

## On domains witnessing increase in information

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**ABSTRACT.** The paper considers algebraic directed-complete partial orders with a semi-regular Scott topology, called regular domains. As is well known, the category of Scott domains and continuous maps is Cartesian closed. This is no longer true, if the domains are required to be regular. Two Cartesian closed subcategories of the regular Scott domains are exhibited: regular dI-domains with stable maps and strongly regular Scott domains with continuous maps. Here a Scott domains is strongly regular if all of its compact open subsets are regular open. If one considers only embeddings as morphisms, then both categories are closed under the construction of dependent products and sums. Moreover, they are  $\omega$ -cocomplete and their object classes are closed under several constructions used in programming language semantics. It follows that recursive domain equations can be solved and models of typed and untyped lambda calculi can be constructed. Both kinds of domains can be used in giving meaning to programming language constructs.

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### 1. INTRODUCTION

Domains and domain-theoretic models of the lambda calculus were discovered by Dana Scott [12] in the fall of 1969 when he was working with Christopher Strachey at Oxford on the semantics of programming languages. In developing his approach he built on well understood ideas from recursive function theory using higher type functionals. But while people in this area were mainly interested in total functions, he started right away to consider also partial objects. Similar work was done by Yuri L. Ershov [6] in Russia at nearly the same time. He too was motivated by the theory of functionals in computability theory.

A domain is a structure modelling the notion of approximation and of computation. A computation performed using an algorithm proceeds in discrete steps. After each step there is more information available about the result of the computation. In this way the result obtained after each step can be seen as an approximation of the final result.

To be a bit more formal, a domain is a structure having one binary relation  $\sqsubseteq$ , a partial order, often called *information order*, with the intended meaning that  $x \sqsubseteq y$  just in case  $x$  is an approximation of  $y$  or  $y$  contains at least as much information as  $x$ . We also require that a domain should include a least element modelling no information. To guarantee the existence of the result of a computation, that is, of a consistent set of approximations, every directed subset of a domain must have a least upper bound. Directed subsets contain consistent information. The information contained in a finite set of elements is consistent, if the set is bounded from above.

In any step of a computation a computing device can process at most a finite amount of information. Domain elements containing only finite information are called compact. We shall consider only algebraic domains in this paper. This means that every point is the least upper bound of all compact elements below it.

As has already been said, for elements  $x$  and  $y$  of a domain,  $x \sqsubseteq y$  means that  $y$  contains at least the information contained in  $x$ . In this paper we are interested in domains with the property that if the information contained in a compact element can be enlarged, then it can be enlarged in at least two essentially different ways. This means that for any two distinct compact elements  $x$  and  $y$  with  $x \sqsubseteq y$  there is a further compact element  $z$  above  $x$  which is inconsistent with  $y$ . It follows in this case that  $x \sqsubset y$  not only indicates that  $y$  properly includes the information contained in  $x$ , but that there is also a witness for this, namely a point  $z$  with  $x \sqsubseteq z$  such that  $y$  and  $z$  are not bounded from above. This excludes situations in which  $x \sqsubset y$ , but  $x$  uniquely determines  $y$ .

Each domain comes with a canonical order-consistent topology, the Scott topology. It turns out that the just mentioned requirement on the information order is equivalent to the fact that the Scott topology is semi-regular. Therefore, we call domains satisfying the above condition regular. It is the aim of this paper to show that large subclasses of regular domains have sufficient closure properties so that they can be used for defining the meaning of programming language constructs.

As we shall see, both the separated and the coalesced sum of families of (at least two) regular domains are regular again. The same holds for the product and—in the case of finite families—for the smash product. Moreover, the category of regular domains with embedding/projections as morphisms is  $\omega$ -cocomplete: the inverse limit of any  $\omega$ -cochain of regular domains is regular again. In addition, this category is closed under the construction of dependent sums.

An important closure property needed for the interpretation of programming languages that allow procedures taking functions as inputs is closure under the function space construction. As is well known, the class of algebraic domains is not closed under this construction. Therefore, one usually restricts oneself to the subclass of Scott domains, when giving meaning to programming language constructs. Unfortunately, the Scott domain of continuous maps between two regular Scott domains need not be regular again. However, we shall present two subclasses of the regular Scott domains that are closed under the corresponding function space construction.

First, we consider the subclass of regular dI-domains. dI-domains have been introduced by Gérard Berry [4] in order to show that certain continuous maps are not lambda definable. With stable maps as morphisms their category is Cartesian closed. The same is true for the full subcategory of regular dI-domains. We show more generally that the category of dI-domains and rigid embedding/projections is closed under the construction of dependent products. Since the dI-domains are also closed under the other constructions mentioned above, regular dI-domains can be used in giving meaning to functional programs. Recursive domain equations have a solution [13] and stable regular models for typed and untyped lambda calculi can be constructed [2].

The second way we take is to strengthen the regularity requirement. The stronger condition is such that not only all basic Scott open sets are regular open, but all finite unions of such sets. These are exactly the compact open subsets of the domain. We call domains satisfying the stronger requirement strongly regular. As we shall see, this class is closed under nearly all the domain constructions mentioned earlier. Moreover, we show that the category of strongly regular Scott domains and embedding/projections is closed under the construction of dependent products. It follows that the category of strongly regular Scott domains with continuous maps as morphisms is Cartesian closed. Thus, strongly regular Scott domains too are well suited for defining the semantics of programming languages. Again, recursive domain equations have a solution and strongly regular models for typed and untyped lambda calculi can be constructed.

The paper is organized as follows. In Section 2 regular domains are introduced. An example is given showing that the category of regular Scott domains with continuous maps is not Cartesian closed. In the next two sections, two Cartesian closed subcategories are exhibited.

First, in Section 3, regular dI-domains are considered and then, in Section 4, strongly regular Scott domains. In both cases it is proved that the categories of these domains with embedding/projections as morphisms are closed under the construction of dependent products. Further closure properties of the regular and the strongly regular domains are studied in Section 5. Some final remarks appear in Section 6.

## 2. REGULAR DOMAINS

Let  $(D, \sqsubseteq)$  be a partial order with smallest element  $\perp$ . We write  $x \sqsubset y$ , if  $x \sqsubseteq y$  and  $x \neq y$ . For a subset  $S$  of  $D$ ,  $\downarrow S = \{x \in D \mid (\exists y \in S)x \sqsubseteq y\}$  and  $\uparrow S = \{x \in D \mid (\exists y \in S)y \sqsubseteq x\}$ , respectively, are the *lower* and the *upper set* generated by  $S$ . The subset  $S$  is called *consistent* if it has an upper bound.  $S$  is *directed*, if it is nonempty and every pair of elements in  $S$  has an upper bound in  $S$ .  $D$  is a *directed-complete* partial order (cpo) if every directed subset  $S$  of  $D$  has a least upper bound  $\bigsqcup S$  in  $D$ , and  $D$  is *bounded-complete* if every consistent subset has a least upper bound in  $D$ . In particular, if a pair  $\{x, y\}$  is consistent this is denoted by  $x \uparrow y$ . In a bounded-complete cpo any consistent pair  $\{x, y\}$  has a least upper bound which is written as  $x \sqcup y$ .

An element  $x$  of a cpo  $D$  is *compact* if for any directed subset  $S$  of  $D$  the relation  $x \sqsubseteq \bigsqcup S$  always implies the existence of an element  $u \in S$  with  $x \sqsubseteq u$ . We write  $\mathcal{K}(D)$  for the set of compact elements of  $D$ . If for every  $y \in D$  the set  $\downarrow\{y\} \cap \mathcal{K}(D)$  is directed and  $y = \bigsqcup \downarrow\{y\} \cap \mathcal{K}(D)$ , the cpo  $D$  is said to be *algebraic* and  $\omega$ -*algebraic*, if, in addition,  $\mathcal{K}(D)$  is countable. Note that instead of algebraic cpo we also say *domain*. Standard references for domain theory and its applications are [8, 7, 1, 14, 2].

