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On the Legendre differential equation with uncertainties at the regular-singular point 1: $\mathrm{L}^p(\Omega)$ random power series solution and approximation of its statistical moments.

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Summary

In this paper, we construct two linearly independent response processes to the random Legendre differential equation on $(-1,1)\cup(1,3)$, consisting of $\mathrm{L}^p(\Omega)$ convergent random power series around the regular-singular point 1. A theorem on the existence and uniqueness of $\mathrm{L}^p(\Omega)$ solution to the random Legendre differential equation on the intervals $(-1,1)$ and $(1,3)$ is obtained. The hypotheses assumed are simple: initial conditions in $\mathrm{L}^p(\Omega)$ and random input A in $\mathrm{L}^\infty(\Omega)$ (this is equivalent to A having absolute moments that grow at most exponentially). Thus, this paper extends the deterministic theory to a random framework. Uncertainty quantification for the solution stochastic process is performed by truncating the random series and taking limits in $\mathrm{L}^p(\Omega)$. In the numerical experiments, we approximate its expectation and variance for certain forms of the differential equation. The reliability of our approach is compared with Monte Carlo simulations and gPC expansions.

KEYWORDS:
Random Legendre differential equation, Regular-singular point, Random power series, $\mathrm{L}^p(\Omega)$ random calculus, Uncertainty quantification

1 | INTRODUCTION

The random Legendre differential equation (1),

$$\begin{cases}(1-t^2)\ddot{X}(t)-2t\dot{X}(t)+A(A+1)X(t)=0,\\ X(t_0)=Y_0,\\ \dot{X}(t_0)=Y_1,\end{cases}\tag{1}$$

where A, Y_0 and Y_1 are random variables in a common underlying complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, has been already studied at the regular point $t=0$ with initial condition at $t_0=0$. In[1], a mean square power series solution to (1) was constructed on $(-1/\mathrm{e},1/\mathrm{e})$, being e the Euler constant. This result was extended in our recent contribution[2], where the solution has been constructed on the whole domain $(-1,1)$ with weaker assumptions on the random input coefficients. A common hypotheses in both works[1,2] was that the random variable A has statistical absolute moments that increase at most exponentially, equivalently, that A is a bounded random variable, see[2, Lemma 2.2]. This assumption of boundedness for A will be essential in our subsequent development.

The aim of this paper is to continue extending the classical deterministic results for the Legendre differential equation to the random setting by taking advantage of the so-called $\mathrm{L}^p(\Omega)$, $1 \leq p \leq \infty$, random calculus. For the sake of clarity, we recall that if $(H, \mathcal{A}, \mu)$ is a measure space, $\mathrm{L}^p(H)$ ($1 \leq p < \infty$) is the set of real valued measurable functions $f : H \to \mathbb{R}$ such that $\|f\|_{\mathrm{L}^p(H)} = (\int_H |f|^p \,\mathrm{d}\mu)^{1/p} < \infty$. While, as usual, $\mathrm{L}^\infty(H)$ is the set of measurable functions such that $\|f\|_{\mathrm{L}^\infty(H)} = \inf\{\sup\{|f(x)| : x \in H \backslash N\} : \mu(N) = 0\} < \infty$. As in our particular context we are interested in dealing with random variables and stochastic processes, we take $H = \Omega$ and $\mu = \mathbb{P}$. The reader can check, for example, the references[3,4],[5, Ch. IV],[6] for further details. It is worth pointing out that the $\mathrm{L}^p(\Omega)$ random calculus has been widely used to study both theoretical and numerically random differential equations[7,8,9,10,11].

Our goal in this paper is to construct a fundamental set around the regular-singular point $t = 1$, as well as to quantify reliable approximations to the main statistical functions of the solution stochastic process to the initial value problem (1). To the best of our knowledge, this is the first contribution in the extant literature where the regular-singular point case is addressed for a random second-order differential equation. In this sense, we want to point out that the subsequent approach may be useful to study other important random second-order linear differential equations around regular-singular points. This kind of differential equations are met in many physical and engineering problems[12,13,14]. In particular, the Legendre differential equation is very useful for treating the boundary value problems exhibiting spherical symmetry.

The organization of the present paper is as follows. In Section 2, the main results regarding the random initial value problem (1) are stated and proved. In Section 3, we show how to approximate the moments of the solution stochastic process by truncating the corresponding random power series. Section 4 is devoted to illustrating our theoretical findings via different examples of the random initial value problem (1), where approximations of the expectation and the variance functions of the solution stochastic process are computed and compared with Monte Carlo simulations and gPC expansions. Finally, conclusions are drawn in Section 5.

