EP elements in rings with involution

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Abstract: Let R be a unital ring with involution. We first show that the EP elements in R can be characterized by three equations. Namely, let $a \in R$, then a is EP if and only if there exists $x \in R$ such that $(xa)^* = xa$, $xa^2 = a$ and $ax^2 = x$. Any EP element in R is core invertible and Moore-Penrose invertible. We give more equivalent conditions for a core (Moore-Penrose) invertible element to be an EP element. Finally, any EP element is characterized in terms of the n-EP property, which is a generalization of the bi-EP property.

Key words: Core inverse, EP, bi-EP, n-EP.

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1 Introduction

Throughout this paper, R will denote a unital ring with involution, i.e., a ring with a mapping $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$. The notion of core inverse for a complex matrix was introduced by Baksalary and Trenkler [3]. In [17], Rakić et al. generalized the core inverse of a complex matrix to the case of an element in R. More precisely, let $a, x \in R$, if

$$axa = a,\ xR = aR \quad \text{and} \quad Rx = Ra^*,$$

then x is called a *core inverse* of a. If such an element x exists, then it is unique and denoted by $a^{\oplus}$. The set of all core invertible elements in R will be denoted by $R^{\oplus}$. Also, in [17] the authors defined a related pseudo-inverse in a ring with an involution. If $a \in R$, then $a_{\oplus} \in R$ is called a *core dual inverse* of a if

$$aa_{\oplus}a = a,\ a_{\oplus}R = a^*R \quad \text{and} \quad Ra_{\oplus} = Ra.$$

Rakić et al. proved that if a has a core dual inverse (we say that a is core dual invertible), then is unique. We denote by $R_{\oplus}$ the subset of R composed of core dual invertible

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elements. It is elemental to prove that $a \in R^{\#}$ if an only if $a^* \in R_{\#}$, and in this case, one has $(a^{\#})^* = (a^*)_{\#}$. This last observation permits to get results concerning dual core inverses from the corresponding results on dual inverses.

Let $a, x \in R$. If

$$axa = a, \quad xax = x, \quad (ax)^* = ax \quad \text{and} \quad (xa)^* = xa,$$

then x is called a *Moore-Penrose inverse* of a. If such an element x exists, then it is unique and denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements will be denoted by $R^\dagger$.

Let $a \in R$. It can be easily proved that the set of elements $x \in R$ such that:

$$axa = a, \quad xax = x \quad \text{and} \quad ax = xa$$

is empty or a singleton. If the set is a singleton, its unique element is called the *group inverse* of a and denoted by $a^\#$. The set of all group invertible elements will be denoted by $R^\#$. The subset of R composed of invertible elements will be denote by R^{-1}.

A matrix $A \in \mathbb{C}_{n\times n}$ is called an EP (range-Hermitian) matrix if the range equality $\mathcal{R}(A) = \mathcal{R}(A^*)$, where $\mathbb{C}_{n\times n}$ denotes the set of all $n \times n$ matrices over the field of complex numbers and $\mathcal{R}(A)$ stands for the range (column space) of $A \in \mathbb{C}_{n\times n}$. This concept was first introduced by Schwerdtfeger in [18]. An element $a \in R$ is said to be an EP element if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$ [8]. The set of all EP elements will be denoted by R^{EP}. Mosić et al. in [12, Theorem 2.1] gave several equivalent conditions such that an element in R to be an EP element. Patrício and Puystjens in [16, Proposition 2] proved that for a Moore-Penrose invertible element $a \in R$, $a \in R^{\mathrm{EP}}$ if and only if $aR = a^*R$. As for a Moore-Penrose invertible element $a \in R$, $a \in R^{\mathrm{EP}}$ if and only if $aa^\dagger = a^\dagger a$, thus we deduce that $aa^\dagger = a^\dagger a$ if and only if $aR = a^*R$. In [17, Theorem 3.1], Rakić et al. investigated some equivalent conditions such that a (dual) core invertible element in R to be an EP element. Also, they showed that $R^\dagger \cap coreR = R^\dagger \cap R^\#$. Motivated by [6, 12, 16, 17], in this paper, we will give new equivalent characterizations such that an element in R to be an EP element.

We first show that the EP elements in R can be characterized by three equations. That is, let $a \in R$, then $a \in R^{\mathrm{EP}}$ if and only if there exists $x \in R$ such that $(xa)^* = xa$, $xa^2 = a$ and $ax^2 = x$. In [17], Rakić et al. proved that $a \in R^\dagger$ if and only if there exists $x \in R$ such that $axa = a$, $xR = a^*R$ and $Rx = Ra^*$. Inspired by this result, we show that $a \in R^{\mathrm{EP}}$ if and only if there exists $x \in R$ such that

$$axa = a, \ xR = aR \quad \text{and} \quad Rx^* = Ra.$$

In [6, Theorem 16], for an operator $T \in L(X)$, where X is a Banach space, Boasso proved that for a Moore-Penrose invertible operator T, T is an EP operator if and only there exists an invertible operator $P \in L(X)$ such that $T^\dagger = PT$. We generalize this result to the ring case. Moreover, for $a \in R^\dagger$, we show that $a \in R^{\mathrm{EP}}$ if and only if there exists a (left) invertible element v such that $a^\dagger = va$. Similarly, for $a \in R^{\#}$, then $a \in R^{\mathrm{EP}}$ if and only if there exists a (left) invertible element s such that $a^{\#} = sa$.

In [17], Rakić et al. proved that $a \in R^{\mathrm{EP}}$ if and only if $a \in R^\dagger \cap R^\#$ with $a^\dagger = a^{\#}$. Also, it is proved that $a \in R^{\mathrm{EP}}$ if and only if $a \in R^{\#}$ with $a^\# = a^{\#}$. In [14, Theorem 2.1], Mosić and Djordjević proved that $a \in R^{\text{EP}}$ if and only if $a \in R^{\#} \cap R^{\dagger}$ with $a^n a^{\dagger} = a^{\dagger} a^n$ for all choices $n \geqslant 1$. This result also can be found in [7, Theorem 2.4] by Chen. Motivated by [14, 17], we will give more new equivalent conditions under which a core invertible element is an EP element. And we define the concept of n-EP as a generalization of bi-EP. As a application, we will use n-EP property to give an equivalent characterization of the EP elements in R.

2 New characterizations of EP elements by equations

In this section, we first show that any EP element in R can be characterized by three equations. Let us begin with an auxiliary lemma.

Lemma 2.1. [11, Theorem 7.3] *Let $a \in R$. Then $a \in R^{\text{EP}}$ if and only if $a \in R^{\#}$ with $(aa^{\#})^* = aa^{\#}$.*

It is well known that the group inverse of an element in a ring can be characterized by three equations and the Moore-Penrose inverse of an element in a ring can be characterized by four equations. In the following theorem, we show that an EP element in a ring can be described by three equations.