Let us first consider some examples. For a set  $A$  such that  $\perp \notin A$  set  $A_\perp = A \cup \{\perp\}$  and order it by the flat ordering  $x \sqsubseteq y$  if  $x = \perp$  or  $x = y$ . Obviously, all elements of  $A_\perp$  are compact. Thus,  $A_\perp$  is a domain.

A further example of a domain is the set  $\mathcal{P}(\omega)$  of sets of natural numbers, ordered by set inclusion. Here, the finite sets are exactly the compact elements.

The cpo of *lazy natural numbers* is the set

$$\omega_L = \{s^n(\perp) \mid n \in \omega\} \cup \{s^n(0) \mid n \in \omega\} \cup \{s^\omega(\perp)\}$$

with  $s^0(\perp) = \perp$  and  $s^0(0) = 0$ , ordered by

$$x \sqsubseteq y \Leftrightarrow x = \perp \vee x = y \vee [x = s^m(\perp) \wedge (\exists z)y = s^m(z)].$$

Note that the last case includes  $y = s^\omega(\perp)$ , setting  $s^\omega(\perp) = s^m(s^\omega(\perp))$ . In this case all elements except  $s^\omega(\perp)$  are compact and  $s^\omega(\perp)$  is the least upper bound of all  $s^m(\perp)$  with  $m \in \omega$ .

Other examples are the *Cantor domain*, that is the set of all finite and infinite sequences of 0's and 1's, ordered by the prefix ordering, and the *Baire domain*, that is the set of all finite and infinite sequences of natural numbers, also ordered by the prefix ordering. In both cases the finite sequences are the compact elements.

The *product*  $D \times E$  of two cpo's  $D$  and  $E$  is the Cartesian product of the underlying sets ordered coordinatewise. Obviously,  $\mathcal{K}(D \times E) = \mathcal{K}(D) \times \mathcal{K}(E)$ .

As is well known, on each cpo there is a canonical topology: the *Scott topology*. A subset  $X$  is open, if it is upwards closed with respect to  $\sqsubseteq$  and intersects each directed subset of  $D$  of which it contains the least upper bound. In case that  $D$  is algebraic, this topology is generated by the sets  $\uparrow\{z\}$  with  $z \in \mathcal{K}(D)$ .

Let us now introduce the class of domains we want to study in this paper.

**Definition 2.1.** A domain  $D$  is regular if for all  $u, v \in \mathcal{K}(D)$

$$u \not\sqsubseteq v \Rightarrow (\exists z \in \mathcal{K}(D)) v \sqsubseteq z \wedge u \not\uparrow z.$$

Note that all domains mentioned in the examples above are regular, except in the case of domains  $A_\perp$  where  $A$  is a singleton. Moreover, the product of two regular domains is regular again. If  $D$  is bounded-complete, the above condition can be simplified.

**Lemma 2.2.** Let  $D$  be a bounded-complete domain. Then  $D$  is regular if and only if for all  $u, v \in \mathcal{K}(D)$

$$v \sqsubseteq u \Rightarrow (\exists z \in \mathcal{K}(D)) v \sqsubseteq z \wedge u \not\uparrow z.$$

*Proof.* The “only if” part is obvious. For the “if” part only the case that  $u$  and  $v$  are not comparable with respect to the partial order has to be considered. If  $u \not\uparrow v$  take  $z = v$ . In the opposite case the least upper bound  $x$  of  $u$  and  $v$  exists, which is compact as well. Thus, there is some  $z \in \mathcal{K}(D)$  such that  $v \sqsubseteq z$  and  $x \not\uparrow z$ . It follows that also  $u \not\uparrow z$ .  $\square$

This shows for two elements  $u$  and  $v$  of a regular bounded-complete domain containing finite information that if  $u$  contains more information than  $v$ , then there is a witness for this, that is, an element containing at least as much information as  $v$ , but the information contained in which is inconsistent with the information contained in  $u$ .

As we shall show next, the regularity of a domain, which was defined via the partial order, can also be expressed in topological terms. For a subset  $X$  of  $D$  let  $\text{int}(X)$ ,  $\text{cl}(X)$  and  $\text{ext}(X)$ , respectively, be the interior, the closure and the exterior of  $X$  with respect to the Scott topology. Then  $X$  is *regular open* if  $X = \text{int}(\text{cl}(X))$ . Moreover,  $D$  is *semi-regular* if all basic open sets  $\uparrow\{z\}$  with  $z \in \mathcal{K}(D)$  are regular open. Note that for  $z \in \mathcal{K}(D)$ ,  $\text{ext}(\uparrow\{z\}) = \bigcup \{\uparrow\{z'\} \mid z' \in \mathcal{K}(D) \wedge z' \not\uparrow z\}$ .

**Lemma 2.3.** Let  $D$  be a domain with compact element  $u$ . Then the following two statements are equivalent:

- (1)  $(\forall v \in \mathcal{K}(D)) [u \not\sqsubseteq v \Rightarrow (\exists z \in \mathcal{K}(D)) v \sqsubseteq z \wedge u \not\uparrow z]$ .
- (2) The set  $\uparrow\{u\}$  is regular open.

*Proof.* In order to see that (1) implies (2), assume that there is some  $v \in \text{int}(\text{cl}(\uparrow\{u\})) \setminus \uparrow\{u\}$ . Then  $u \not\sqsubseteq v$ . Because of (1) there exists some  $z \in \mathcal{K}(D)$  with  $v \sqsubseteq z$  such that  $\{u, z\}$  is not consistent. It follows that  $z \in \text{int}(\text{cl}(\uparrow\{u\}))$ . Hence  $z \in \text{cl}(\uparrow\{u\})$ , which implies that  $\uparrow\{z\}$  intersects  $\uparrow\{u\}$ . Thus,  $u$  and  $z$  have a common upper bound, contradicting what has been said about  $u$  and  $z$  before.

For the proof that (1) is a consequence of (2) let  $v \in \mathcal{K}(D)$  such that  $u \not\sqsubseteq v$ . Since  $\uparrow\{u\}$  is regular open, it follows that  $v \in \text{ext}(\uparrow\{u\})$ , which means that  $\{u, v\}$  cannot be consistent.  $\square$

**Corollary 2.4.** A domain is regular if and only if its Scott topology is semi-regular.

Maps between cpo's are not only required to preserve information but also computations: the value under a map of the result of a computation should not contain more information than what is obtained from the approximating objects.

**Definition 2.5.** Let  $D$  and  $E$  be cpo's. A map  $f: D \rightarrow E$  is said to be Scott continuous if it is monotone and for any directed subset  $S$  of  $D$ ,

$$f(\bigsqcup S) = \bigsqcup f(S).$$

It is well known that Scott continuity coincides with topological continuity. Therefore, in what follows we use the shorter term *continuous* instead of *Scott continuous*.

Denote the collection of all continuous maps from  $D$  to  $E$  by  $[D \rightarrow E]$ . Endowed with the *pointwise order*, that is,  $f \sqsubseteq_p g$  if  $f(x) \sqsubseteq g(x)$  for all  $x \in D$ , it is a cpo again. Since  $[D \rightarrow E]$  need not be algebraic, even if  $D$  and  $E$  are, one considers subclasses of domains which have this property, when using domains in programming language semantics, *e.g.* Scott domains.

**Definition 2.6.** A Scott domain is a bounded-complete  $\omega$ -algebraic cpo.

As is widely known, the category **SD** of Scott domains and continuous maps is Cartesian closed: the one-point domain  $\{\perp\}$  is the terminal object, the domain product is the categorical product and the space of continuous maps between two Scott domains is the categorical exponent.