2 | RANDOM LEGENDRE DIFFERENTIAL EQUATION AT THE REGULAR SINGULAR POINT 1

As it has been previously indicated, our goal in this section is to provide an analogous analysis to[1,2] for the regular-singular point $t = 1$.

By the deterministic theory on the Legendre differential equation, if A is constant, then a deterministic fundamental set $\{\phi_1(t), \phi_2(t)\}$ is given by

$$\phi_1(t) = \sum_{n=0}^{\infty} c_n (t-1)^n, \quad |t-1| < 2,$$

where $\{c_n\}_{n=0}^{\infty}$ is defined by the recursive relation

$$c_0 = 1, \quad c_{n+1} = \frac{(n+1-A)(A-n)}{2(n+1)^2} c_n, \quad n = 0, 1, 2, \dots,$$

and

$$\phi_2(t) = \phi_1(t) \log|t-1| + \sum_{n=1}^{\infty} d_n (t-1)^n, \quad |t-1| < 2,$$

where $\{d_n\}_{n=1}^{\infty}$ is defined as follows:

$$d_1 = \frac{-c_0 - 4c_1}{2}, \quad d_{n+1} = -\frac{(n+1+A)(n-A)d_n + 4(n+1)c_{n+1} + (2n+1)c_n}{2(n+1)^2}.$$

To construct rigorously this fundamental set of independent solutions, $\{\phi_1(t), \phi_2(t)\}$, we can apply the so-called Fröbenius method[15, Th. 7].

In our setting, we first randomize this fundamental set of solutions. Therefore, we consider the following two stochastic processes

$$X_1(t) = \sum_{n=0}^{\infty} C_n (t-1)^n, \quad X_2(t) = X_1(t) \log|t-1| + \sum_{n=1}^{\infty} D_n (t-1)^n, \tag{2}$$

where the coefficients $\{C_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$ are random variables defined in our complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ recursively as follows:

$$C_0 = 1, \quad C_{n+1} = \frac{(n+1-A)(A-n)}{2(n+1)^2} C_n, \quad n = 0, 1, 2, \dots, \tag{3}$$

and
$$D_1 = \frac{-C_0 - 4C_1}{2},\ D_{n+1} = -\frac{(n+1+A)(n-A)D_n + 4(n+1)C_{n+1} + (2n+1)C_n}{2(n+1)^2}. \tag{4}$$
Notice that, as usual in the notation, we have distinguished deterministic and random magnitudes by using lower and upper case, respectively.

In the following Proposition 1, we will see that both series (2) converge in $\mathrm{L}^\infty(\Omega)$ on $(-1,1)\cup(1,3)$ (i.e., radius of convergence 2, so retaining their deterministic counterpart), in particular, pointwise on $\omega \in \Omega$ for $t \in (-1,1)\cup(1,3)$. This $\mathrm{L}^\infty(\Omega)$ convergence will imply that the stochastic processes $X_1(t)$ and $X_2(t)$ are solutions to the random Legendre differential equation on the domain $(-1,1)\cup(1,3)$ in the $\mathrm{L}^\infty(\Omega)$ sense, since the random power series can be differentiated in the $\mathrm{L}^\infty(\Omega)$ sense term by term.

Proposition 1. If $A \in \mathrm{L}^\infty(\Omega)$, then both series defined by (2)–(4) converge in $\mathrm{L}^\infty(\Omega)$ on the interval $(-1,1)\cup(1,3)$. In particular, the stochastic processes $X_1(t)$ and $X_2(t)$, given by (2)–(4), are solutions to the random Legendre differential equation on the domain $(-1,1)\cup(1,3)$ in the $\mathrm{L}^\infty(\Omega)$ sense.