Theorem 2.2. *Let $a \in R$. Then $a \in R^{\text{EP}}$ if and only if there exists $x \in R$ such that*

$$(xa)^* = xa, \quad xa^2 = a \quad \text{and} \quad ax^2 = x. \tag{2.1}$$

Proof. Suppose $a \in R^{\text{EP}}$. Let $x = a^{\dagger} = a^{\#}$, then $(xa)^* = (a^{\dagger}a)^* = a^{\dagger}a = xa$, $xa^2 = a^{\#}a^2 = a$ and $ax^2 = a(a^{\#})^2 = a^{\#} = x$. Conversely, if there exists $x \in R$ such that $(xa)^* = xa$, $xa^2 = a$ and $ax^2 = x$, then $a(x^2a) = (ax^2)a = xa = x(xa^2) = (x^2a)a$, $a(x^2a)a = (xa)a = xa^2 = a$, and $(x^2a)a(x^2a) = (xa)(x^2a) = x(ax^2)a = x^2a$. These three equalities prove that $a \in R^{\#}$, $a^{\#} = x^2a$, and $aa^{\#} = xa$. By Lemma 2.1, we get $a \in R^{\text{EP}}$. $\square$

For an idempotent p in a ring R, every $a \in R$ can be written as

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p)$$

or in the matrix form

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$ and $a_{22} = (1-p)a(1-p)$.

Let us observe that pRp and $(1-p)R(1-p)$ are subrings whose unities are p and $1-p$, respectively. Also, we notice that if $p = p^*$, then the above matrix representation preserves the involution. The term *projection* will be reserved for a Hermitian idempotent.

Suppose in this paragraph that $a \in R$ is an EP element. If we denote $p = aa^{\dagger} = a^{\dagger}a$, since $ap = pa = a$ and $a^{\dagger}p = pa^{\dagger} = a^{\dagger}$, then the matrix representations of a and $a^{\dagger}$ with respect to the Hermitian idempotent p are

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad a^{\dagger} = \begin{bmatrix} {}^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.2}$$

respectively.

Recall that a ring R is *prime* if for any two elements a and b of R, $arb = 0$ for all r in R implies that either $a = 0$ or $b = 0$.

In next theorem, the set of elements $x \in R^{\mathrm{EP}}$ satisfying (2.1) is described.

Theorem 2.3. *Let $a \in R$. If a is EP, then $\{x \in R : (xa)^* = xa, xa^2 = a, ax^2 = x\} = \{a^\dagger + aa^\dagger y(1 - aa^\dagger) : y \in R\}$. Moreover, if R is a prime ring, then $\{x \in R : (xa)^* = xa, xa^2 = a, ax^2 = x\} = \{a^\dagger\}$ if and only if $a = 0$ or a is invertible.*

Proof. Suppose a is an EP element. We use the matrix representations of a and $a^\dagger$ with respect to the projection $p = aa^\dagger$ given in (2.2). Let $x = \left[\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}\right]$ be the representation of any $x \in R$ with respect to p. From $xa^2 = a$, we get

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & v \\ w & z \end{bmatrix} \begin{bmatrix} a^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ua^2 & 0 \\ wa^2 & 0 \end{bmatrix}.$$

Since a is EP and $aa^\dagger = p = a^\dagger a$, then a is invertible in pRp and its inverse is $a^\dagger$. Hence from $a = ua^2$ and $0 = wa^2$, we obtain $u = a^\dagger$ and $0 = w$, respectively. Now, from $x = ax^2$ we have

$$\begin{bmatrix} a^\dagger & v \\ 0 & z \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^\dagger & v \\ 0 & z \end{bmatrix} x = \begin{bmatrix} p & av \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^\dagger & v \\ 0 & z \end{bmatrix} = \begin{bmatrix} a^\dagger & v + avz \\ 0 & 0 \end{bmatrix},$$

which implies $z = 0$. Therefore,

$$x = \begin{bmatrix} a^\dagger & v \\ 0 & 0 \end{bmatrix} = a^\dagger + v,$$

that is $\{x \in R : (xa)^* = xa, xa^2 = a, ax^2 = x\} \subseteq \{a^\dagger + aa^\dagger y(1 - aa^\dagger) : y \in R\}$.

Let us prove the opposite inclusion. We have that $aa^\dagger = a^\dagger a$ since a is EP. Then

$$\begin{aligned} &[a^\dagger + aa^\dagger y(1 - aa^\dagger)]a = a^\dagger a \text{ is Hermitian,} \\ &[a^\dagger + aa^\dagger y(1 - aa^\dagger)]a^2 = a^\dagger a^2 = a, \\ &a[a^\dagger + aa^\dagger y(1 - aa^\dagger)]^2 = [aa^\dagger + ay(1 - aa^\dagger)][a^\dagger + aa^\dagger y(1 - aa^\dagger)] = a^\dagger + aa^\dagger y(1 - aa^\dagger). \end{aligned}$$

Suppose that R is prime ring. If $a = 0$, then $\{x \in R : (xa)^* = xa, xa^2 = a, ax^2 = x\} = \{0\}$. If a is invertible, then $\{x \in R : (xa)^* = xa, xa^2 = a, ax^2 = x\} = \{a^{-1}\}$. If $\{x \in R : (xa)^* = xa, xa^2 = a, ax^2 = x\}$ is a singleton, then $aa^\dagger y(1 - aa^\dagger) = 0$ for all $y \in R$, by using that R is prime, then $aa^\dagger = 0$ or $1 - aa^\dagger = 0$. The first of the previous alternatives is equivalent to $a = 0$ and the second one (since a is EP) is equivalent to the invertibility of a. □

We will also use the following notations: $aR = \{ax : x \in R\}$, $Ra = \{xa : x \in R\}$, ${}^\circ a = \{x \in R : xa = 0\}$, $a^\circ = \{x \in R : ax = 0\}$ and $[a, b] = ab - ba$. The following lemma will be useful in the sequel.

Lemma 2.4. [19] *Let $a, b \in R$. Then:*

(1) *$aR \subseteq bR$ implies ${}^\circ b \subseteq {}^\circ a$ and the converse is valid whenever b is regular;*

(2) $Ra \subseteq Rb$ *implies* $b^\circ \subseteq a^\circ$ *and the converse is valid whenever* b *is regular.*

Theorem 2.5. *Let* $a \in R$. *Then the following are equivalent:*

(1) $a \in R^{\mathrm{EP}}$;

(2) *there exists* $x \in R$ *such that* $axa = a$, $xR = aR$ *and* $Rx^* = Ra$;

(3) *there exists* $x \in R$ *such that* $axa = a$, $xR = aR$ *and* $Rx^* \subseteq Ra$;

(4) *there exists* $x \in R$ *such that* $xax = x$, $xR = aR$ *and* $Rx^* = Ra$;

(5) *there exists* $x \in R$ *such that* $xax = x$, $xR = aR$ *and* $Ra \subseteq Rx^*$;

(6) *there exists* $x \in R$ *such that* $axa = a$, ${}^\circ x = {}^\circ a$ *and* $(x^*)^\circ = a^\circ$;

(7) *there exists* $x \in R$ *such that* $axa = a$, ${}^\circ x = {}^\circ a$ *and* $a^\circ \subseteq (x^*)^\circ$;

(8) *there exists* $x \in R$ *such that* $xax = x$, ${}^\circ x = {}^\circ a$ *and* $(x^*)^\circ = a^\circ$;

(9) *there exists* $x \in R$ *such that* $xax = x$, ${}^\circ x = {}^\circ a$ *and* $(x^*)^\circ \subseteq a^\circ$.

Proof. (1) $\Rightarrow$ (2): Let $x = a^\dagger = a^\#$, then $axa = a$, $x = a(a^\#)^2$, $x^* = (a^\dagger)^* = (a^\dagger a a^\dagger)^* = (a^\dagger)^* a^\dagger a$, and $xa^2 = a^\# a^2 = a = aa^\dagger a = aa^*(a^\dagger)^* = aa^*x^*$. Thus $xR = aR$ and $Rx^* = Ra$.