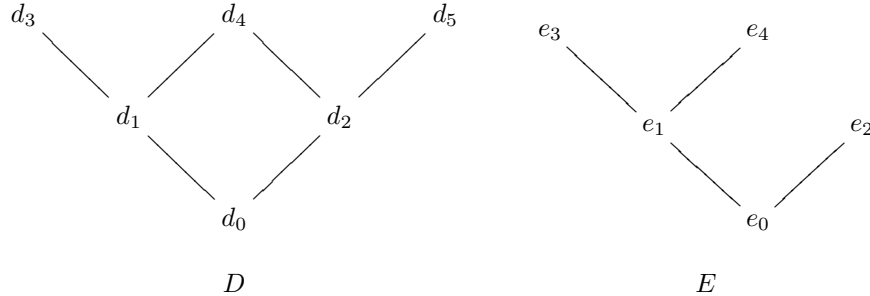
For two Scott domains  $D$  and  $E$  and elements  $d \in \mathcal{K}(D)$  and  $e \in \mathcal{K}(E)$  define the *step function*  $(d \searrow e): D \rightarrow E$  by

$$(d \searrow e)(x) = \begin{cases} e & \text{if } d \sqsubseteq x, \\ \perp & \text{otherwise.} \end{cases}$$

Then the compact elements of  $[D \rightarrow E]$  are exactly the maps of the form  $\bigsqcup \{(d_i \searrow e_i) \mid i \leq n\}$ , where  $d_0, \dots, d_n \in \mathcal{K}(D)$  and  $e_0, \dots, e_n \in \mathcal{K}(E)$  so that whenever  $I \subseteq \{0, \dots, n\}$  is such that  $\{d_i \mid i \in I\}$  is consistent, then so is  $\{e_i \mid i \in I\}$ .

The next example shows that in general the function space  $[D \rightarrow E]$  of two regular Scott domains  $D$  and  $E$  need not be regular again.

**Example 2.7.** Consider the Scott domains  $D$  and  $E$  defined by the following diagrams:



and define  $f_1, f_2 \in [D \rightarrow E]$  by  $f_1 = (d_2 \searrow e_1)$  and  $f_2 = (d_1 \searrow e_1) \sqcup (d_5 \searrow e_1)$ . Then both maps are compact. Obviously,  $f_1$  and  $f_2$  are not comparable with respect to the pointwise order, but have a common upper bound, that is,  $f_2 \in \text{cl}(\uparrow\{f_1\})$ . We want to show now that  $\uparrow\{f_2\}$  is included in  $\text{cl}(\uparrow\{f_1\})$ , which implies that  $\uparrow\{f_1\}$  is not regular open. Assume to this end that there is some  $f \in [D \rightarrow E]$  with  $f_2 \sqsubseteq_p f$  and  $f_1 \not\uparrow f$ . Then  $e_1 \sqsubseteq f(y)$ , if  $d_1 \sqsubseteq y$  or  $y = d_5$ . Moreover, there is some  $z \in D$  such that  $f_1(z)$  and  $f(z)$  have no common upper bound. Then  $d_2 \sqsubseteq z$ , since for  $z \in D$  with  $d_2 \not\sqsubseteq z$  we have that  $f_1(z) = e_0 \sqsubseteq f(z)$ . Moreover  $z \notin \{d_4, d_5\}$ . Thus  $z = d_2$ , which implies that  $\{z, d_1\}$  and hence also  $\{f(z), f(d_1)\}$  are consistent. It follows that  $f(z) \neq e_2$ , from which we obtain that  $f_1(z)$  and  $f(z)$  have a common upper bound, in contradiction to our assumption. Hence  $\uparrow\{f_2\}$  is a subset of  $\text{cl}(\uparrow\{f_1\})$ .

In the following sections we introduce two subclasses of the regular Scott domains which are closed under the construction of function spaces. As we shall see moreover, they are also closed under several other constructions used in denotational semantics. To this end we need the following definitions.

**Definition 2.8.** *An embedding/projection  $(f, g)$  from a cpo  $D$  to a cpo  $E$  is a pair of continuous maps  $f: D \rightarrow E$  and  $g: E \rightarrow D$  such that  $g \circ f = \text{id}_D$ , the identity map on  $D$ , and  $f \circ g \sqsubseteq_p \text{id}_E$ . The map  $f$  is called embedding and  $g$  projection.*

Normally, we write an embedding/projection as  $(f^L, f^R)$ . Note that the map  $f^R$  is uniquely determined by  $f^L$ , and vice versa [13]. Embeddings are one-to-one and preserve compactness as well as least upper bounds of finite consistent sets of domain elements [11]. Moreover, both the embedding and the projection are strict, that is, they map least elements onto least elements.

Suppose  $(f^L, f^R)$  is an embedding/projection from  $C$  to  $D$  and  $(g^L, g^R)$  is an embedding/projection from  $D$  to  $E$ . Then the composition of  $(f^L, f^R)$  and  $(g^L, g^R)$  is defined by

$$(g^L, g^R) \circ (f^L, f^R) = (g^L \circ f^L, f^R \circ g^R).$$

Let  $\mathbf{SD}^{\text{ep}}$  denote the category of Scott domains and embedding/projections.

Any partially ordered set may be viewed as a category. For a cpo  $D$ , this is done by letting the objects of the category be the elements of  $D$  and letting the morphism set between objects  $x$  and  $y$  be the one-point set  $\{\iota_{x,y}\}$  precisely when  $x \sqsubseteq y$  and the empty set otherwise.

Let  $F: D \rightarrow \mathbf{SD}^{\text{ep}}$  be a functor. Recall that this means that  $F(x)$  is a domain for each  $x \in D$ , and if  $x \sqsubseteq y$  then  $F(\iota_{x,y})$  is an embedding/projection from  $F(x)$  to  $F(y)$  such that  $F(\iota_{x,x}) = \text{id}_{F(x)}$ , and if  $x \sqsubseteq y$  and  $y \sqsubseteq z$  then  $F(\iota_{x,z}) = F(\iota_{y,z}) \circ F(\iota_{x,y})$ . When  $x \sqsubseteq y$  we shall use the notation  $F_{x,y}$  for  $F(\iota_{x,y})$ .

**Definition 2.9.** *Let  $D$  be a cpo. A functor  $F: D \rightarrow \mathbf{SD}^{\text{ep}}$  is continuous if for any directed subset  $S$  of  $D$ ,  $F(\bigsqcup S)$  is a colimit of the diagram*

$$(\{F(x) \mid x \in S\}, \{F_{x,y} \mid x, y \in S \wedge x \sqsubseteq y\}).$$

A *section* of  $F$  is a map  $f: D \rightarrow \bigcup \{F(x) \mid x \in D\}$  with  $f(x) \in F(x)$ , for all  $x \in D$ .

**Definition 2.10.** Let  $D$  be cpo and  $F: D \rightarrow \mathbf{SD}^{\text{ep}}$  be a continuous functor. An *section*  $f$  of  $F$  is said to be

- (1) monotone if  $F_{x,y}^L(f(x)) \sqsubseteq f(y)$ , for all  $x, y \in D$  with  $x \sqsubseteq y$ .
- (2) continuous if  $f$  is monotone and for every directed subset  $S$  of  $D$ ,

$$f(\bigsqcup S) = \bigsqcup \{F_{x, \bigsqcup S}^L(f(x)) \mid x \in S\}.$$

### 3. REGULAR dI-DOMAINS

In this section we restrict our attention to the subclass of regular dI-domains.

**Definition 3.1.** A *Scott domain*  $D$  is a dI-domain if it satisfies the two axioms  $d$  and  $I$ :

- *Axiom d:* For all  $x, y, z \in D$ , if  $y \uparrow z$  then

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z).$$

- *Axiom I:* For all  $u \in \mathcal{K}(D)$ ,  $\downarrow\{u\}$  is finite.