Proof. First, let us prove that the random power series defining $X_1(t)$ converges in $\mathrm{L}^\infty(\Omega)$, for every $t \in (-1,1)\cup(1,3)$. From (3), we have
$$\|C_n\|_{\mathrm{L}^\infty(\Omega)} \le \frac{(n+\|A\|_{\mathrm{L}^\infty(\Omega)})(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)}{2n^2}\|C_{n-1}\|_{\mathrm{L}^\infty(\Omega)},$$
that is,
$$\frac{\|C_n\|_{\mathrm{L}^\infty(\Omega)}}{\|C_{n-1}\|_{\mathrm{L}^\infty(\Omega)}} \le \frac{(n+\|A\|_{\mathrm{L}^\infty(\Omega)})(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)}{2n^2}.$$
Now if we multiply both sides of this last inequality by r, being $0 < r < 2$, and afterwards we take limits as $n \to \infty$, then we obtain that
$$\limsup_{n\to\infty} \frac{\|C_n\|_{\mathrm{L}^\infty(\Omega)} r^n}{\|C_{n-1}\|_{\mathrm{L}^\infty(\Omega)} r^{n-1}} \le \lim_{n\to\infty} r\frac{(n+\|A\|_{\mathrm{L}^\infty(\Omega)})(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)}{2n^2} = \frac{r}{2} < 1.$$
By applying the d'Alembert's ratio test for numerical series, we derive that the series with general term $\|C_n\|_{\mathrm{L}^\infty(\Omega)} r^n$ is convergent, i.e.,
$$\sum_{n=0}^{\infty} \|C_n\|_{\mathrm{L}^\infty(\Omega)} r^n < \infty,$$
for $0 < r < 2$. This implies that the random power series $X_1(t)$, given by (2), has radius of convergence 2 in the Banach space $(\mathrm{L}^\infty(\Omega), \|\cdot\|_{\mathrm{L}^\infty(\Omega)})$.

Now let us check that
$$\sum_{n=1}^{\infty} \|D_n\|_{\mathrm{L}^\infty(\Omega)} r^n < \infty,$$
for $0 < r < 2$. Since $\sum_{n=0}^{\infty} \|C_n\|_{\mathrm{L}^\infty(\Omega)} r^n < \infty$, the sequence $\{\|C_n\|_{\mathrm{L}^\infty(\Omega)} r^n\}_{n=0}^{\infty}$ is bounded by a number $M > 0$. That is, $\|C_n\|_{\mathrm{L}^\infty(\Omega)} \le M/r^n$, $n \ge 0$. Using this inequality in (4), we obtain
$$\begin{aligned}\|D_{n+1}\|_{\mathrm{L}^\infty(\Omega)} &\le \frac{(n+\|A\|_{\mathrm{L}^\infty(\Omega)})(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)\|D_n\|_{\mathrm{L}^\infty(\Omega)} + (2n+1)\frac{M}{r^n} + 4(n+1)\frac{M}{r^{n+1}}}{2(n+1)^2}\\ &\le \frac{(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)^2\|D_n\|_{\mathrm{L}^\infty(\Omega)} + (2n+1)\frac{M}{r^n} + 4(n+1)\frac{M}{r^{n+1}}}{2(n+1)^2}.\end{aligned} \tag{5}$$
Let us define the following sequence of positive numbers $\{H_n : n \ge 1\}$:
$$H_1 = \|D_1\|_{\mathrm{L}^\infty(\Omega)},$$
$$H_{n+1} = \frac{(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)^2 H_n + (2n+1)\frac{M}{r^n} + 4(n+1)\frac{M}{r^{n+1}}}{2(n+1)^2}, \quad n = 1,2,\ldots.$$
From inequality (5), it is evident that the sequence $\{H_n : n \ge 1\}$ majorizes $\{\|D_n\|_{\mathrm{L}^\infty(\Omega)} : n \ge 1\}$, that is,
$$\|D_n\|_{\mathrm{L}^\infty(\Omega)} \le H_n, \quad n \ge 1.$$
Now, let us define
$$K_n = \max_{1\le k\le n} H_k r^k.$$

We obtain

$$H_{n+1}r^{n+1} \leq \frac{r(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)^2}{2(n+1)^2}H_n r^n + \frac{r(2n+1)M+4(n+1)M}{2(n+1)^2}$$
$$\leq \frac{r(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)^2}{2(n+1)^2}K_n + \frac{r(2n+1)M+4(n+1)M}{2(n+1)^2}.$$

Observe

$$\lim_{n\to\infty} \frac{r(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)^2}{2(n+1)^2} = \frac{r}{2} < 1, \quad \lim_{n\to\infty} \frac{r(2n+1)M+4(n+1)M}{2(n+1)^2} = 0. \tag{6}$$

Then, for $0<\epsilon<1-r/2$ arbitrary but fixed, using the limits in (6), we can choose $n_0 = n_0(r, \|A\|_{\mathrm{L}^\infty(\Omega)}, M)$ such that, for all $n \geq n_0$,

$$\frac{r(n+\|A\|_{\mathrm{L}^\infty(\Omega)}+1)^2}{2(n+1)^2} < 1-\epsilon, \quad \frac{r(2n+1)M+4(n+1)M}{2(n+1)^2} < 1.$$