(2) $\Rightarrow$ (3) and (6) $\Rightarrow$ (7) are clear.

(2) $\Rightarrow$ (6) and (3) $\Rightarrow$ (7) are obvious by Lemma 2.4.

(7) $\Rightarrow$ (1): Suppose there exists $x \in R$ such that $axa = a$, ${}^\circ x = {}^\circ a$ and $a^\circ \subseteq (x^*)^\circ$. Since $(1-ax)a = 0$, then $1-ax \in {}^\circ a = {}^\circ x$, hence $(1-ax)x = 0$. Since $a(1-xa) = 0$, then $1-xa \in a^\circ \subseteq (x^*)^\circ$, hence $x^*(1-xa) = 0$, i.e., $x = (xa)^*x$. We get $xa = (xa)^*xa$, hence xa is Hermitian. Finally, $x = xax$ implies $1-xa \in {}^\circ x = {}^\circ a$, whence $xa^2 = a$ Therefore $a \in R^{\mathrm{EP}}$ by Theorem 2.2.

The implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (8) $\Rightarrow$ (9) are similar to (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (6) $\Rightarrow$ (7).

(9) $\Rightarrow$ (1): There exists $x \in R$ such that $xax = x$, ${}^\circ x = {}^\circ a$ and $(x^*)^\circ \subseteq a^\circ$. It is obvious that $(x^*)^\circ \subseteq a^\circ$ is equivalent to ${}^\circ x \subseteq {}^\circ(a^*)$. We have $a = xa^2$ since $(1-xa)x = 0$ implies $(1-xa)a = 0$. Similarly, we have $a^* = xaa^*$ since $(1-xa)x = 0$ implies $(1-xa)a^* = 0$. Thus $(xa)^* = a^*x^* = xaa^*x^* = xa(xa)^*$, that is $(xa)^* = xa$. By $a^* = xaa^*$ and $(xa)^* = xa$, we have $a^* = xaa^* = (xa)^*a^* = (axa)^*$, that is $a = axa$. Hence by ${}^\circ x = {}^\circ a$, we have $(1-ax)a = 0$ implies $(1-ax)x = 0$, which gives $x = ax^2$. Therefore $a \in R^{\mathrm{EP}}$ by Theorem 2.2. $\square$

Theorem 2.6. *Let* $a \in R^{\mathrm{EP}}$ *and denote* $p = aa^\dagger$. *Then*

(1) $\{x \in R : axa = a, x \in aR\} = \{a^\dagger + py(1-p) : y \in R\}$;

(2) $\{x \in R : xax = x, xR = aR\} = \{a^\dagger + pz(1-p) : z \in R\}$;

(3) $\{x \in R : axa = a, {}^\circ a \subseteq {}^\circ x\} = \{a^\dagger + py'(1-p) : y' \in R\}$;

(4) $\{x \in R : xax = x, {}^\circ a = {}^\circ x\} = \{a^\dagger + pz'(1-p) : z' \in R\}$.

Furthermore, if R is prime, then any of the above subsets is a singleton if and only if $a = 0$ or a is invertible.

Proof. We use the matrix representations of a and $a^\dagger$ with respect to the projection $p = aa^\dagger$ given in (2.2). Let $x = \left[\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}\right]$ be the representation of any x with respect to p.

(1) Let x satisfy $axa = a$ and $x \in aR$. From $axa = a$, we have

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u & v \\ w & z \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} au & av \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} aua & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $a \in R^{\mathrm{EP}}$, we have that a is invertible in pRp and its inverse is $a^\dagger$. Hence $a = aua$ gives $u = a^\dagger$. Since $x \in aR$, we can write

$$\begin{bmatrix} u & v \\ w & z \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix} = \begin{bmatrix} a\xi_1 & a\xi_2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $w = z = 0$. Hence

$$x = \begin{bmatrix} a^\dagger & v \\ 0 & 0 \end{bmatrix} = a^\dagger + px(1-p) \in \{a^\dagger + py(1-p) : y \in R\}.$$

The opposite inclusion is trivial.

(2) Let $x \in R$ satisfy $xax = x$ and $xR = aR$. Since $x \in aR$, by the proof of (1), we have $w = z = 0$. Now, since $a \in xR$, we can write

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix},$$

which implies

$$a = u\delta_1 + v\delta_3, \qquad 0 = u\delta_2 + v\delta_4. \tag{2.3}$$

Now, we use $xax = x$:

$$\begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ua & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} uau & uav \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$u = uau, \qquad v = uav. \tag{2.4}$$

Post-multiply the first equality of (2.4) by δ_1 and the second equality of (2.4) by δ_3 to obtain

$$u\delta_1 = uau\delta_1, \qquad v\delta_3 = uav\delta_3.$$

From (2.3),

$$a = u\delta_1 + v\delta_3 = uau\delta_1 + uav\delta_3 = ua(u\delta_1 + v\delta_3) = ua^2.$$

We get $u = a^\dagger$ because a is invertible in pRp and its inverse is $a^\dagger$. Therefore

$$x = \begin{bmatrix} a^\dagger & v \\ 0 & 0 \end{bmatrix} = a^\dagger + v = a^\dagger + px(1-p) \in \{a^\dagger + pz(1-p) : z \in R\}.$$

For the opposite inclusion, it is easy to check that $[a^\dagger + pz(1-p)]a[a^\dagger + pz(1-p)] = a^\dagger + pz(1-p)$ and $a^\dagger + pz(1-p) \in aR$ in view of $a \in R^{\rm EP}$. From $a = [a^\dagger + pz(1-p)]a^2$, we deduce that $a \in [a^\dagger + pz(1-p)]R$.

The proof of statements (3) and (4) follows from (1) and (2), respectively, since by Lemma 2.4, we obtain that $x \in aR$ is equivalent to ${}^\circ a \subseteq {}^\circ x$ and $xR = aR$ is equivalent to ${}^\circ a = {}^\circ x$, respectively.

The proof of the last affirmation of this theorem has the same proof as the corresponding part of Theorem 2.3. $\square$

By considering that a is EP if and only if a^* is EP and having in mind Theorem 2.2, Theorem 2.3, Theorem 2.5 and Theorem 2.6, we get the following four theorems.

Theorem 2.7. *Let $a \in R$. Then $a \in R^{\rm EP}$ if and only if there exists $y \in R$ such that*

$$(ay)^* = ay, \quad a^2y = a \quad \textit{and} \quad y^2a = y.$$

Theorem 2.8. *Let $a \in R$. If a is EP, then $\{y \in R : (ay)^* = ay, a^2y = a, y^2a = y\} = \{a^\dagger + (1-aa^\dagger)xaa^\dagger : x \in R\}$. Moreover, if R is a prime ring, then $\{y \in R : (ay)^* = ay, a^2y = a, y^2a = y\} = \{a^\dagger\}$ if and only if $a = 0$ or a is invertible.*

Theorem 2.9. *Let $a \in R$. Then the following are equivalent:*

(1) *$a \in R^{\rm EP}$;*

(2) *there exists $y \in R$ such that $aya = a$, $Ry = Ra$ and $y^*R = aR$;*

(3) *there exists $y \in R$ such that $aya = a$, $Ry = Ra$ and $y^*R \subseteq aR$;*

(4) *there exists $y \in R$ such that $yay = y$, $Ry = Ra$ and $y^*R = aR$;*

(5) *there exists $y \in R$ such that $yay = y$, $Ry = Ra$ and $aR \subseteq y^*R$;*

