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On the problem of relaxed indistinguishability operators aggregation

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Abstract

In this paper we focus our attention on exploring the aggregation of relaxed indistinguishability operators. Concretely we characterize, in terms of triangular triplets with respect to a t-norm, those functions that allow to merge a collection of relaxed indistinguishability operators into a single one.

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1. Introduction

In the literature one can find many mathematical tools for classifying objects, one of them is the so-called indistinguishability operators when measures presents some kind of uncertainty. According to the definition provided in [10], given a t-norm $T : [0,1]^2 \to [0,1]$, a T-indistinguishability operator on a (non-empty) set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x, y, z \in X$ the following:

(i) $E(x,x) = 1$

(ii) $E(x,y) = E(y,x)$,

(iii) $T(E(x,y), E(y,z)) \leq E(x,z)$.

We assume that the reader is familiar with the basics of triangular norms (we refer the reader to [6] for a deeper treatment of the topic).

The notion of indistinguishability operators is essentially interpreted as a measure of similarity (in contrast to dissimilarity modeled by pseudo-metrics). Thus, $E(x,y)$ matches up with the degree of indistinguishability between the objects x and y. In fact, the greater $E(x,y)$ the most similar are x and y. In such a way that when $x = y$, then the measure of similarity is exactly $E(x,x) = 1$.

Many times in the problems stated in applied fields the data to be processed is coming from different sources (which can be even of different nature). Thus it is necessary to merge such incoming information in order to get a working conclusion. Of course, in such situations, the pieces of information to be processed is represented by means of numerical values and, hence, the techniques for merging are based on numerical aggregation operators (a recent monograph on the subject is [1]). Sometimes the aggregation method used to yield the working decision imposes that the nature of the merged data be kept (as might be expected in this case each piece of information has the same nature). This is the case when one wants to aggregate a collection of indistinguishability operators defined on the same set in order to provide a new one.

The problem of how to combine a collection of indistinguishability operators into a single one has been addressed in [7] (see also [8]). Concretely, in the preceding reference the notion of indistinguishability aggregation function was given as follows:

A function $F : [0,1]^n \to [0,1]$ ($n \in \mathbb{N}$) is said to be an indistinguishability aggregation function provided that the fuzzy relation $F(E_1,\ldots,E_n) : X \times X \to [0,1]$ given by $F(E_1,\ldots,E_n)(x,y) = F(E_1(x,y),\ldots E_n(x,y))$ is a T-indistinguishability operator for any collection $(E_i)_{i=1}^n$ of T-indistinguishability operators on the nonempty set X.

Recently, a description of the functions that aggregates indistinguishability operators have been given in [4]. In particular a characterization of such functions was withdrawn by means of the notion of triangular triplet with respect to a t-norm. The alluded new notion can be formulated in the following way. Given a t-norm T, a triplet $(a,b,c) \in [0,1]^n$ is said to be a n-dimensional T-triangular triplet whenever $T(a_i,b_i) \leq a_i$, $T(a_i,c_i) \leq b_i$ and $T(b_i,c_i) \leq a_i$ for all $i = 1,\ldots,n$.

Taking into account the above notion the next result supplies the announced characterization.

Theorem 1. *Let T be a t-norm and let $n \in \mathbb{N}$. If $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) *F is a T-indistinguishability operators aggregation function.*
2) *F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets and $F(1,\ldots,1) = 1$.*

In the last decades a lot of generalizations of the concept of pseudo-metrics have been extensively treated in the literature because they have shown to be useful in mathematical modeling in many fields of Computer Science such as Domain Theory, Denotational Semantics, Logic Programming and Asymptotic Complexity of Programs. In particular, the mentioned generalized pseudo-metrics can be retrieved as a particular case of the so-called relaxed metrics which have been introduced by the first time in [2]. Let us recall, following [2], that, given $s \in [0,\infty]$, a s-relaxed pseudo-metric on a nonempty set X is a function $d : X \times X \to [0, s]$ that satisfies for all x, y, z the following:

(i) $d(x, y) = d(y, x)$,

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Note that, when $s = \infty$, s-relaxed pseudo-metrics match up with relaxed pseudo-metric in [2]. According to [2], s-relaxed pseudo-metrics are closely related to indistinguishability operators in the sense that the logical counterpart of the former are a kind of generalized indistinguishability operator called relaxed indistinguishability operator. Motivated by the exposed fact, the relationship between both kind of notions has been explored in [5]. Specifically, techniques for generating one notion from the other have been made explicit in such a way that the classical ones which specify the relationship between indistinguishability operators and pseudo-metrics (see [6]) are retrieved as a particular case.

On account of [2], the notion of generalized indistinguishability operator related to relaxed pseudo-metrics can be formulated as follows:

Let X be a non-empty set and let T be a t-norm. A relaxed T-indistinguishability operator E on X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying the following properties for any $x, y, z \in X$:

(i) $E(x, y) = E(y, x)$,

(ii) $T(E(x, z), E(z, y)) \leq E(x, y)$.

Observe that every T-indistinguishability operator E is a relaxed one which satisfies in addition that $E(x, x) = 1$ for all $x \in X$.

Next we provide an example of relaxed indistinguishability operators that is not a indistinguishability operator.

Example 2. Let Σ be a nonempty alphabet. Denote by Σ^∞ the set of all finite and infinite sequences over Σ. Given $v \in \Sigma^\infty$ denote by $l(v)$ the length of v. Thus $l(v) \in \mathbb{N} \cup \{\infty\}$ for all $v \in \Sigma^\infty$. Moreover, if $\Sigma_F = \{v \in \Sigma^\infty : l(v) \in \mathbb{N}\}$ and $\Sigma_\infty = \{v \in \Sigma^\infty : l(v) = \infty\}$, then $\Sigma^\infty = \Sigma_F \cup \Sigma_\infty$. Define the fuzzy binary relation $E_\Sigma : \Sigma^\infty \times \Sigma^\infty \to [0,1]$ by

$$E_\Sigma(u,v) = 1 - 2^{-l(v,w)}$$

for all $u, v \in \Sigma^\infty$, where $l(v,w)$ denotes the longest common prefix between v and w. Of course we have adopted the convention that $2^{-\infty} = 0$. Then it is not hard to check that E_Σ is a relaxed T_{Min}-indistinguishability operator which is not a T_{Min}-indistinguishability operator. It is clear that E_Σ is not a T_{Min}-indistinguishability operator because $E_\Sigma(u,u) < 1$ for each $x \in \Sigma_F$. In fact $E_\Sigma(u,u) = 1 - \frac{1}{2^{l(u)}}$ for all $u \in \Sigma^\infty$. Moreover, $E_\Sigma(u,u) = 1 \Leftrightarrow u \in \Sigma_\infty$.

Inspired by the fact that the problem of aggregating fuzzy relations and generalized metrics has received considerable attention from the community researching in fuzzy mathematics (see, for instance [3, 6, 7, 8, 9] and the references therein), in this paper we focus our attention on exploring the aggregation of relaxed indistinguishability operators. Concretely we characterize, in terms of triangular triplets with respect to a t-norm, those functions that allow to aggregate a collection of relaxed indistinguishability operators.

2. The aggregation of relaxed indistinguishability operators problem

In order to provide an answer to the posed question about which properties must satisfy a function to merge a collection of relaxed indistinguishability operators into a single one, we extend the notion of indistinguishability aggregation function to our more general context in the following obvious manner.

A function $F : [0,1]^n \to [0,1]$ ($n \in \mathbb{N}$) is said to be a relaxed T-indistinguishability aggregation function provided that the fuzzy relation $F(E_1, \ldots, E_n) : X \times X \to [0,1]$ given by $F(E_1, \ldots, E_n)(x,y) = F(E_1(x,y), \ldots . E_n(x,y))$ is a relaxed T-indistinguishability operator for any collection $(E_i)_{i=1}^n$ of relaxed T-indistinguishability operators on the non-empty set X.