Thus, for $n \geq n_0$,

$$H_{n+1}r^{n+1} \leq (1-\epsilon)K_n + 1.$$

Suppose that $(1-\epsilon)K_n + 1 \leq K_n$, for all $n \geq n_0$. This implies

$$H_{n+1}r^{n+1} \leq K_n,$$

so that $K_{n+1} = K_n$ for $n \geq n_0$. Let $K = K_n$, $n \geq n_0$. Then

$$H_n \leq K/r^n, \quad n \geq n_0,$$

therefore

$$\sum_{n=n_0}^{\infty} H_n r_0^n \leq K \sum_{n=n_0}^{\infty} H_n (r_0/r)^n < \infty,$$

for each $0 < r_0 < r$. As $0<r<2$ is arbitrary, we conclude that

$$\sum_{n=n_0}^{\infty} \|D_n\|_{\mathrm{L}^\infty(\Omega)} r^n \leq \sum_{n=n_0}^{\infty} H_n r^n < \infty,$$

as wanted.

Otherwise, if there is a strictly increasing sequence of natural numbers $\{n_l\}_{l=1}^\infty$ such that $(1-\epsilon)K_{n_l}+1 > K_{n_l}$, for all $l \geq 1$, we arrive at $K_{n_l} < 1/\epsilon$, $l \geq 1$. Since the sequence $\{K_n\}_{n=1}^\infty$ is increasing, we deduce that $K_n < 1/\epsilon$, for all $n \geq 1$. Let $K = 1/\epsilon$, so that

$$H_n \leq K/r^n, \quad n \geq 1.$$

The same reasoning as in the previous paragraph applies in this case, and we are done.

□

Theorem 1. Let $1 \leq p \leq \infty$ and $t_0 \in I$, where I is either $(-1,1)$ or $(1,3)$. Given two initial conditions $X(t_0) = Y_0$ and $\dot{X}(t_0) = Y_1$ that belong to $\mathrm{L}^p(\Omega)$ and if $A \in \mathrm{L}^\infty(\Omega)$, then there exists a unique response process $X(t)$ in the $\mathrm{L}^p(\Omega)$ sense to (1) on I. This solution process $X(t)$ has the form

$$X(t) = A_1 X_1(t) + A_2 X_2(t), \tag{7}$$

where

$$A_1 = \frac{Y_0 \dot{X}_2(t_0) - Y_1 X_2(t_0)}{W(X_1,X_2)(t_0)}, \quad A_2 = \frac{Y_1 X_1(t_0) - Y_0 \dot{X}_1(t_0)}{W(X_1,X_2)(t_0)}, \tag{8}$$

and $W(X_1,X_2)(t_0)$ is the Wronskian of the pair $\{X_1(t_0), X_2(t_0)\}$, where

$$W(X_1,X_2)(t_0) = X_1(t_0)\dot{X}_2(t_0) - X_2(t_0)\dot{X}_1(t_0)$$
$$= \begin{cases} \frac{-2}{|1-t^2|}, & t \in (-1,1), \\ \frac{2}{|1-t^2|}, & t \in (1,3). \end{cases} \tag{9}$$

Proof. At each outcome $\omega \in \Omega$, the pair $\{X_1(t)(\omega), X_2(t)(\omega)\}$ is a fundamental set in the sample path sense (because $X_1(t)$ and $X_2(t)$ are linearly independent $\mathrm{L}^\infty(\Omega)$ solutions), so we can compute the Wronskian pointwise on ω by using the deterministic Liouville's formula [16, Prop. 2.15]: if $t \in I$, then

$$W(X_1, X_2)(t)(\omega) = C_I(\omega) \mathrm{e}^{\int \frac{2t}{1-t^2}\,\mathrm{d}t} = \frac{C_I(\omega)}{|1-t^2|},$$

for a certain random variable $C_I(\omega)$ that depends on I. To obtain C_I at each $\omega \in \Omega$, notice that

$$\begin{aligned} C_I &= |1-t^2| W(X_1, X_2)(t) \\ &= |1-t^2| \left\{ X_1(t) \left(\dot{X}_1(t) \log|t-1| + \frac{X_1(t)}{t-1} + \sum_{n=1}^{\infty} n D_n (t-1)^{n-1} \right) - X_2(t)\dot{X}_1(t) \right\} \\ &\to \begin{cases} -2, & \text{if } t \to 1^-, \\ 2, & \text{if } t \to 1^+. \end{cases} \end{aligned}$$

This proves (9).