Note that in a bounded-complete domain any two elements  $x$  and  $y$  have a greatest lower bound  $x \sqcap y$ . Moreover, observe that by Axiom I all elements of a dI-domain below a compact element are also compact.

The elements of a dI-domain can be generated by a special kind of compact elements.

**Definition 3.2.** Let  $D$  be a bounded-complete domain. An element  $u \in D$  is a *complete prime* if for any consistent subset  $S$  of  $D$  the relation  $u \sqsubseteq \bigsqcup S$  always implies the existence of an element  $x \in S$  with  $u \sqsubseteq x$ .

The complete primes in a dI-domain  $D$  turn out to be those compact elements that have a unique element below them in the domain order [15]. As follows from a result of Winskel, every element in  $D$  is the least upper bound of all complete primes below it.

In the context of dI-domains it is usual to work with embedding/projections that satisfy an additional requirement.

**Definition 3.3.** An *embedding/projection*  $(f, g)$  from a dI-domain  $D$  to a dI-domain  $E$  is *rigid* if for all  $x \in D$  and  $y \in E$  with  $y \sqsubseteq f(x)$ ,  $f \circ g(y) = y$ . The map  $f$  is called *rigid embedding* and  $g$  *rigid projection*.

Note that rigid embeddings preserve greatest lower bounds of finite sets of domain elements. Moreover, they map complete primes onto complete primes. As a consequence of this, rigid projections preserve the least upper bounds of finite consistent sets of domain elements. We denote the category of dI-domains and rigid embedding/projections by  $\mathbf{DI}^{\text{ep}}$ .

For the following definition observe that for any bounded-complete domain  $D$  and elements  $x, y, z \in D$  such that  $x, y \sqsubseteq z$ ,  $(x \sqcap y, \iota_{x \sqcap y, x}, \iota_{x \sqcap y, y})$  is a pullback of  $(\iota_{x, z}, \iota_{y, z})$  in  $D$  viewed as a category.



**Definition 3.4.** Let  $D$  be a bounded-complete domain,  $F: D \rightarrow \mathbf{DI}^{\text{rep}}$  a functor and  $f$  a section of  $F$ .

- (1) The functor  $F$  is stable if it is continuous and for all  $x, y, z \in D$  with  $x, y \sqsubseteq z$ ,  $(F(x \sqcap y), F_{x \sqcap y, x}, F_{x \sqcap y, y})$  is a pullback of  $(F_{x, z}, F_{y, z})$ .
- (2) Let  $F$  be stable. The section  $f$  of  $F$  is stable if it is continuous and for all  $x, y \in D$  with  $x \uparrow y$ ,

$$f(x \sqcap y) = F_{x \sqcap y, x}^R(f(x)) \sqcap F_{x \sqcap y, y}^R(f(y)).$$

For the remainder of this section we assume that  $D$  is a dI-domain and  $F: D \rightarrow \mathbf{DI}^{\text{rep}}$  a stable functor.

**Proposition 3.5.** Let  $f$  be a stable section of  $F$ . Then for any  $x \in D$  and  $v \in \mathcal{K}(F(x))$  such that  $v \sqsubseteq f(x)$ , there are  $\bar{u} \in \mathcal{K}(D)$  and  $\bar{v} \in \mathcal{K}(F(\bar{u}))$  with the following properties:

- (1)  $\bar{u} \sqsubseteq x$ .
- (2)  $v = F_{\bar{u}, x}^L(\bar{v})$ .
- (3)  $\bar{v} \sqsubseteq f(\bar{u})$ .
- (4)  $(\forall y, z \in D)[\bar{u}, y \sqsubseteq z \wedge F_{\bar{u}, z}^L(\bar{v}) \sqsubseteq F_{y, z}^L(f(y)) \Rightarrow \bar{u} \sqsubseteq y]$ .

*Proof.* Let  $x \in D$  and  $v \in \mathcal{K}(F(x))$  with  $v \sqsubseteq f(x)$ . Since

$$f(x) = \bigsqcup \{ F_{u, x}^L(f(u)) \mid u \in \mathcal{K}(D) \wedge u \sqsubseteq x \},$$

there is some  $\hat{u} \in \mathcal{K}(D)$  with  $v \sqsubseteq F_{\hat{u}, x}^L(f(\hat{u}))$ . Because  $\downarrow\{\hat{u}\}$  is finite, there is a minimal such  $\hat{u}$ , say  $\bar{u}$ . Thus  $v \sqsubseteq F_{\bar{u}, x}^L(f(\bar{u}))$ . Observe that  $F_{\bar{u}, x}$  is rigid. Therefore  $v = F_{\bar{u}, x}^L \circ F_{\bar{u}, x}^R(v)$ . Set  $\bar{v} = F_{\bar{u}, x}^R(v)$ . Then  $\bar{v} \in F(\bar{u})$  and  $\bar{v} \sqsubseteq f(\bar{u})$ . Moreover,  $\bar{v}$  is compact.

Now, let  $y, z \in D$  such that  $\bar{u}, y \sqsubseteq z$  and  $F_{\bar{u}, z}^L(\bar{v}) \sqsubseteq F_{y, z}^L(f(y))$ . Set  $v_y = F_{y, z}^R \circ F_{\bar{u}, z}^L(\bar{v})$ . Then  $v_y \in F(y)$ . Moreover, it follows as above that  $F_{\bar{u}, z}^L(\bar{v}) = F_{y, z}^L(v_y)$ . Since  $F$  is stable, we have that  $(F(\bar{u} \sqcap y), F_{\bar{u} \sqcap y, \bar{u}}, F_{\bar{u} \sqcap y, y})$  is a pullback of  $(F_{\bar{u}, z}, F_{y, z})$ . Therefore, there is some  $\hat{v} \in F(\bar{u} \sqcap y)$  with  $F_{\bar{u} \sqcap y, \bar{u}}^L(\hat{v}) = \bar{v}$  and  $F_{\bar{u} \sqcap y, y}^L(\hat{v}) = v_y$ . It follows that  $\hat{v} \sqsubseteq F_{\bar{u} \sqcap y, \bar{u}}^R(f(\bar{u})) \sqcap F_{\bar{u} \sqcap y, y}^R(f(y))$ . Because  $f$  is stable as well, we obtain that  $\hat{v} \sqsubseteq f(\bar{u} \sqcap y)$  and hence that  $v \sqsubseteq F_{\bar{u} \sqcap y, x}^L(f(\bar{u} \sqcap y))$ . Then  $\bar{u} = \bar{u} \sqcap y$ , by the minimality of  $\bar{u}$ , which implies that  $\bar{u} \sqsubseteq y$ .  $\square$

This result justifies the following definition.

**Definition 3.6.** Let  $f$  be a stable section of  $F$ . The set

$$\begin{aligned} \text{Tr}(f) = \{ (u, v) \mid u \in \mathcal{K}(D) \wedge v \in \mathcal{K}(F(u)) \wedge v \sqsubseteq f(u) \\ \wedge (\forall y, z \in D)[u, y \sqsubseteq z \wedge F_{u, z}^L(v) \sqsubseteq F_{y, z}^L(f(y)) \Rightarrow u \sqsubseteq y] \} \end{aligned}$$

is called trace of  $f$ .

**Lemma 3.7.** Let  $f$  be a stable section of  $F$ . Then the trace of  $f$  has the following properties:

- (1) If  $(u_0, v_0), \dots, (u_n, v_n) \in \text{Tr}(f)$  such that  $\{u_0, \dots, u_n\}$  has least upper bound  $\bar{u}$ , then  $\{F_{u_0, \bar{u}}^L(v_0), \dots, F_{u_n, \bar{u}}^L(v_n)\}$  has a least upper bound, say  $\bar{v}$ , and  $(\bar{u}, \bar{v}) \in \text{Tr}(f)$ .