The next result yields a first approach to the description of relaxed T-indistinguishability aggregation function as follows:

Proposition 3. *Let T be a t-norm and let $F : [0,1]^n \to [0,1]$ be a relaxed T-indistinguishability aggregation function. Then F satisfies*

$$T(F(a), F(b)) \leq F(T(a_1, b_1), \ldots, T(a_n, b_n))$$

for all $a, b \in [0,1]^n$.

Notice that the preceding result allows to discard those functions that are not useful to merge relaxed indistinguishability operators. The next example illustrates that fact.

Example 4. Define the function $F : [0,1]^3 \to [0,1]$ by $F(a) = a_1 \cdot a_2 + a_3$ for all $a \in [0,1]^3$. Then we have that

$$0.24 = F(0.2, 0.2, 0.2) < T_{Min}(F(0.2, 0.4, 0.2), F(0.4, 0.2, 0.4)) = 0.28.$$

Therefore, by Proposition 3, we conclude that F is not a relaxed T_M-indistinguishability operator aggregation function.

The following result, whose easy proof we omit, can be obtained immediately from Proposition 3 when we assume monotony for the relaxed T-indistinguishability aggregation function.

Corollary 5. *Let T be a t-norm and let $F : [0,1]^n \to [0,1]$ be an increasing relaxed T-indistinguishability aggregation function. Then F satisfies*

$$T(F(a), F(b)) \leq F(\min\{a_1, b_1\}, \ldots, \min\{a_n, b_n\}) \leq F(a + b)$$

for all $a, b \in [0,1]^n$ such that $a + b \in [0,1]^n$.

The next result provides a converse of Proposition 3.

Proposition 6. *Let T be a t-norm. If a function $F : [0,1]^n \to [0,1]$ is increasing and satisfies $T(F(a), F(b)) \leq F(T(a_1, b_1), \ldots, T(a_n, b_n))$ for all $a, b \in [0,1]^n$, then F is a relaxed T-indistinguishability aggregation function.*

In the light of Proposition 6, as a natural question one can wonder if the converse of Proposition 6 is true in general. However, the next example shows that there are relaxed T-indistinguishability aggregation functions that are not increasing.

Example 7. Consider the function $F : [0,1]^2 \to [0,1]$ defined by

$$F(a) = \begin{cases} 1 & \text{if } a = (1,1) \\ 0 & \text{if } a = (\frac{1}{2}, \frac{1}{2}) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

for all $a \in [0,1]^2$. It is not hard to see that F is a relaxed T_D-indistinguishability operator aggregation function. Clearly F satisfies that

$$T_D(F(a), F(b)) \leq F(T_D(a_1, b_1), T_D(a_2, b_2)).$$

Nevertheless, F is not monotone, since $F(\frac{1}{2}, \frac{1}{2}) \leq F(0,0)$.

Taking into account the information about relaxed indistinguishability operators aggregation function yielded by Propositions 3 and 6, we state the relationship between the properties assumed in the statement of the aforesaid results and the transformation of triangle triplets.

Proposition 8. *Let T be a t-norm, $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$ be a function. Then, among the below assertions,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)*:*

1) F is increasing and satisfies $T(F(a), F(b)) \leq F(T(a_1, b_1), \ldots, T(a_n, b_n))$ for all $a, b \in [0,1]^n$.
2) F is a relaxed T-indistinguishability operators aggregation function.
3) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.
4) F satisfies $T(F(a), F(b)) \leq F(T(a_1, b_1), \ldots, T(a_n, b_n))$ for all $a, b \in [0,1]^n$.

Of course Example 7 shows that there are functions satisfying the condition in the assertion 4) in the above result which are not increasing. In the light of this handicap we clarify which are those functions that aggregate relaxed indistinguishability operators in the below result.

Theorem 9. *Let T be a t-norm and let $n \in \mathbb{N}$. If $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F is a relaxed T-indistinguishability operators aggregation function.

2) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.

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