^*, centered at x with radius $\frac{1}{2^n}$. The collection of these balls will be denoted by $\mathcal{G} = \{U^*_{xn} : x \in X; n \in \mathbb{N}\}$.

2.1. **Completion of a fractal structure.** The completion of a fractal structure is constructed from the following extension of X introduced in [1].

Let $G_n = \{U^*_{xn} : x \in X\}$. Note that G_n is a partition of X. Then we can define the projection $\rho_n : X \to G_n$ by $\rho_n(x) = U^*_{xn}$, and the bonding maps $\phi_n : G_{n+1} \to G_n$ given by $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$. We will denote by $\widetilde{X} = \varprojlim G_n = \{(g_1, g_2, ...) \in \prod_{n=1}^{\infty} G_n : \phi(g_{n+1}) = g_n, \forall n \in \mathbb{N}\}$. Now, the map $\rho : X \to \widetilde{X}$ defined as $\rho(x) = (\rho_n(x))_{n \in \mathbb{N}}$ is an embedding of X into $\widetilde{X}$.

Using the previous extension, we can introduce the bicompletion of a fractal structure following [2]. Given $\mathbf{\Gamma}$ a fractal structure, we define level n of the extended fractal structure $\widetilde{\mathbf{\Gamma}}$ as $\widetilde{\Gamma}_n = \{\widetilde{A} : A \in \Gamma_n\}$, where $\widetilde{A} = \{(\rho_k(x_k))_{k \in \mathbb{N}} \in \widetilde{X} : x_n \in A\}$ for each $A \in \Gamma_n$ and $n \in \mathbb{N}$.

We will denote by $\widetilde{U}^*_{xn} = \{y \in \widetilde{X} : \widetilde{d}^*(x,y) \leq \frac{1}{2^n}\}$, where $\widetilde{d}^*$ is the ultrametric induced by $\widetilde{\mathbf{\Gamma}}$ on $\widetilde{X}$. Following a similar notation, we will denote the collection of these balls by $\widetilde{\mathcal{G}} = \{\widetilde{U}^*_{xn} : x \in X; n \in \mathbb{N}\} = \{\widetilde{U}^*_{xn} : x \in \widetilde{X}; n \in \mathbb{N}\}$.

Note that $(\widetilde{X}, \widetilde{d}^*)$ is a complete ultrametric space.

3. Defining a probability measure on $\widetilde{X}$

In this section we show how to define a probability measure on $\widetilde{X}$ by defining it on $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ (this section is further developed in [3]). From now on, we will assume that $\tau(d^*)$ is separable, and hence $(\widetilde{X}, \widetilde{d}^*)$ is a Polish ultrametric space.

Let ω be a pre-measure $\omega : \mathcal{G} \to [0,1]$. We will say that ω satisfies the mass distribution conditions if:

(1) $\sum\{\omega(U^*_{x1}) : U^*_{x1} \in G_1\} = 1$.
(2) $\omega(U^*_{xn}) = \sum\{\omega(U^*_{y,n+1}) : U^*_{y,n+1} \in G_{n+1}; y \in U^*_{xn}\}$ for each $U^*_{xn} \in G_n$ and each $n \in \mathbb{N}$.

Note that ω can be extended to $\widetilde{\mathcal{G}}$ by letting $\widetilde{\omega}(\widetilde{U}^*_{xn}) = \omega(U^*_{xn})$, for each $x \in X$ and $n \in \mathbb{N}$. It follows that $\widetilde{\omega}$ also satisfies the mass distribution conditions.

It is proved in [3] that $\widetilde{\omega}$ can be extended to a probability measure μ on the Borel sigma-algebra of $(\widetilde{X}, \widetilde{d}^*)$.

There is an alternative way of defining the pre-measure ω using Γ_n instead of G_n. We refer the interested reader to [3].

4. Cumulative distribution function on a Polish ultrametric space

In this section we elaborate a theory of a cumulative distribution function on a Polish ultrametric space (this section is further developed in [4]). In this section we assume that (X,d) is a Polish ultrametric space (that is, d is a separable complete ultrametric).

First, we define an order in X from the collection of balls $G_n = \{B_{xn} : x \in X\}$, where $B_{xn} = \{y \in X : d(x, y) \leq 2^{-n}\}$ is the closed ball of radius 2^{-n}. Note that G_n is countable since d is separable.

We can enumerate $G_1 = \{g_1, g_2, \ldots\}$. Now we enumerate G_2 such that $g_i = g_{i1} \cup g_{i2} \cup \cdots$ for each $g_i \in G_1$, and define the lexicographical order in G_2. Recursively, we define an order in G_n for each $n \in \mathbb{N}$.