- (2) If  $(u, v) \in \text{Tr}(f)$ ,  $x, z \in D$  and  $v' \in F(x)$  such that  $u, x \sqsubseteq z$  and  $v' \sqsubseteq F_{x,z}^R \circ F_{u,z}^L(v)$ , then there exists  $u' \sqsubseteq u$  so that  $F_{x,z}^L(v') \in F_{u',z}^L(F(u'))$  and  $(u', F_{u',z}^R \circ F_{x,z}^L(v')) \in \text{Tr}(f)$ .
- (3) If  $(u, v), (u', v') \in \text{Tr}(f)$  and for some  $z \in D$  with  $u, u' \sqsubseteq z$ ,  $F_{u,z}^L(v) = F_{u',z}^L(v')$ , then  $u = u'$  and hence  $v = v'$ .

*Proof.* (1) If  $\{u_0, \dots, u_n\}$  has least upper bound  $\bar{u}$ , then we have for  $i \leq n$  that  $F_{u_i, \bar{u}}^L(v_i) \sqsubseteq f(\bar{u})$ . Hence,  $\{F_{u_0, \bar{u}}^L(v_0), \dots, F_{u_n, \bar{u}}^L(v_n)\}$  has a least upper bound  $\bar{v}$ . Both,  $\bar{u}$  and  $\bar{v}$  are compact. It remains to show that  $(\bar{u}, \bar{v}) \in \text{Tr}(f)$ .

We have already observed that  $\bar{v} \sqsubseteq f(\bar{u})$ . Now, let  $y, z \in D$  with  $\bar{u}, y \sqsubseteq z$  and  $F_{\bar{u}, z}^L(\bar{v}) \sqsubseteq F_{y,z}^L(f(y))$ . Then we have for  $i \leq n$  that  $u_i \sqsubseteq z$  and  $F_{u_i, z}^L(v_i) \sqsubseteq F_{y,z}^L(f(y))$ . Since  $(u_i, v_i) \in \text{Tr}(f)$  it follows that  $u_i \sqsubseteq y$ , for  $i \leq n$ , which implies that  $\bar{u} \sqsubseteq y$ .

(2) Let  $(u, v) \in \text{Tr}(f)$  and  $x, z \in D$  with  $u, x \sqsubseteq z$ . Moreover, let  $v' \in F(x)$  such that  $v' \sqsubseteq F_{x,z}^R \circ F_{u,z}^L(v)$ . Then  $F_{x,z}^L(v') \sqsubseteq F_{u,z}^L(v)$ . Since  $v$  is compact, we have that also  $F_{u,z}^L(v)$  is compact. Hence,  $F_{x,z}^L(v')$  is compact as well. Note that

$$F_{x,z}^L(v') \sqsubseteq F_{u,z}^L(v) \sqsubseteq F_{u,z}^L(f(u)) \sqsubseteq f(z).$$

Therefore, by Proposition 3.5, there is some  $(u', \bar{v}) \in \text{Tr}(f)$  such that  $u' \sqsubseteq z$  and  $F_{x,z}^L(v') = F_{u',z}^L(\bar{v})$ . Because  $u, u' \sqsubseteq z$  and  $F_{u',z}^L(\bar{v}) = F_{x,z}^L(v') \sqsubseteq F_{u,z}^L(f(u))$ , it follows that  $u' \sqsubseteq u$ .

(3) If  $(u, v), (u', v') \in \text{Tr}(f)$  and for some  $z \in D$  with  $u, u' \sqsubseteq z$ ,  $F_{u,z}^L(v) = F_{u',z}^L(v')$ , then  $v' \sqsubseteq f(u')$  and thus  $F_{u,z}^L(v) = F_{u',z}^L(v') \sqsubseteq F_{u',z}^L(f(u'))$ . Since  $(u, v) \in \text{Tr}(f)$ , it follows that  $u \sqsubseteq u'$ . In the same way we obtain that  $u' \sqsubseteq u$ . Therefore  $u = u'$  and hence  $F_{u,z}^L(v) = F_{u,z}^L(v')$ , which implies that  $v = v'$ , as  $F_{u,z}^L$  is one-to-one.  $\square$

The next lemma shows how a stable section can be recovered from its trace.

**Lemma 3.8.** (1) *A stable section  $f$  of  $F$  can be computed from its trace in the following way:*

$$f(x) = \bigsqcup \{ F_{u,x}^L(v) \mid u \sqsubseteq x \wedge (u, v) \in \text{Tr}(f) \}.$$

- (2) *Every set of pairs  $(u, v)$  with  $u \in \mathcal{K}(D)$  and  $v \in \mathcal{K}(F(u))$  satisfying Properties (1)-(3) of Lemma 3.7 is a trace of the stable section of  $F$  defined by the formula above.*

*Proof.* (1) Because of Property (1) of Lemma 3.7 the set of all  $F_{u,x}^L(v)$  with  $(u, v) \in \text{Tr}(f)$  and  $u \sqsubseteq x$  is directed and thus has a least upper bound. Obviously, the right-hand side of the formula is less than or equal to the left-hand side. If  $v \in \mathcal{K}(F(x))$  with  $v \sqsubseteq f(x)$ , then, by Proposition 3.5, there is some  $(u', v') \in \text{Tr}(f)$  with  $u' \sqsubseteq x$  and  $v = F_{u',x}^L(v')$ . It follows that  $v$  is less than or equal to the right-hand side of the formula. Since  $f(x)$  is the least upper bound of all such  $v$ , the same is true for  $f(x)$ .

(2) We have to show that if  $X$  satisfies Properties (1)-(3) of Lemma 3.7, then the map  $f$  defined by

$$f(x) = \bigsqcup \{ F_{u,x}^L(v) \mid u \sqsubseteq x \wedge (u, v) \in X \}.$$

is a stable section of  $F$  with  $\text{Tr}(f) = X$ .

The map  $f$  is well defined, since the set  $\{ F_{u,x}^L(v) \mid u \sqsubseteq x \wedge (u, v) \in X \}$  is directed by Property 3.7(1) and therefore has a least upper bound.

Obviously,  $f$  is monotone. To see that it is continuous, let  $S$  be a directed subset of  $D$  and let  $v \in \mathcal{K}(F(\bigsqcup S))$  with  $v \sqsubseteq f(\bigsqcup S)$ . It follows that there is some  $(\bar{u}, \bar{v}) \in X$  so that  $\bar{u} \sqsubseteq \bigsqcup S$  and  $v \sqsubseteq F_{\bar{u}, \bigsqcup S}^L(\bar{v})$ . Since  $\bar{u}$  is compact as well, there is some  $x \in S$  with  $\bar{u} \sqsubseteq x$ . Thus  $F_{\bar{u}, x}^L(\bar{v}) \sqsubseteq f(x)$ , which implies that  $v \sqsubseteq F_{x, \bigsqcup S}^L(f(x))$ . Since  $f(\bigsqcup S)$  is the least upper bound of all such  $v$ , we have that  $f(\bigsqcup S) \sqsubseteq \bigsqcup \{ F_{x, \bigsqcup S}^L(f(x)) \mid x \in S \}$ . The converse inequality is obvious, by monotonicity.