(6) *there exists $y \in R$ such that $aya = a$, $y^\circ = a^\circ$ and ${}^\circ(y^*) = {}^\circ a$;*

(7) *there exists $y \in R$ such that $aya = a$, $y^\circ = a^\circ$ and ${}^\circ a \subseteq {}^\circ(y^*)$;*

(8) *there exists $y \in R$ such that $yay = y$, $y^\circ = a^\circ$ and ${}^\circ(y^*) = {}^\circ a$;*

(9) *there exists $y \in R$ such that $yay = y$, $y^\circ = a^\circ$ and ${}^\circ(y^*) \subseteq {}^\circ a$.*

Theorem 2.10. *Let $a \in R^{EP}$ and denote $p = aa^\dagger$. Then*

(1) *$\{y \in R : aya = a, y \in Ra\} = \{a^\dagger + (1-p)xp : x \in R\}$;*

(2) *(2) $\{y \in R : yay = y, Ry = Ra\} = \{a^\dagger + (1-p)zp : z \in R\}$;*

(3) *$\{y \in R : aya = a, a^\circ \subseteq y^\circ\} = \{a^\dagger + (1-p)x'p : x' \in R\}$;*

(4) *$\{y \in R : yay = y, a^\circ = y^\circ\} = \{a^\dagger + (1-p)z'p : z' \in R\}$.*

Furthermore, if R is prime, then any of the above subsets is a singleton if and only if $a = 0$ or a is invertible.

We will characterize when $a \in R$ is EP by another subset of three equations.

Theorem 2.11. *Let $a \in R$. Then $a \in R^{\mathrm{EP}}$ if and only if there exists $x \in R$ such that*

$$a^2x = a, \quad ax = xa \quad \text{and} \quad (ax)^* = ax. \tag{2.5}$$

Proof. If a is EP, by taking $x = a^\dagger = a^\#$, we get (2.5). Conversely, assume that exists $x \in R$ such that (2.5) is satisfied. We shall show that $a \in R^{\mathrm{EP}}$ and $a^\# = ax^2$. Since $ax = xa$, we get $a(ax^2) = (ax^2)a$, but in addition, $a(ax^2) = (a^2x)x = ax$, which leads to $a(ax^2)a = a^2x = a$ and $(ax^2)a(ax^2) = (ax^2)ax = (a^2x)x^2 = ax^2$. Since $aa^\# = a^2x^2 = ax$ is Hermitian, the conclusion follows from Lemma 2.1. □

We have seen that if $a \in R$ is EP, then $\{x \in R : a^2x = a, ax = xa, (ax)^* = ax\}$ is not empty. In next theorem we describe this last set.

Theorem 2.12. *Let $a \in R$. If a is EP, then $\{x \in R : a^2x = a, ax = xa, (ax)^* = ax\} = \{a^\dagger + (1 - aa^\dagger)y(1 - aa^\dagger) : y \in R\}$. Moreover, if R is a prime ring, then $\{x \in R : a^2x = a, ax = xa, (ax)^* = ax\} = \{a^\dagger\}$ if and only if a is invertible.*

Proof. Suppose that a is an EP element. We use the matrix representations of a and $a^\dagger$ with respect to the projection $p = aa^\dagger$ given in (2.2). Let $x = \left[\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}\right]$ be the representation of any $x \in R$ with respect to p.

Let $x \in R$ satisfy $a^2x = a$, $ax = xa$ and $(ax)^* = ax$. From $a^2x = a$, we get

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} u & v \\ w & z \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},$$

which leads to $a^2u = a$ and $a^2v = 0$. Since a is EP and $aa^\dagger = p = a^\dagger a$, then a is invertible in pRp and its inverse is $a^\dagger$. Hence from $a = a^2u$ and $0 = a^2v$, we obtain $u = a^\dagger$ and $0 = v$, respectively. Now, from $ax = xa$ we have

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^\dagger & 0 \\ w & z \end{bmatrix} = \begin{bmatrix} a^\dagger & 0 \\ w & z \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & z \end{bmatrix},$$

which implies $0 = wa$, and taking into account that a is invertible in pRp, we have $0 = w$. Therefore,

$$x = \begin{bmatrix} a^\dagger & 0 \\ 0 & z \end{bmatrix} = a^\dagger + z,$$

that is $\{x \in R : a^2x = a, ax = xa, (ax)^* = ax\} \subseteq \{a^\dagger + (1 - aa^\dagger)y(1 - aa^\dagger) : y \in R\}$.

Let us prove the opposite inclusion. We have $aa^\dagger = a^\dagger a$ since a is EP. Now,

$$\begin{aligned}
&a[a^\dagger + (1 - aa^\dagger)y(1 - aa^\dagger)] = aa^\dagger \text{ is Hermitian}, \\
&a^2[a^\dagger + (1 - aa^\dagger)y(1 - aa^\dagger)] = a^2a^\dagger = a, \\
&a[a^\dagger + (1 - aa^\dagger)y(1 - aa^\dagger)] = aa^\dagger = a^\dagger a = [a^\dagger + (1 - aa^\dagger)y(1 - aa^\dagger)]a.
\end{aligned}$$

Suppose that R is a prime ring. If a is invertible, then $\{x \in R : a^2x = a, ax = xa, (ax)^* = ax\} = \{a^{-1}\}$. If $\{x \in R : a^2x = a, ax = xa, (ax)^* = ax\}$ is a singleton, then $(1 - aa^\dagger)y(1 - aa^\dagger) = 0$ for all $y \in R$. By using that R is prime, we get $1 - aa^\dagger = 0$, which (since a is EP) is equivalent to the invertibility of a. □

Theorem 2.13. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{\mathrm{EP}}$;

(2) $^\circ(a^2) \subseteq {}^\circ a$ *and there exists* $x \in R$ *such that* $xa^2 = a$ *and* $(xa)^* = xa$;

(3) $(a^2)^\circ \subseteq a^\circ$ *and there exists* $x \in R$ *such that* $a^2x = a$ *and* $(ax)^* = ax$.

Furthermore, under these equivalences one has that the set of elements x satisfying (2) *is* $\{a^\dagger + y(1 - aa^\dagger) : y \in R\}$ *and the set of elements x satisfying* (3) *is* $\{a^\dagger + (1 - aa^\dagger)z : z \in R\}$. *If R is prime and a is EP, then the sets of x satisfying (2) or (3) is $\{a^\dagger\}$.*

Proof. (1) $\Rightarrow$ (2): The inclusion $^\circ(a^2) \subseteq {}^\circ a$ is evident from $a \in R^\#$. For the remaining, it is sufficient to take $x = a^\dagger = a^\#$.

(2) $\Rightarrow$ (1): Since $(ax - 1)a^2 = a(xa^2) - a^2 = a^2 - a^2 = 0$, we get $ax - 1 \in {}^\circ(a^2) \subseteq {}^\circ a$, hence $axa = a$. From

$$ax^2a^2 = ax(xa^2) = axa = a = xa^2$$

we get $ax^2 - x \in {}^\circ(a^2) \subseteq {}^\circ a$, hence $ax^2a = xa$. Now, we prove $a^\# = x^2a$ by the definition of the group inverse,

$$a(x^2a) = ax^2a = xa, \qquad (x^2a)a = x(xa^2) = xa;$$

$$a(x^2a)a = xa^2 = a; \qquad (x^2a)a(x^2a) = x(xa^2)x^2a = xax^2a = x(ax^2a) = x^2a.$$

Now, $a^\# = x^2a$ and $aa^\# = ax^2a = xa$ is Hermitian. Hence (1) follows from Lemma 2.1.