This order induces an order in X given by $x \leq_n y$ if and only if $B_{xn} \leq B_{yn}$ in G_n.

Finally we can define a new order in X given by $x \leq y$ if and only if $x \leq_n y$ for each $n \in \mathbb{N}$.

Definition 2. The cumulative distribution function (in short, cdf) of a probability measure μ on a Polish ultrametric space X is a function $F : X \to [0, 1]$ defined by $F(x) = \mu(\leq x)$, where $(\leq x) = \{y \in X : y \leq x\}$.

Proposition 3. *Let F be the cdf of a probability measure μ on a Polish ultrametric space X. Then:*

(1) F is non-decreasing.
(2) F is right τ_d-continuous.
(3) $\lim_{x\to\infty} F(x) = 1$ (this means that for each $\varepsilon > 0$ and $x \in X$ there exists $y \in X$ with $x \leq y$ and such that $1 - F(y) < \varepsilon$).

5. Distribution function of a probability measure constructed from a fractal structure

In this section we show how to use the theory of a cdf on a Polish ultrametric space in the completion of a space with a fractal structure (this section is further developed in [5]). By using the probability measure constructed from a pre-measure satisfying the mass distribution conditions, we will be able to prove some results of the theory of a cdf on a Polish ultrametric space.

First, we show that the cdf of a probability measure constructed from a pre-measure ω satisfying the mass distribution conditions can be described by just using the pre-measure.

Theorem 4. *Let* $\mathbf{\Gamma}$ *be a fractal structure on a set* X*,* ω *a pre-measure on* $\mathcal{G}$ *(or* $\widetilde{\mathcal{G}}$*) satisfying the mass distribution conditions,* μ *the extension of* ω *to a probability measure on the Borel* σ*-algebra of* $(\widetilde{X}, \widetilde{d}^*)$ *and* F *be the cdf of* μ*. Then* $F(x) = \lim h_n^+(x)$*, for each* $x \in \widetilde{X}$*, where* $h_n^+(x) = \sum\{\widetilde{\omega}(g) : g \in \widetilde{G}_n; g \leq_n \widetilde{U}^*_{xn}\}$*, for each* $x \in \widetilde{X}$ *and* $n \in \mathbb{N}$*.*

Next, we prove that any function on $\widetilde{X}$ satisfying the properties of Proposition 3 is in fact the cumulative distribution function of a probability measure on $\widetilde{X}$ defined with the help of a fractal structure.

Theorem 5. *Let* $F : \widetilde{X} \to [0, 1]$ *be a non-decreasing, right* $\tau_{\widetilde{d}^*}$*-continuous function such that* $\lim_{x\to\infty} F(x) = 1$*. Then there exists a pre-measure* $\omega : \mathcal{G} \to [0, 1]$*, satisfying the mass distribution conditions, such that* F *is the cdf of* μ*, where* μ *is the extension of* $\widetilde{\omega}$ *to the Borel* σ*-algebra of* $(\widetilde{X}, \widetilde{d}^*)$*.*

As a consequence of the previous result, we can prove a similar one in the general context of Polish ultrametric spaces.

Theorem 6. *Let* X *be a Polish ultrametric space and let* $F : X \to [0, 1]$ *be a non-decreasing, right* τ_d*-continuous function such that* $\lim_{x\to\infty} F(x) = 1$*. Then* F *is the cdf of a probability measure* μ *on* X*.*

By using the previous result, we can give a decomposition theorem for a cdf.

Given a cdf F of a probability measure μ on a Polish ultrametric space, we can define $F_-(x) = \mu(< x)$, where $(< x) = \{y \in X : y < x\}$.

Lemma 7. *Let* F *be the cdf of a probability measure* μ *on a Polish ultrametric space.* $F = F_-$ *is equivalent to* $\mu(\{x\}) = 0$ *for each* $x \in X$*. Moreover, if* $F = F_-$ *then* F *is continuous.*

In the decomposition theorem, we will use the condition $F = F_-$ instead of the continuity of F in order to get the uniqueness of the decomposition.

Theorem 8. *Let* X *be a Polish ultrametric space and let* $F : X \to [0, 1]$ *be a cdf. Then* F *can be decomposed as a convex sum* $F = \alpha G + (1 - \alpha)H$ *with* $0 \leq \alpha \leq 1$*, where* G *is a step cdf, and* H *is a cdf satisfying that* $H_- = H$*. Moreover, the decomposition is unique.*

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