Note also that the random variables A_1 and A_2 defined by (8) belong to $\mathrm{L}^p(\Omega)$, because $X_1(t_0), \dot{X}_1(t_0), X_2(t_0), \dot{X}_2(t_0) \in \mathrm{L}^\infty(\Omega)$ and $Y_0, Y_1 \in \mathrm{L}^p(\Omega)$. Since $X_1(t)$ and $X_2(t)$ are L^∞ solutions on $(-1,1) \cup (1,3)$ by Proposition 1, and $A_1, A_2 \in \mathrm{L}^p(\Omega)$, from (7) we derive that $X(t)$ is an L^p solution to (1) on I.

To demonstrate the uniqueness, we use [3, Th. 5.1.2]. Rewrite (1) as a first-order linear differential equation

$$\dot{Z}(t) = B(t) Z(t),$$

where

$$Z(t) = \begin{pmatrix} X(t) \\ \dot{X}(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 1 \\ \frac{A(A+1)}{1-t^2} & \frac{-2t}{1-t^2} \end{pmatrix}.$$

We say that $Z = (Z_1, Z_2)$ belongs to $\mathrm{L}_2^p(\Omega)$ if

$$\|Z\|_{\mathrm{L}_2^p(\Omega)} := \max\{\|Z_1\|_{\mathrm{L}^p(\Omega)}, \|Z_2\|_{\mathrm{L}^p(\Omega)}\} < \infty.$$

Consider the random matrix norm

$$|||B||| := \max_i \sum_j \|b_{ij}\|_{\mathrm{L}^\infty(\Omega)}.$$

If $Z, Z' \in \mathrm{L}_2^p(\Omega)$, then

$$\|B(t)Z - B(t)Z'\|_{\mathrm{L}_2^p(\Omega)} \le |||B(t)||| \cdot \|Z - Z'\|_{\mathrm{L}_2^p(\Omega)},$$

where

$$\int_a^b |||B(t)|||\,\mathrm{d}t = \int_a^b \frac{\|A\|_{\mathrm{L}^\infty(\Omega)}(\|A\|_{\mathrm{L}^\infty(\Omega)}+1) + 2|t|}{1-t^2}\,\mathrm{d}t < \infty$$

for each $a < b$ that belong to I. Then the assumptions of [3, Th. 5.1.2] hold.

□

To finish this section, we would like to comment that the hypothesis $A \in \mathrm{L}^\infty(\Omega)$ is not restrictive in practice, as any unbounded random variable can be truncated in a support as large as we want [17].

3 | APPROXIMATION OF THE MOMENTS OF THE RESPONSE PROCESS: EXPECTATION AND VARIANCE

Apart from determining the solution stochastic process $X(t)$ to the Legendre random differential equation (1), a main goal is also to construct reliable approximations regarding its statistical behaviour. This latter information is mainly summarized by the mean and the variance functions. The mean or expectation function, $\mathbb{E}[X(t)]$, provides a measure of the average behaviour of the process at each time instant t, while the variance, $\mathbb{V}[X(t)]$, quantifies the dispersion or variability of the process around the mean $\mathbb{E}[X(t)]$.

As the solution stochastic process $X(t)$ has been constructed via an infinite series, it is natural to approximate both its mean and its variance by considering truncations of that series to keep the computational burden affordable.

Now, we will show how under the conditions of Theorem 1, the moments of $X(t)$ can be approximated up to order p. We consider the truncation

$$X^N(t) = A_1^N X_1^N(t) + A_2^N X_2^N(t), \tag{10}$$

for $N \geq 1$, where

$$X_1^N(t) = \sum_{n=0}^{N} C_n (t-1)^n, \quad X_2^N(t) = X_1^N(t) \log|t-1| + \sum_{n=1}^{N} D_n (t-1)^n, \tag{11}$$

and

$$A_1^N = \frac{Y_0 \dot{X}_2^N(t_0) - Y_1 X_2^N(t_0)}{W(X_1, X_2)(t_0)}, \quad A_2^N = \frac{Y_1 X_1^N(t_0) - Y_0 \dot{X}_1^N(t_0)}{W(X_1, X_2)(t_0)}, \tag{12}$$

being $W(X_1, X_2)(t_0)$ the Wronskian computed in expression (9). As a consequence of Proposition 1, $X_1^N(t) \to X_1(t)$ and $X_2^N(t) \to X_2(t)$ in $\mathrm{L}^\infty(\Omega)$ as $N \to \infty$. Since $Y_0, Y_1 \in \mathrm{L}^p(\Omega)$, one has that $A_1^N \to A_1$ and $A_2^N \to A_2$ in $\mathrm{L}^p(\Omega)$ as $N \to \infty$. This implies that $X^N(t) \to X(t)$ in $\mathrm{L}^p(\Omega)$ as $N \to \infty$ too. In particular, the statistical moments up to order p of $X^N(t)$ tend to those of $X(t)$.