For the verification of stability it suffices to show for  $x, y, z \in D$  with  $x, y \sqsubseteq z$  that  $F_{x \sqcap y, x}^R(f(x)) \sqcap F_{x \sqcap y, y}^R(f(y)) \sqsubseteq f(x \sqcap y)$ . Let to this end  $(u, v), (u', v') \in X$  with  $u \sqsubseteq x$  and  $u' \sqsubseteq y$ . Since the binary operator  $\sqcap$  is continuous [7], we only have to verify that

$$F_{x \sqcap y, x}^R \circ F_{u,x}^L(v) \sqcap F_{x \sqcap y, y}^R \circ F_{u',y}^L(v') \sqsubseteq f(x \sqcap y).$$

Set  $\bar{v} = F_{x \sqcap y, x}^R \circ F_{u,x}^L(v) \sqcap F_{x \sqcap y, y}^R \circ F_{u',y}^L(v')$ . Then  $\bar{v} \sqsubseteq F_{x \sqcap y, x}^R \circ F_{u,x}^L(v), F_{x \sqcap y, y}^R \circ F_{u',y}^L(v')$ . Hence, by Property 3.7(2), there is some  $\bar{u} \sqsubseteq u$  such that

$$(3.1) \quad F_{x \sqcap y, x}^L(\bar{v}) \in F_{\bar{u}, x}^L(F(\bar{u}))$$

and  $(\bar{u}, F_{\bar{u}, x}^R \circ F_{x \sqcap y, x}^L(\bar{v})) \in X$ . Moreover, there is some  $\bar{u}' \sqsubseteq u'$  such that

$$(3.2) \quad F_{x \sqcap y, y}^L(\bar{v}) \in F_{\bar{u}', y}^L(F(\bar{u}'))$$

and  $(\bar{u}', F_{\bar{u}', y}^R \circ F_{x \sqcap y, y}^L(\bar{v})) \in X$ . Note that  $\bar{u} \sqsubseteq x \sqsubseteq z$  and  $\bar{u}' \sqsubseteq y \sqsubseteq z$ . In addition

$$\begin{aligned} F_{\bar{u}, z}^L \circ F_{\bar{u}, x}^R \circ F_{x \sqcap y, x}^L(\bar{v}) &= F_{x, z}^L \circ F_{\bar{u}, x}^L \circ F_{\bar{u}, x}^R \circ F_{x \sqcap y, x}^L(\bar{v}) \\ &= F_{x, z}^L \circ F_{x \sqcap y, x}^L(\bar{v}) && \text{(by (3.1))} \\ &= F_{x \sqcap y, z}^L(\bar{v}) \\ &= F_{\bar{u}', z}^L \circ F_{\bar{u}', y}^R \circ F_{x \sqcap y, y}^L(\bar{v}) && \text{(by (3.2))} \end{aligned}$$

By Property 3.7(3) we therefore have that  $\bar{u} = \bar{u}'$ . Thus  $\bar{u} \sqsubseteq x \sqcap y$ . It follows that

$$F_{\bar{u}, x}^R \circ F_{x \sqcap y, x}^L(\bar{v}) = F_{\bar{u}, x \sqcap y}^R \circ F_{x \sqcap y, x}^R \circ F_{x \sqcap y, x}^L(\bar{v}) = F_{\bar{u}, x \sqcap y}^R(\bar{v}).$$

Since  $(\bar{u}, F_{\bar{u}, x \sqcap y}^R(\bar{v})) \in X$  and  $\bar{v} \in F_{\bar{u}, x \sqcap y}^L(F(\bar{u}))$ , by (3.1), we obtain that  $\bar{v} = F_{\bar{u}, x \sqcap y}^L \circ F_{\bar{u}, x \sqcap y}^R(\bar{v}) \sqsubseteq f(x \sqcap y)$ .

Next, we show that  $\text{Tr}(f)$  contains  $X$ . If  $(u, v) \in X$  then  $v \sqsubseteq f(u)$ . Let  $y, z \in D$  such that  $u, y \sqsubseteq z$  and  $F_{u, z}^L(v) \sqsubseteq F_{y, z}^L(f(y))$ . Since  $F_{u, z}^L(v)$  is compact, it follows from the definition of  $f$  that there is some  $(\hat{u}, \hat{v}) \in X$  with  $\hat{u} \sqsubseteq y$  and

$F_{u,z}^L(v) \sqsubseteq F_{y,z}^L \circ F_{\hat{u},y}^L(\hat{v})$ . By Property 3.7(2) we now obtain that there exists a  $u' \sqsubseteq \hat{u}$  so that

$$(3.3) \quad F_{u,z}^L(v) \in F_{u',z}^L(F(u'))$$

and  $(u', F_{u',z}^R \circ F_{u,z}^L(v)) \in X$ . Thus, we have that  $(u, v), (u', F_{u',z}^R \circ F_{u,z}^L(v)) \in X$ ,  $u, u' \sqsubseteq z$  and, because of (3.3),  $F_{u',z}^L \circ F_{u',z}^R \circ F_{u,z}^L(v) = F_{u,z}^L(v)$ . Therefore  $u = u'$ , by Property 3.7(3). Since  $u' \sqsubseteq \hat{u} \sqsubseteq y$  it follows that  $(u, v) \in \text{Tr}(f)$ .

Finally, we have to verify that also  $X$  contains  $\text{Tr}(f)$ . Let  $(u, v) \in \text{Tr}(f)$ . Then  $v \sqsubseteq f(u)$ . As above it follows that there is some  $u' \sqsubseteq u$  so that  $(u', F_{u',u}^R(v)) \in X$  and  $v \in F_{u',u}^L(F(u'))$ . Then  $v \sqsubseteq F_{u',u}^L(f(u'))$ . As a consequence of the minimality condition in the definition of a trace we obtain that  $u \sqsubseteq u'$ . Thus  $u = u'$ , which means that  $(u, v) \in X$ .  $\square$

Define  $\Pi_D F$  to be the set of all stable sections of  $F$  and order it by the *stable ordering*, that is

$$f \sqsubseteq_s g \Leftrightarrow \text{Tr}(f) \subseteq \text{Tr}(g).$$

Then  $\Pi_D F$  with the stable ordering is the *dependent product of  $F$  over  $D$* .

Our next goal is to show that  $\Pi_D F$  is a dI-domain. To this end we need the following lemma.

**Lemma 3.9.** *Let  $f$  be a stable section of  $F$  and  $I \subseteq \text{Tr}(f)$ . Moreover, let*

$$I' = \{ (x, y) \in \text{Tr}(f) \mid (\exists (u, v) \in I) x \sqsubseteq u \wedge F_{x,u}^L(y) \sqsubseteq v \}$$

and

$$X = \{ (x, y) \in \text{Tr}(f) \mid (\exists (u_0, v_0), \dots, (u_n, v_n) \in I') x = \bigsqcup_{i \leq n} u_i \wedge y = \bigsqcup_{i \leq n} F_{u_i, x}^L(v_i) \}.$$

Then the following three statements hold:

- (1) *The set  $X$  has Properties (1)-(3) of Lemma 3.7.*
- (2) *The stable section  $g$  defined by  $X$  according to Lemma 3.8(2) is the smallest stable section  $h$  of  $F$  with  $h \sqsubseteq_s f$  and  $I \subseteq \text{Tr}(h)$ .*
- (3) *If the set  $I$  is finite, so is  $X$ .*

*Proof.* (1) Property 3.7(1) holds by construction and 3.7(3) is inherited from  $\text{Tr}(f)$ . It remains to show that also 3.7(2) holds. Let  $(u, v) \in X$  and for  $x, z \in D$  with  $u, x \sqsubseteq z$  let  $v' \in F(x)$  such that  $v' \sqsubseteq F_{x,z}^R \circ F_{u,z}^L(v)$ . Since  $X$  is contained in  $\text{Tr}(f)$  there is some  $u' \sqsubseteq u$  so that  $F_{x,z}^L(v') \in F_{u',z}^L(F(u'))$  and  $(u', F_{u',z}^R \circ F_{x,z}^L(v')) \in \text{Tr}(f)$ . We show that  $(u', F_{u',z}^R \circ F_{x,z}^L(v')) \in X$ .