The proof of (1) $\Leftrightarrow$ (3) is similar to the proof of (1) $\Leftrightarrow$ (2).

Now, let us prove the last part of the theorem. Recall that a is EP. If $x \in R$ satisfies $xa^2 = a$, then $(x - a^\dagger)aa^\dagger = xaa^\dagger - a^\dagger = xa^2(a^\dagger)^2 - a^\dagger = a(a^\dagger)^2 - a^\dagger = 0$. Hence $x - a^\dagger = (x - a^\dagger)(1 - aa^\dagger)$, which yields $x \in \{a^\dagger + y(1 - aa^\dagger) : y \in R\}$. Reciprocally, it is evident that for any $y \in R$ one has that $[a^\dagger + y(1 - aa^\dagger)]a^2 = a$ and $[a^\dagger + y(1 - aa^\dagger)]a$ is Hermitian. The affirmation concerning the primality of R has the same proof as the corresponding in previous Theorem 2.12. $\square$

3 When a core invertible element is an EP element

Any EP element is core invertible, but when a core invertible element is EP? In this section we answer this question. Let us start this section with a lemma.

Lemma 3.1. [17, Theorem 2.14] *An element $a \in R$ is core invertible if and only if there exists $x \in R$ such that*

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a \quad \text{and} \quad ax^2 = x.$$

Under this equivalence, one has that $x = a^{\oplus}$.

Let us recall the following result.

Theorem 3.2. *Let $a \in R$.*

(1) [5, Proposition 8.24] *a is group invertible if and only if exists an idempotent* $p \in R$ *such that* $ap = pa = 0$ *and* $a + p$ *is invertible. Under this equivalence, we have* $p = 1 - aa^{\#}$ *and* $a^{\#} = (a+p)^{-1} - p$.

(2) [4, Theorem 2.1] *a is EP if and only if exists a projection* $p \in R$ *such that* $ap = pa = 0$ *and* $a + p$ *is invertible. Under this equivalence, we have* $p = 1 - aa^{\dagger}$ *and* $a^{\dagger} = (a+p)^{-1} - p$.

In fact, the second item of previous result was stated for unital C^*-algebras, but as one can easily check, its proof remains valid for unital rings with an involution. We give a similar characterization of the core invertibility.

Theorem 3.3. *Let* $a \in R$. *The following affirmations are equivalent:*

(1) *a is core invertible.*

(2) *Exists a projection p such that* $pa = 0$ *and* $a(1-p)$ *is invertible in the subring* $(1-p)R(1-p)$.

(3) *Exists a projection p such that* $pa = 0$ *and* $a(1-p) + p$ *is invertible.*

(4) *Exists a projection p such that* $pa = 0$ *and* $a + p$ *is invertible.*

Under this equivalence, one has this projection p is unique and $p = 1 - aa^{\oplus}$. *In addition,*

$$(a(1-p))^{-1}_{(1-p)R(1-p)} = a^{\oplus}, \quad (a(1-p)+p)^{-1} = p + a^{\oplus}, \quad (a+p)^{-1} = p - a^{\oplus}ap + a^{\oplus}.$$

Proof. (1) $\Rightarrow$ (2): Let us represent a with respect to the projection $p = 1 - aa^{\oplus}$. We use the notation $\overline{p} = 1 - p = aa^{\oplus}$. Observe that from Lemma 3.1 we have $pa = (1 - aa^{\oplus})a = 0$ and $a^{\oplus}p = a^{\oplus}(1 - aa^{\oplus}) = 0$. Therefore,

$$a = \begin{bmatrix} pap & pa\overline{p} \\ \overline{p}ap & \overline{p}a\overline{p} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ap & a\overline{p} \end{bmatrix} \quad \text{and} \quad a^{\oplus} = \begin{bmatrix} pa^{\oplus}p & pa^{\oplus}\overline{p} \\ \overline{p}a^{\oplus}p & \overline{p}a^{\oplus}\overline{p} \end{bmatrix} = \begin{bmatrix} 0 & pa^{\oplus} \\ 0 & \overline{p}a^{\oplus} \end{bmatrix}. \tag{3.1}$$

Now, $a(a^{\oplus})^2 = a^{\oplus}$ leads to $\overline{p}a^{\oplus} = a^{\oplus}$ and $pa^{\oplus} = 0$. Hence

$$a^{\oplus} = \begin{bmatrix} 0 & 0 \\ 0 & a^{\oplus} \end{bmatrix} \in \overline{p}R\overline{p}.$$

From $a^{\oplus}a^2 = a$ we have $a^{\oplus}a^2aa^{\oplus} = aa^{\oplus}$, i.e., $a^{\oplus}a\overline{p} = \overline{p}$. Furthermore, $a\overline{p}a^{\oplus} = aa^{\oplus} = \overline{p}$. Therefore, $a\overline{p} \in \overline{p}R\overline{p}$ is invertible in the subring $\overline{p}R\overline{p}$ and its inverse is $a^{\oplus}$.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): Let $p \in R$ be a projection such that $pa = 0$. The representation of a with respect to p is the same as in (3.1). Now we have

$$a(1-p) = \begin{bmatrix} 0 & 0 \\ 0 & a(1-p) \end{bmatrix}, \quad a(1-p) + p = \begin{bmatrix} p & 0 \\ 0 & a(1-p) \end{bmatrix}, \quad a + p = \begin{bmatrix} p & 0 \\ ap & a(1-p) \end{bmatrix}.$$

Taking into account that p is invertible in the subring pRp (in fact, p is the unity), evidently we have that $a(1-p) \in ((1-p)R(1-p))^{-1} \Leftrightarrow a(1-p) + p \in R^{-1} \Leftrightarrow a + p \in R^{-1}$.

(2) $\Rightarrow$ (1): Let $x \in (1-p)R(1-p)$ be the inverse of $a(1-p)$ in $(1-p)R(1-p)$. This means that $a(1-p)x = xa(1-p) = 1-p$. Observe that $x \in (1-p)R(1-p)$ implies $(1-p)x = x(1-p) = x$, and therefore, $ax = xa(1-p) = 1-p$. We will prove that $x = a^{\oplus}$ by using Lemma 3.1. Let us recall that we can use $pa = 0$ and $(1-p)a = a$ by hypothesis.

$$axa = (ax)a = (1-p)a = a.$$

$$xax = x(ax) = x(1-p) = x.$$

$$ax = 1-p \text{ is Hermitian.}$$

$$xa^2 = xa(1-p)a = (1-p)a = a.$$

$$ax^2 = (ax)x = (1-p)x = x.$$

Now, we shall prove the uniqueness. Assume that q is another projection such that $qa = 0$ and $a(1-q)$ is invertible in $(1-q)R(1-q)$. By the proof of (2) $\Rightarrow$ (1) we get that the inverse of $a(1-q)$ in $(1-q)R(1-q)$ is $a^{\oplus}$, in particular $a^{\oplus} \in (1-q)R(1-q)$ and $a(1-q)a^{\oplus} = 1-q$. By using also Lemma 3.1 we get

$$(1-q)aa^{\oplus} = a(1-q)a^{\oplus}aa^{\oplus} = a(1-q)a^{\oplus} = 1-q. \tag{3.2}$$

Since $a^{\oplus} \in (1-q)R(1-q)$, exists $u \in R$ such that $a^{\oplus} = u(1-q)$. Now,

$$aa^{\oplus}(1-q) = au(1-q)^2 = au(1-q) = aa^{\oplus}.$$

Apply involution and use Lemma 3.1 in this last equality to get $(1-q)aa^{\oplus} = aa^{\oplus}$. From this last equality and (3.2) we obtain $aa^{\oplus} = 1-q$. In other words, we have proved the uniqueness of such q.