For $p \geq 2$ arbitrary but fixed, then we can approximate the average of $X(t)$, $\mathbb{E}[X(t)]$, and the variance of $X(t)$, $\mathbb{V}[X(t)]$, by using

$$\mathbb{E}[X(t)] = \lim_{N\to\infty} \mathbb{E}[X^N(t)], \quad \mathbb{V}[X(t)] = \lim_{N\to\infty} \mathbb{V}[X^N(t)],$$

see[3, Th. 4.2.1, Th. 4.3.1].

4 | NUMERICAL EXPERIMENTS

In this section we illustrate our theoretical findings by means of several numerical examples performed in the software Mathematica®. We will choose specific probability distributions for the input random variables A, Y_0 and Y_1 and then we will approximate the expectation and the variance of the response stochastic process $X(t)$ by using different orders of truncation N in expressions (10)–(12). The reliability of the obtained results will be shown by comparing them with the results provided by the following two techniques for uncertainty quantification:

- Monte Carlo simulations[18], which consist in obtaining a number m of realizations for the random input parameters, say

$$A^{(1)}, \ldots, A^{(m)},$$
$$Y_0^{(1)}, \ldots, Y_0^{(m)},$$
$$Y_1^{(1)}, \ldots, Y_1^{(m)},$$

  and then solving each one of the corresponding deterministic Legendre differential equations,

$$\begin{cases} (1-t^2)\ddot{X}^{(i)}(t) - 2t\dot{X}^{(i)}(t) + A^{(i)}(A^{(i)}+1)X^{(i)}(t) = 0, \\ X^{(i)}(t_0) = Y_0^{(i)}, \\ \dot{X}^{(i)}(t_0) = Y_1^{(i)}, \end{cases}$$

  which gives a realization $X^{(i)}(t)$ (a sample path) of $X(t)$, for $1 \leq i \leq m$. The expectation and variance of $X(t)$ can be approximated as follows:

$$\mathbb{E}[X(t)] \approx \mu_m(t) = \frac{1}{m} \sum_{i=1}^{m} X^{(i)}(t)$$

  and

$$\mathbb{V}[X(t)] \approx \frac{1}{m-1} \sum_{i=1}^{m} \left(X^{(i)}(t) - \mu_m(t) \right)^2.$$

  Monte Carlo simulations require many realizations or simulations of $X(t)$ to get accurately its statistics (the error convergence rate is inversely proportional to the square root of the number m of realizations), therefore the computational cost of Monte Carlo simulations is higher than our method based on random series.

- A variation of generalized Polynomial Chaos (gPC) expansions for continuous stochastic systems with dependent and jointly absolutely continuous random inputs, which is a method described in [19] and applied in [20]. We consider the canonical bases

$$C_1^m = \{1, A, A^2, \dots, A^m\},$$
$$C_2^m = \{1, Y_0, Y_0^2, \dots, Y_0^m\},$$
$$C_3^m = \{1, Y_1, Y_1^2, \dots, Y_1^m\},$$

and the vector of random input coefficients, $\zeta = (A, Y_0, Y_1)$. We consider a simple tensor product to construct a basis of monomials of degree less than or equal to m:

$$\Xi^p = \{\phi_0(\zeta), \phi_1(\zeta), \dots, \phi_p(\zeta)\},$$

where

$$\phi_0 = 1, \quad p = \binom{m+3}{3}, \quad \phi_i(\zeta) = A^{i_1} Y_0^{i_2} Y_1^{i_3},$$

where $i_1 + i_2 + i_3 \le m$ and $i \leftrightarrow (i_1, i_2, i_3)$ in a bijective manner.

We impose a solution to the random initial value problem (1) of the form

$$X^p(t) = \sum_{i=0}^{p} \tilde{X}_i^p(t) \phi_i(\zeta),$$

where $\tilde{X}_i^p(t)$, $0 \le i \le p$, are deterministic functions to be found. The deterministic differential equation satisfied by the coefficients is the following:

$$R \frac{\mathrm{d}^2}{\mathrm{d}t^2} \tilde{X}^p(t) - \frac{2t}{1-t^2} R \frac{\mathrm{d}}{\mathrm{d}t} \tilde{X}^p(t) + N \tilde{X}^p(t) = 0,$$

with initial conditions

$$R \tilde{X}^p(t_0) = v,$$

$$R \frac{\mathrm{d}}{\mathrm{d}t} \tilde{X}^p(t_0) = w,$$

where

$$\tilde{X}^p(t) = \left(\tilde{X}_0^p(t), \dots, \tilde{X}_p^p(t) \right)^{\mathrm{T}},$$