Because  $(u, v) \in X$ , there are  $(u_0, v_0), \dots, (u_n, v_n) \in I'$  so that  $u = \bigsqcup_{i \leq n} u_i$  and  $v = \bigsqcup_{i \leq n} F_{u_i, u}^L(v_i)$ . Define  $u'_i = u_i \sqcap u'$ , for  $i \leq n$ . Then  $u' = \bigsqcup_{i \leq n} u'_i$ . As  $F_{x,z}^L(v') \in F_{u',z}^L(F(u'))$  there exists  $\hat{v} \in F(u')$  with  $F_{x,z}^L(v') = F_{u',z}^L(\hat{v})$ . Then  $\hat{v} = F_{u',z}^R \circ F_{x,z}^L(v')$ . For  $i \leq n$  set  $\tilde{v}_i = F_{u_i, u}^L(v_i) \sqcap F_{u', u}^L(\hat{v})$ . It follows that  $\tilde{v}_i \sqsubseteq F_{u_i, u}^L(v_i), F_{u', u}^L(\hat{v})$ . Therefore  $F_{u_i, u}^L \circ F_{u_i, u}^R(\tilde{v}_i) = \tilde{v}_i = F_{u', u}^L \circ F_{u', u}^R(\tilde{v}_i)$ . Since

$(F(u'_i), F_{u'_i, u'}, F_{u'_i, u_i})$  is a pullback of  $(F_{u', u}, F_{u_i, u})$ , we obtain that there is some  $v'_i \in F(u'_i)$  with  $F_{u'_i, u'}^L(v'_i) = F_{u', u}^R(\tilde{v}_i)$  and  $F_{u'_i, u_i}^L(v'_i) = F_{u_i, u}^R(\tilde{v}_i)$ . Then

$$F_{u'_i, u_i}^L(v'_i) = F_{u_i, u}^R(\tilde{v}_i) \sqsubseteq F_{u_i, u}^R \circ F_{u_i, u}^L(v_i) = v_i.$$

Thus  $(u'_i, v'_i) \in I'$ . Moreover,  $F_{u'_i, u'}^L(v'_i) = F_{u', u}^R(\tilde{v}_i) \sqsubseteq F_{u', u}^R \circ F_{u', u}^L(\hat{v}) = \hat{v}$  and

$$\begin{aligned} \bigsqcup_{i \leq n} F_{u'_i, u'}^L(v'_i) &= \bigsqcup_{i \leq n} F_{u', u}^R(\tilde{v}_i) \\ &= F_{u', u}^R(\bigsqcup_{i \leq n} \tilde{v}_i) \\ &= F_{u', u}^R(\bigsqcup_{i \leq n} F_{u_i, u}^L(v_i) \sqcap F_{u', u}^L(\hat{v})) \\ &= F_{u', u}^R(F_{u', u}^L(\hat{v}) \sqcap (\bigsqcup_{i \leq n} F_{u_i, u}^L(v_i))) \\ &= F_{u', u}^R(F_{u', u}^L(\hat{v}) \sqcap v) \\ &= F_{u', u}^R \circ F_{u', u}^L(\hat{v}) \\ &= \hat{v}. \end{aligned}$$

Here, we used that

$$\begin{aligned} F_{u', u}^L(\hat{v}) &= F_{u', u}^L \circ F_{u', u}^R \circ F_{u', u}^R \circ F_{u', u}^L(\hat{v}) \sqsubseteq F_{u', u}^R \circ F_{u', u}^L(\hat{v}) \sqsubseteq \\ &F_{u', u}^R \circ F_{u', u}^L(v) \sqsubseteq v. \end{aligned}$$

It follows that  $(u', \hat{v}) \in X$ .

(2) Properties 3.7(1)-(3) imply for any stable section  $h$  of  $F$  with  $I \subseteq \text{Tr}(h) \subseteq \text{Tr}(f)$  that  $X \subseteq \text{Tr}(h)$ .

(3) Since embeddings are one-to-one and Axiom I holds in  $D$  and all  $F(x)$  with  $x \in D$ , it follows that if  $I$  is finite, so is  $I'$ . Obviously  $\|X\| \leq 2^{\|I'\|}$ . Therefore, also  $X$  is finite in this case.  $\square$

**Theorem 3.10.** *Let  $D$  be a dI-domain and  $F: D \rightarrow \mathbf{DI}^{\text{rep}}$  a stable functor. Then  $\Pi_D F$  is a dI-domain. The compact elements in  $\Pi_D F$  are exactly the sections with finite trace.*

*Proof.* As is easily seen,  $\Pi_D F$  is a cpo. If  $S$  is a subset of  $\Pi_D F$  with upper bound  $f$ , then it follows with Lemma 3.9 that the section generated by  $\bigcup \{ \text{Tr}(h) \mid h \in S \}$  is the least upper bound of  $S$ . Thus,  $\Pi_D F$  is bounded-complete. Obviously, the sections of  $F$  with finite trace are the compact elements. Therefore,  $\Pi_D F$  is a Scott domain and Axiom I holds. With Lemmas 3.7 and 3.8 it is readily verified for two section  $f$  and  $g$  that  $\text{Tr}(f) \cap \text{Tr}(g)$  is the trace of  $f \sqcap g$ . By using the two lemmas once more we then obtain that Axiom d is satisfied as well.  $\square$

In the special case of a constant functor  $F$ , say  $F(x) = E$  for  $x \in D$ , one has the well known fact that the space  $[D \rightarrow_s E]$  of stable maps from  $D$  to  $E$  is a dI-domain again. Indeed, the category of dI-domains and stable maps is Cartesian closed [7].

Now, let  $\mathbf{RDI}^{\text{rep}}$  be the category of regular dI-domains and rigid embedding/projections. We will show that for a stable functor  $F: D \rightarrow \mathbf{RDI}^{\text{rep}}$  the dependent product domain  $\Pi_D F$  is also regular.

**Theorem 3.11.** *Let  $D$  be a dI-domain and  $F: D \rightarrow \mathbf{RDI}^{\text{rep}}$  a stable functor. Then  $\Pi_D F$  is a regular dI-domain.*

*Proof.* We only have to verify the condition in Lemma 2.2. Let  $f, g \in \mathcal{K}(\Pi_D F)$ . Since  $\text{Tr}(f)$  is finite, we have that  $f(\uparrow\{u\})$  is finite as well. Hence there is some  $x \in \uparrow\{u\}$  so that for all  $z \in D$  with  $x \sqsubseteq z$ ,  $F_{x,z}^L(f(x)) = f(z)$ . Obviously, we can assume that  $x$  is compact. By the formula in Lemma 3.8(1) we moreover obtain that there is some  $(\hat{u}, \hat{v}) \in \text{Tr}(f)$  with  $\hat{u} \sqsubseteq x$  and  $f(x) = F_{\hat{u},x}^L(\hat{v})$ . Because

$$F_{\hat{u},x}^L(\hat{v}) \sqsubseteq F_{\hat{u},x}^L(f(\hat{u})) \sqsubseteq f(x) = F_{\hat{u},x}^L(\hat{v}),$$

we also have that  $f(\hat{u}) = \hat{v}$ . Let  $y = f(x)$ .

Now, suppose that  $f \sqsubset g$ . Then  $\text{Tr}(f) \not\subseteq \text{Tr}(g)$ . Thus, there is some  $(u, v) \in \text{Tr}(g) \setminus \text{Tr}(f)$ .

**Claim 3.12.**  $F_{u,x}^L(v) \not\subseteq y$ .

*Proof.* Assume that  $v \sqsubseteq F_{u,x}^R \circ F_{\hat{u},x}^L(\hat{v})$ . By Property 3.7(2) there is then some  $u' \sqsubseteq \hat{u}$  so that  $F_{u,x}^L(v) \in F_{u',x}^L(F(u'))$  and  $(u', F_{u',x}^R \circ F_{u,x}^L(v)) \in \text{Tr}(f)$ . It follows that  $(u', F_{u',x}^R \circ F_{u,x}^L(v)) \in \text{Tr}(g)$ . Since  $(u, v) \in \text{Tr}(g)$  as well and  $F_{u',x}^L \circ F_{u',x}^R \circ F_{u,x}^L(v) = F_{u,x}^L(v)$ , we obtain with Property 3.7(3) that  $u = u'$  and  $F_{u',x}^R \circ F_{u',x}^L(v) = v$ . Hence  $(u, v) \in \text{Tr}(f)$ , a contradiction.