The expresion of () $\square$

Lemma 3.4. [17, Theorem 3.1] *Let $a \in R$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$*;*
(2) $a \in R^{\dagger}$ *and* $[a, a^{\dagger}] = 0$*;*
(3) $a \in R^{\oplus}$ *and* $[a, a^{\oplus}] = 0$*;*
(4) $a \in R^{\oplus}$ *and* $a^{\#} = a^{\oplus}$*;*
(5) $a \in R^{\dagger} \cap R^{\#}$ *and* $a^{\dagger} = a^{\oplus}$.

Lemma 3.5. [17, Theorem 2.18] *Let $a \in R^{\oplus}$. Then $a^{\oplus} \in R^{\mathrm{EP}}$ and $(a^{\oplus})^{\oplus} = (a^{\oplus})^{\dagger} = (a^{\oplus})^{\#} = a^2a^{\oplus}$. Moreover, if $a \in R^{\dagger}$, then $(a^{\dagger})^{\oplus} = (a^{\oplus}a)^*a$.*

Lemma 3.6. [16] *Let $a \in R$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$*;*
(2) $a \in R^{\#}$ *and* $aR = a^*R$*;*
(3) $a \in R^{\#}$ *and* $Ra = Ra^*$.

In the following theorem, we show that the equality $aR = a^*R$ in Lemma 3.6 can be replaced by weaker inclusions $aR \subseteq a^*R$ or $a^*R \subseteq aR$.

Theorem 3.7. *Let $a \in R$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $a \in R^{\#}$ *and* $aR \subseteq a^*R$;
(3) $a \in R^{\#}$ *and* $Ra \subseteq Ra^*$;
(4) $a \in R^{\#}$ *and* $a^*R \subseteq aR$;
(5) $a \in R^{\#}$ *and* $Ra^* \subseteq Ra$.

Proof. (1) $\Rightarrow$ (2)-(5) is obvious by Lemma 3.6.

(2) $\Rightarrow$ (1): By $aR \subseteq a^*R$, we have $a = a^*r$ for some $r \in R$, then $a = (aa^{\#}a)^*r = (a^{\#}a)^*a^*r = (a^{\#}a)^*a$. Thus $a^{\#}a = aa^{\#} = (a^{\#}a)^*aa^{\#} = (a^{\#}a)^*a^{\#}a$, which gives $(a^{\#}a)^* = a^{\#}a$. Therefore $a \in R^{EP}$ by the definition of EP element.

(3)-(5) $\Rightarrow$ (1) is similar to (2) $\Rightarrow$ (1). □

Theorem 3.8. *Let $a \in R$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $a \in R^{\oplus}$ *and* $(a^{\oplus}a)^* = a^{\oplus}a$;
(3) $a \in R^{\oplus}$ *and* $(a^{\oplus})^{\oplus} = a$;
(4) $a \in R^{\oplus}$ *and* $(a^{\oplus})^{\dagger} = a$;
(5) $a \in R^{\oplus}$ *and* $(a^{\oplus})^{\#} = a$;
(6) $a \in R^{\dagger} \cap R^{\#}$ *and* $(a^{\dagger})^{\oplus} = a$;
(7) $a \in R^{\dagger} \cap R^{\#}$ *and* $(a^{\dagger})^{\oplus} = (a^{\oplus})^{\dagger}$;
(8) $a \in R^{\oplus}$ *and* $ap = 0$, *where* $p = 1 - aa^{\oplus}$.

Proof. (1) $\Leftrightarrow$ (2): Suppose $a \in R^{\mathrm{EP}}$. Then by Lemma 3.4, we have $a^{\dagger} = a^{\oplus}$. Thus $(a^{\dagger}a)^* = a^{\dagger}a$ implies $(a^{\oplus}a)^* = a^{\oplus}a$. Conversely, suppose $a \in R^{\oplus}$ and $(a^{\oplus}a)^* = a^{\oplus}a$. By Lemma 3.1, we have $aa^{\oplus}a = a$, $a^{\oplus}aa^{\oplus} = a^{\oplus}$ and $(aa^{\oplus})^* = aa^{\oplus}$. Thus by the definition of Moore-Penrose inverse, we have $a^{\dagger} = a^{\oplus}$. Hence by Lemma 3.4, we have $a \in R^{\mathrm{EP}}$.

(1) $\Leftrightarrow$ (3): Suppose $a \in R^{\mathrm{EP}}$. Then by Lemma 3.4, we have $[a, a^{\oplus}] = 0$. By Lemma 3.5, we have $(a^{\oplus})^{\oplus} = a^2a^{\oplus}$. Thus $(a^{\oplus})^{\oplus} = a^2a^{\oplus} = a(aa^{\oplus}) = a(a^{\oplus}a) = a$. Conversely, suppose $(a^{\oplus})^{\oplus} = a$. By Lemma 3.5, we have $(a^{\oplus})^{\oplus} = a^2a^{\oplus}$. Then $a = a^2a^{\oplus}$. Thus

$$a^{\oplus}a = a^{\oplus}a^2a^{\oplus} = aa^{\oplus}.$$

Therefore $a \in R^{\mathrm{EP}}$ by Lemma 3.4.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) is clear by Lemma 3.5.

(1) $\Rightarrow$ (6): By Lemma 3.4 and Lemma 3.5, we have $(a^{\dagger})^{\oplus} = (a^{\oplus}a)^*a = (a^{\dagger}a)^*a = a^{\dagger}a^2 = a^{\#}a^2 = a$.

(6) $\Rightarrow$ (7): Suppose that $a \in R^{\oplus}$ and $(a^{\dagger})^{\oplus} = a$. Then by Lemma 3.1, we have $a^{\dagger}a^2 = a$ and $a(a^{\dagger})^2 = a^{\dagger}$. Thus $a^{\dagger}a = a(a^{\dagger})^2a = a^{\dagger}a^2a^{\dagger} = aa^{\dagger}$. By Lemma 3.4, we have $a \in R^{\mathrm{EP}}$. Then $a^{\oplus} = a^{\#}$. By Lemma 3.5, we have $(a^{\oplus})^{\dagger} = a^2a^{\oplus} = a^2a^{\#} = a$. Thus by $(a^{\dagger})^{\oplus} = a$, we have $(a^{\dagger})^{\oplus} = (a^{\oplus})^{\dagger}$.

(7) $\Rightarrow$ (1): Suppose that $a \in R^{\oplus}$ and $(a^{\dagger})^{\oplus} = (a^{\oplus})^{\dagger}$. Then by Lemma 3.5, we have $(a^{\oplus})^{\dagger} = a^2a^{\oplus}$ and $(a^{\dagger})^{\oplus} = (a^{\oplus}a)^*a$. Thus by $(a^{\dagger})^{\oplus} = (a^{\oplus})^{\dagger}$, we have

$$a^2a^{\oplus} = (a^{\oplus}a)^*a. \tag{3.3}$$

Taking involution on (3.3), we have $a^*a^{\oplus}a = (aa^{\oplus})^*a^* = aa^{\oplus}a^*$. Thus $a^{\oplus}a = (a^{\oplus})^*a^*a^{\oplus}a = (a^{\oplus})^*aa^{\oplus}a^* = aa^{\oplus}$. That is $[a, a^{\oplus}] = 0$, therefore $a \in R^{\mathrm{EP}}$ by Lemma 3.4.