$$v = \left(\mathbb{E}[Y_0 \phi_0(\zeta)], \dots, \mathbb{E}[Y_0 \phi_p(\zeta)] \right)^{\mathrm{T}}$$

and

$$w = \left(\mathbb{E}[Y_1 \phi_0(\zeta)], \dots, \mathbb{E}[Y_1 \phi_p(\zeta)] \right)^{\mathrm{T}},$$

and R and N are the following matrices of size $(p+1) \times (p+1)$:

$$R = \begin{pmatrix} \mathbb{E}[(\phi_0(\zeta))^2] & \cdots & \mathbb{E}[\phi_0(\zeta)\phi_p(\zeta)] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\phi_p(\zeta)\phi_0(\zeta)] & \cdots & \mathbb{E}[(\phi_p(\zeta))^2] \end{pmatrix}$$

and

$$N = \frac{1}{1-t^2} \begin{pmatrix} \mathbb{E}[A(A+1)(\phi_0(\zeta))^2] & \cdots & \mathbb{E}[A(A+1)\phi_0(\zeta)\phi_p(\zeta)] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[A(A+1)\phi_p(\zeta)\phi_0(\zeta)] & \cdots & \mathbb{E}[A(A+1)(\phi_p(\zeta))^2] \end{pmatrix},$$

respectively.

This system of differential equations can be solved via standard numerical techniques, such as the Runge-Kutta algorithm. This method provides mean square approximations to $X(t)$ with spectral convergence rate in general, although numerical errors may arise for large orders m of bases, due to ill-conditioning of the matrix R for large p.

It can be shown that the expectation and the variance of solution stochastic process $X(t)$ can be approximated using the following finite sums:

$$\mathbb{E}[X(t)] \approx \mathbb{E}[X^p(t)] = \sum_{i=0}^{p} \tilde{X}_i^p(t) e_i,$$

$$\mathbb{V}[X(t)] \approx \mathbb{V}[X^p(t)] = \sum_{i,j=0}^{p} \tilde{X}_i^p(t) \tilde{X}_j^p(t) (R_{ij} - e_i e_j),$$

where $e_i = \mathbb{E}[\phi_i(\zeta)]$.

Now, we show two examples. We want to highlight that in the first example we assume that the input random data of the Legendre differential equation are assumed to be statistically dependent (with a joint probability distribution), while in the second example the corresponding random data are independent. In this manner we show that our theoretical results are able to consider both situations in practice.

Example 1. We assume the following joint probability distribution for the random input data:

$$(A, Y_0, Y_1) \sim \text{Dirichlet}(5, 1, 2, 3),$$

with initial conditions at $t_0 = 0$. Since A, Y_0 and Y_1 are bounded random variables, Theorem 1 ensures that $X(t)$ is an $\mathrm{L}^\infty(\Omega)$ solution to random initial value problem (1) on the interval $(-1, 1)$.

In Table 1 and Table 2, approximations of $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$, respectively, are performed with order of truncations $N = 25$ and $N = 26$. The results are compared with Monte Carlo simulations and gPC expansions. Observe that there is stabilization of the results for $t \geq 0$, while for $t < 0$, specially for t near -1, larger truncation orders may be required. This is because the initial conditions are located at 0 and the power series are centered around 1, so better approximations are expected around those points. The results obtained agree with Monte Carlo simulations and gPC expansions. Notice that more simulations are required in the Monte Carlo method due to its slowness of convergence. On the other hand, gPC expansions converge quickly due to spectral convergence.

t	$\mathbb{E}[X^{25}(t)]$	$\mathbb{E}[X^{26}(t)]$	MC 500,000	gPC $m = 4$	gPC $m = 5$
−0.9	−0.174008	−0.174833	−0.184978	−0.184818	−0.184818
−0.5	−0.0140125	−0.0140143	−0.0140897	−0.0140191	−0.0140191
0	0.0909091	0.0909091	0.090896	0.0909091	0.0909091
0.5	0.180172	0.180172	0.180222	0.180172	0.180172
0.9	0.281694	0.281694	0.281855	0.281694	0.281694

TABLE 1 Approximation of the expectation of the solution stochastic process. Example 1.

t	$\mathbb{V}[X^{25}(t)]$	$\mathbb{V}[X^{26}(t)]$	MC 500,000	gPC $m = 4$	gPC $m = 5$
−0.9	0.0241268	0.0242201	0.0253679	0.0253799	0.0253799
−0.5	0.0107702	0.0107703	0.0107672	0.0107704	0.0107704
0	0.00688705	0.00688705	0.00689240	0.00688705	0.00688705
0.5	0.00844482	0.00844482	0.00845436	0.00844483	0.00844483
0.9	0.0262702	0.0262702	0.0262890	0.0262703	0.0262703

TABLE 2 Approximation of the variance of the solution stochastic process. Example 1.