(8) $\Leftrightarrow$ (1): If a is an EP element, then $ap = 0$ is clear by $aa^{\oplus} = a^{\oplus}a$. Conversely, assume that (8) holds. Then $a = a^2 a^{\oplus} = a(aa^{\oplus})^* = a(a^{\oplus})^* a^* \in Ra^*$. As a core invertible element is group invertible, thus $a \in R^{\mathrm{EP}}$ by Theorem 3.7. $\square$

Example 3.9. If $a \in R$ is core invertible element and there exists a projection $p \in R$ such that $ap = 0$, we could not get that $a \in R^{\mathrm{EP}}$. Considering the following counterexample. Let R be the ring of all 2×2 matrices over in field $\mathbb{F}$ with transpose as involution. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^{\#} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $A^{\oplus} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $AP = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$and $P^2 = P = P^*$. But $A^{\#} \neq A^{\oplus}$, therefore A is not EP.

Theorem 3.10. *Let $a \in R$. Then the following are equivalent:*
(1) *$a \in R^{\mathrm{EP}}$;*
(2) *$a \in R^{\oplus}$ and $aR \subseteq a^*R$;*
(3) *$a \in R^{\oplus}$ and $[a^{\oplus}, (a^{\oplus}a)^*a] = 0$.*

Proof. (1) $\Leftrightarrow$ (2) is easy to see that by Lemma 3.1 and Theorem 3.7.

(1) $\Rightarrow$ (3): Suppose a is EP, then $a \in R^{\oplus}$ and $a^{\#} = a^{\dagger} = a^{\oplus}$. Thus $[a^{\oplus}, (a^{\oplus}a)^*a] = [a^{\#}, (a^{\dagger}a)^*a] = [a^{\#}, a^{\dagger}a^2] = [a^{\#}, a^{\#}a^2] = [a^{\#}, a] = 0$.

(3) $\Rightarrow$ (2): Suppose $[a^{\oplus}, (a^{\oplus}a)^*a] = 0$, then

$$a^{\oplus}(a^{\oplus}a)^*a = (a^{\oplus}a)^*aa^{\oplus}. \tag{3.4}$$

Taking involution $*$ on (3.4), we can get $a^*a^{\oplus}a(a^{\oplus})^* = (a^{\oplus})^*a^*a^{\oplus}a$, which gives

$$a^*a^{\oplus}a(a^{\oplus})^* = (a^{\oplus})^*a^*a^{\oplus}a = (aa^{\oplus})^*a^{\oplus}a = a(a^{\oplus})^2a = a^{\oplus}a.$$

Therefore $a^*a^{\oplus}a(a^{\oplus})^*a = a^{\oplus}a^2 = a$. That is $aR \subseteq a^*R$. Therefore the condition (2) is satisfied. $\square$

In [6, Theorem 16], for an operator $T \in L(X)$, where X is a Banach space, Boasso proved that for a Moore-Penrose invertible operator T, T is an EP operator if and only if there exists an invertible operator $P \in L(X)$ such that $T^{\dagger} = PT$. Inspired by this result, we get the following theorem.

Theorem 3.11. *Let $a \in R^{\oplus}$. Then the following are equivalent:*
(1) *$a \in R^{\mathrm{EP}}$;*
(2) *there exists a unit $u \in R$ such that $a^{\oplus} = ua$;*
(3) *there exists a left invertible element $v \in R$ such that $a^{\oplus} = va$;*
(4) *there exists an element $b \in R$ such that $a^{\oplus} = ba$.*

Proof. (1) $\Rightarrow$ (2): Suppose $a \in R^{\mathrm{EP}}$, then $a \in R^{\oplus}$ and $a^{\oplus} = a^{\#}$. Let $u = (a^{\#})^2 + 1 - aa^{\#}$. Since $u(a^2 + 1 - aa^{\#}) = (a^2 + 1 - aa^{\#})u = 1$, thus u is a unit. And $ua = ((a^{\#})^2 + 1 - aa^{\#})a = a^{\#} = a^{\oplus}$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) We know that $Ra^{\oplus} = Ra^*$ by the definition of the core inverse. From $a^{\oplus} = ba$ we get $Ra^{\oplus} \subseteq Ra$. Thus $Ra^* = Ra^{\oplus} \subseteq Ra$. As a core invertible element is group invertible, we deduce that $a \in R^{\mathrm{EP}}$ by Theorem 3.7. $\square$

4 When a Moore-Penrose invertible element is an EP element

Since any EP element is Moore-Penrose invertible, it is natural to ask when a Moore-Penrose invertible element is an EP element. The concept of bi-EP was introduced by Hartwig and Spindelböck in [18] for complex matrices. They proved that for a complex $A \in \mathbb{C}_{n\times n}$, if A is group invertible, then A is an EP matrix if and only if A is bi-EP. We give a generalization of the above result in Theorem 4.3. In this section, we give the definition of n-EP, which is a generalization of bi-EP. We show that any n-EP element is an EP element whenever this element is group invertible.

Definition 4.1. [18] *An element $a \in R$ is called bi-EP if $a \in R^\dagger$ and $[aa^\dagger, a^\dagger a] = 0$.*

Definition 4.2. *Let n be a positive integer. An element $a \in R$ is called n-EP if $a \in R^\dagger$ and $[a^n a^\dagger, a^\dagger a^n] = 0$.*

Note that 1-EP is coincide with bi-EP.

In [14, Theorem 2.1], Mosić and Djordjević proved that $a \in R^{\mathrm{EP}}$ if and only if $a \in R^\# \cap R^\dagger$ and $a^n a^\dagger = a^\dagger a^n$ for some $n \geqslant 1$. This result also can be found in [7, Theorem 2.4] by Chen. In the following theorem, we give a generalization of this result.

Theorem 4.3. *Let $a \in R$ and n be a positive integer. Then $a \in R^{\mathrm{EP}}$ if and only if $a \in R^\dagger \cap R^\#$ and a is n-EP.*

Proof. Suppose $a \in R^{\mathrm{EP}}$. Then $[a, a^\dagger] = 0$, which gives $[a^n a^\dagger, a^\dagger a^n] = 0$. That is a is n-EP.

Conversely, suppose that $a \in R^\dagger \cap R^\#$ and a is n-EP. Then we have

$$a^\dagger a^{2n} a^\dagger = a^n (a^\dagger)^2 a^n. \tag{4.1}$$

Pre-multiplication and post-multiplication of (4.1) by a respectively now yields $a^{2n}a^\dagger = a^{n+1}(a^\dagger)^2 a^n$, and $a^\dagger a^{2n} = a^n (a^\dagger)^2 a^{n+1}$. Thus

$$a^{2n-1}a^\dagger = a^\# a^{2n} a^\dagger = a^\# a^{n+1}(a^\dagger)^2 a^n = a^n (a^\dagger)^2 a^n. \tag{4.2}$$

$$a^\dagger a^{2n-1} = a^\dagger a^{2n} a^\# = a^n (a^\dagger)^2 a^{n+1} a^\# = a^n (a^\dagger)^2 a^n. \tag{4.3}$$

By (4.2) and (4.3), we have $a^{2n-1}a^\dagger = a^\dagger a^{2n-1}$. Hence by [14, Theorem 2.1], we have $a \in R^{\mathrm{EP}}$. □

Theorem 4.4. *Let $a \in R^\dagger$. Then the following are equivalent:*
(1) *$a \in R^{\mathrm{EP}}$;*
(2) *there exists a unit $u \in R$ such that $a^\dagger = ua$;*
(3) *there exists a left invertible element $v \in R$ such that $a^\dagger = va$.*

Proof. (1) ⇒ (2): Suppose $a \in R^{\mathrm{EP}}$, then $a \in R^\dagger$ and $a^\dagger = a^\#$. Let $u = (a^\#)^2 + 1 - aa^\#$. Since $u(a^2+1-aa^\#) = (a^2+1-aa^\#)u = 1$, thus u is a unit. And $ua = ((a^\#)^2+1-aa^\#)a = a^\# = a^\dagger$.

(2) ⇒ (3) is clear.

(3) ⇒ (1) Suppose there exists a left invertible element $v \in R$ such that $a^\dagger = va$. Then $1 = tv$ for some $t \in R$ and $ta^\dagger = tva = a$. Thus $Ra^\dagger \subseteq Ra$ and $Ra \subseteq Ra^\dagger$. Since $Ra^\dagger = Ra^*$ and $Ra^* = Ra^\dagger$, we deduce that $Ra^* \subseteq Ra$ and $Ra \subseteq Ra^*$. Therefore $Ra = Ra^*$, that is a is an EP element. □

Remark 4.5. In Theorem 3.11 (4), we proved that for a core invertible element $a \in R$, $a \in R^{\mathrm{EP}}$ if and only if there exists an element $b \in R$ such that $a^{\textcircled{\#}} = ba$. The following example shows that this affirmation can not be obtained for a Moore-Penrose invertible element.

Recall that An infinite matrix M is said to be *bi-finite* if it is both row-finite and column-finite.

Example 4.6. Let R be the ring of all bi-finite matrices over in field $\mathbb{F}$ with transpose as involution and $e_{i,j}$ be the matrix in R with 1 in the (i,j) position and 0 elsewhere. Let $A = \sum\limits_{i=1}^{\infty} e_{i+1,i}$ and $B = A^*$, then $AB = \sum\limits_{i=2}^{\infty} e_{i,i}$, $BA = I$. So $A^\dagger = B$ and $A^\dagger = A^\dagger BA = B^2 A$. It is easy to check that B^2 is not left invertible and A is not EP (since $AB \neq BA$).

Proposition 4.7. *Let $a \in R^\dagger$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $[a^\dagger a, a] = [a^\dagger, aa^\dagger] = 0$;
(3) $[a^\dagger a, a] = [a, aa^\dagger] = 0$;
(4) $[a^\dagger a, a^\dagger] = [a^\dagger, aa^\dagger] = 0$;
(5) $[a^\dagger a, a^\dagger] = [a, aa^\dagger] = 0$.

Proof. (1) $\Rightarrow$ (2)-(3): If $a \in R^{\mathrm{EP}}$, then $aa^\dagger = a^\dagger a$. Thus (2) and (3) are obvious.

(2) $\Rightarrow$ (1) Assume that (2) holds. Observe that $[a^\dagger a, a]$ implies that $a = a^\dagger a^2 \in a^* R$ and $[a^\dagger, aa^\dagger]$ implies that $a^\dagger = a(a^\dagger)^2 \in aR$, that is $a^*R \subseteq aR$ since $a^\dagger R = a^* R$. Thus, $aR = a^*R$, i.e., a is EP.

(3) $\Rightarrow$ (1): Assume that (2) holds. Observe that $[a^\dagger a, a]$ implies that $a = a^\dagger a^2 \in a^* R$ and $[a, aa^\dagger]$ implies that $a = a^2 a^\dagger \in Ra^\dagger$, that is $Ra \subseteq Ra^\dagger$ since $Ra^\dagger = Ra^*$. We deduce that $Ra \subseteq Ra^*$, which is equivalent to $a^*R \subseteq aR$. Thus, $aR = a^*R$, i.e., a is EP.

The equivalence between (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) is similar to the proof of the equivalence between (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). $\square$

Example 4.8. The condition $[a^\dagger a, a^\dagger] = 0$ in Proposition 4.7 does not imply that a is an EP element in general. Let R, A and B be same as Example 4.6 , then $AB = \sum\limits_{i=2}^{\infty} e_{i,i}$, $BA = I$. So $A^\dagger = B$ and $[A^\dagger A, A^\dagger] = 0$. But A is not EP by $AB \neq BA$.

Lemma 4.9. [16] *Let $a \in R$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $a \in R^\dagger$ *and* $aR = a^*R$;
(3) $a \in R^\dagger$ *and* $Ra = Ra^*$.

Example 4.8 also shows that the equality $aR = a^*R$ in Lemma 4.9 cannot be replaced by inclusions $aR \subseteq a^*R$ or $a^*R \subseteq aR$.

Theorem 4.10. *Let $a \in R^\dagger$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $aR = a^2R$ *and* $[a^\dagger a, a^\dagger] = 0$;
(3) $aR = a^2R$ *and* $[a^\dagger a, a] = 0$;
(4) $aR = a^2R$ *and* $aR \subseteq a^\dagger R$;
(5) $aR = a^2R$ *and* $aR \subseteq a^*R$.

Proof. (1) $\Rightarrow$ (2)-(5): For a Moore-Penrose invertible element $a \in R$, $a \in R^{EP}$ if and only if $aa^\dagger = a^\dagger a$. Then (2)-(5) hold.

(2) $\Rightarrow$ (1): Assume that (2) holds. Observe that $[a^\dagger a, a^\dagger] = 0$ implies $a^\dagger = (a^\dagger)^2 a$. Thus $Ra^* \subseteq Ra$ (Since $Ra^\dagger = Ra^*$). That is $a^* = ra$ for some $r \in R$. We deduce that $a^* = ra = raa^\dagger a = a^* a^\dagger a$, applying involution on that last equality we obtain $a = a^\dagger a^2$. Therefore $a \in R^{\mathrm{EP}}$ by Theorem 3.7.

(3) $\Rightarrow$ (1): It is clear that $[a^\dagger a, a] = 0$ implies $a = a^\dagger a^2$. Thus $a \in R^{\mathrm{EP}}$ by Theorem 3.7 and the proof of (2) $\Rightarrow$ (1).

(5) $\Rightarrow$ (1): As $aR \subseteq a^* R$ is equivalent to $Ra^* \subseteq Ra$, we get $a \in R^{\mathrm{EP}}$ by the proof of (2) $\Rightarrow$ (1).

(4) $\Leftrightarrow$ (5): It is clear by $a^* R = a^\dagger R$. □

Similarly, we have the following theorem.

Theorem 4.11. *Let $a \in R^\dagger$. Then the following are equivalent:*
(1) $a \in R^{\mathrm{EP}}$*;*
(2) $Ra = Ra^2$ *and* $[aa^\dagger, a^\dagger] = 0$*;*
(3) $Ra = Ra^2$ *and* $[aa^\dagger, a] = 0$*;*
(4) $Ra = Ra^2$ *and* $Ra \subseteq Ra^\dagger$*;*
(5) $Ra = Ra^2$ *and* $Ra \subseteq Ra^*$*.*

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