Example 2. In this example, we assume that A, Y_0 and Y_1 are independent random variables with the following probability distributions:

$$A \sim \text{Beta}(2, 1), \quad Y_0 \sim \text{Normal}(1, 1), \quad Y_1 \sim \text{Exponential}(2).$$

The initial conditions are set at the time instant $t_0 = 1.2$. Since Y_0 and Y_1 have absolute moments of any order $1 \leq p < \infty$ and A is a bounded random variable, Theorem 1 tells us that $X(t)$ is an L^p solution to (1) for each $1 \leq p < \infty$ on $(1, 3)$.

Table 3 and Table 4 show approximations for the mean and variance of $X(t)$ at orders of truncation $N = 25$ and $N = 26$. The results obtained are compared with Monte Carlo simulations and gPC expansions. We observe that stabilization of the approximations occur for $t \leq 2.5$. For $t = 2.9$, a larger order of truncation N is needed, because the initial time $t_0 = 1.2$ and the center point 1 are far from $t = 2.9$. The approximations for the expectation and variance agree with Monte Carlo simulations and gPC expansions. The estimates performed by the gPC method are accurate due to spectral convergence, whereas more realizations for the Monte Carlo method are needed to achieve higher accuracy.

t	$\mathbb{E}[X^{25}(t)]$	$\mathbb{E}[X^{26}(t)]$	MC 500,000	gPC $m = 4$	gPC $m = 5$
1.2	1	1	0.996418	1	1
1.5	1.14660	1.14660	1.14286	1.14660	1.14660
2	1.37833	1.37833	1.37402	1.37833	1.37833
2.5	1.59841	1.59841	1.59347	1.59841	1.59841
2.9	1.76891	1.76723	1.76260	1.76803	1.76803

TABLE 3 Approximation of the expectation of the solution stochastic process. Example 2.

t	$\mathbb{V}[X^{25}(t)]$	$\mathbb{V}[X^{26}(t)]$	MC 500,000	gPC $m = 4$	gPC $m = 5$
−0.9	1	1	1.00313	1	1
−0.5	1.12843	1.12843	1.13185	1.12843	1.12843
0	1.53899	1.53899	1.54307	1.53899	1.53899
0.5	2.06676	2.06676	2.07152	2.06676	2.06676
0.9	2.55896	2.55733	2.56339	2.55810	2.55810

TABLE 4 Approximation of the variance of the solution stochastic process. Example 2.

5 | CONCLUSIONS

In this paper, we have constructed a fundamental set $\{X_1(t), X_2(t)\}$ of $\mathrm{L}^p(\Omega)$ solutions ($1 \leq p \leq \infty$) to the Legendre differential equation with uncertainties on the domain $(-1, 1) \cup (1, 3)$ via random power series centered at the regular-singular point $t = 1$. It has been assumed that the initial conditions at the point $t_0 \in (-1, 1) \cup (1, 3)$, Y_0 and Y_1, belong to $\mathrm{L}^p(\Omega)$, and that the random input A is a bounded random variable (which is equivalent to A having absolute moments that grow at most exponentially). Under these hypotheses, a theorem on existence and uniqueness of $\mathrm{L}^p(\Omega)$ solution $X(t)$ to the random Legendre differential equation on $(-1, 1)$ and $(1, 3)$ has been proved. This result is an extension of already published contributions, which constructed an $\mathrm{L}^p(\Omega)$ power series solution to the random Legendre differential equation on $(-1, 1)$ around the regular point 0. In order to perform uncertainty quantification for $X(t)$, we have proposed a truncation method to approximate $X(t)$ by simpler processes $X^N(t)$ in the $\mathrm{L}^p(\Omega)$ sense, so that the moments of $X(t)$ up to order p can be approximated by those of $X^N(t)$. In particular, if $p \geq 2$, the expectation and variance of $X(t)$ can be approximated.

The numerical experiments have shown examples in which we have approximated both statistics of $X(t)$. These examples have been devised to consider two important situations from a practical standpoint, namely, when the input random data are assumed to be dependent and independent random variables. The results obtained have been compared with Monte Carlo simulations and gPC expansions, showing full agreement.

Finally, we would like to point out that our method could be extended to other important second-order random differential equations with a regular-singular point.

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CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interests regarding the publication of this article.

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