This shows that  $v \not\subseteq F_{u,x}^R \circ F_{\hat{u},x}^L(\hat{v})$ , which implies that  $F_{u,x}^L(v) \not\subseteq F_{\hat{u},x}^L(\hat{v}) = y$ .  $\square$

**Claim 3.13.**  $(u', v') \in \text{Tr}(f) \wedge u' \uparrow x \Rightarrow u' \sqsubseteq x \wedge F_{u',x}^L(v') \sqsubseteq y$ .

*Proof.* With  $u' \uparrow x$  we also have that  $u' \uparrow \hat{u}$ . Then  $(u' \sqcup \hat{u}, F_{u',u' \sqcup \hat{u}}^L(v')) \sqcup F_{\hat{u},u' \sqcup \hat{u}}^L(\hat{v}) \in \text{Tr}(f)$ , by 3.7(1). Hence

$$\begin{aligned} F_{\hat{u},u' \sqcup x}^L(\hat{v}) &\sqsubseteq F_{u' \sqcup \hat{u},u' \sqcup x}^L(F_{u',u' \sqcup \hat{u}}^L(v') \sqcup F_{\hat{u},u' \sqcup \hat{u}}^L(\hat{v})) \\ &\sqsubseteq F_{u' \sqcup \hat{u},u' \sqcup x}^L(f(u' \sqcup \hat{u})) \\ (3.4) \quad &\sqsubseteq f(u' \sqcup x) \\ &= F_{\hat{u},u' \sqcup x}^L(\hat{v}), \end{aligned}$$

which implies that  $F_{u',u' \sqcup \hat{u}}^L(v') \sqcup F_{\hat{u},u' \sqcup \hat{u}}^L(\hat{v}) = F_{\hat{u},u' \sqcup \hat{u}}^L(\hat{v})$ . With 3.7(3) we thus have that  $\hat{u} = u' \sqcup \hat{u}$ , that is,  $u' \sqsubseteq \hat{u} \sqsubseteq x$ . Therefore, we obtain from (3.4) that  $F_{u',x}^L(v') \sqsubseteq F_{\hat{u},x}^L(\hat{v}) = y$ .  $\square$

Since  $F(x)$  is regular and  $F_{u,x}^L(v) \not\subseteq y$  there is some  $\bar{v} \in \mathcal{K}(F(x))$  so that  $y \sqsubseteq \bar{v}$  and  $F_{u,x}^L(v) \nmid \bar{v}$ . Define

$$\begin{aligned} X &= \text{Tr}(f) \cup \{(x, \tilde{v}) \mid \tilde{v} \in \mathcal{K}(F(x)) \wedge \tilde{v} \sqsubseteq \bar{v} \\ &\quad \wedge (\nexists (\check{u}, \check{v}) \in \text{Tr}(f)) \check{u} \sqsubseteq x \wedge F_{\check{u},x}^L(\check{v}) = \tilde{v}\}. \end{aligned}$$

If we have checked that  $X$  has Properties 3.7(1)-(3), we know that  $X$  is the trace of a section  $\bar{f} \in \mathcal{K}(\Pi_D F)$  with  $f \sqsubseteq_s \bar{f}$ . Suppose that  $g, \bar{f} \sqsubseteq_s h$ , for some  $h \in \Pi_D F$ . Then  $F_{u,x}^L(v) \sqsubseteq g(x) \sqsubseteq h(x)$  and  $\bar{v} = \bar{f}(x) \sqsubseteq h(x)$ , which is impossible as  $F_{u,x}^L(v) \nmid \bar{v}$ . Thus  $g \nmid \bar{f}$ .

It remains to check that  $X$  has the properties of Lemma 3.7.

(1) Obviously, it is sufficient to consider the case that  $n = 1$ . Let  $(u_0, v_0), (u_1, v_1) \in X$  so that  $\{u_0, u_1\}$  is consistent. If  $(u_0, v_0), (u_1, v_1) \in \text{Tr}(f)$  we are done. So, without restriction, let us assume that  $(u_0, v_0) \notin \text{Tr}(f)$ . Then  $u_0 = x$  and  $v_0 \sqsubseteq \bar{v}$ . If  $(u_1, v_1) \notin \text{Tr}(f)$  too, then  $u_1 = x$  and  $v_1 \sqsubseteq \bar{v}$ . In the opposite case we obtain with Claim 3.13 that  $u_1 \sqsubseteq x$  and  $F_{u_1,x}^L(v_1) \sqsubseteq \bar{v}$ . Therefore, in both cases  $v_0 \sqcup F_{u_1,x}^L(v_1)$  exists below  $\bar{v}$ .

In order to see that  $(x, v_0 \sqcup F_{u_1,x}^L(v_1)) \in X \setminus \text{Tr}(f)$  assume that there is some  $(\check{u}, \check{v}) \in \text{Tr}(f)$  such that  $\check{u} \sqsubseteq x$  and  $F_{\check{u},x}^L(\check{v}) = v_0 \sqcup F_{u_1,x}^L(v_1)$ . Then  $v_0 \sqsubseteq F_{\check{u},x}^L(\check{v})$ . With Property 3.7(2) it follows that there is some  $u' \sqsubseteq \check{u}$  so that  $v_0 \in F_{u',x}^L(F(u'))$  and  $(u', F_{u',x}^R(v_0)) \in \text{Tr}(f)$ . Hence, we have that  $u' \sqsubseteq x$  and  $F_{u',x}^L \circ F_{u',x}^R(v_0) = v_0$ , contradicting the fact that  $(u_0, v_0) \in X \setminus \text{Tr}(f)$ .

(2) We only have to consider the case that  $(x, \tilde{v}) \in X \setminus \text{Tr}(f)$ ,  $z, z' \in D$  and  $v' \in F(z)$  such that  $x, z \sqsubseteq z'$  and  $v' \sqsubseteq F_{z,z'}^R \circ F_{x,z}^L(\tilde{v})$ . Then  $F_{x,z'}^R \circ F_{z,z'}^L(v') \sqsubseteq \tilde{v} \sqsubseteq \bar{v}$ .

If there is some  $(\check{u}, \check{v}) \in \text{Tr}(f)$  with  $\check{u} \sqsubseteq x$  and  $F_{\check{u},x}^L(\check{v}) = F_{x,z'}^R \circ F_{z,z'}^L(v')$ , it follows that  $\check{u}, z \sqsubseteq z'$ . Moreover  $F_{z,z'}^L(v') \sqsubseteq F_{x,z'}^L(\tilde{v})$ . Since the embeddings are rigid, we therefore obtain that

$$F_{\check{u},z'}^L(\check{v}) = F_{x,z'}^L \circ F_{\check{u},x}^L(\check{v}) = F_{x,z'}^L \circ F_{x,z'}^R \circ F_{z,z'}^L(v') = F_{z,z'}^L(v').$$

Hence  $F_{z,z'}^L(v') \in F_{\check{u},z'}^L(F(\check{u}))$ . As  $\check{v} = F_{\check{u},z'}^R \circ F_{z,z'}^L(v')$ , we have that  $(\check{u}, F_{\check{u},z'}^R \circ F_{z,z'}^L(v')) \in \text{Tr}(f)$ . But  $\text{Tr}(f)$  is contained in  $X$ .

In the opposite case we obtain that  $(x, F_{x,z'}^R \circ F_{z,z'}^L(v')) \in X \setminus \text{Tr}(f)$ . Note here that we have already seen that  $F_{z,z'}^L(v') \sqsubseteq F_{x,z'}^L(\tilde{v})$ , which implies that  $F_{z,z'}^L(v') \in F_{x,z'}^L(F(x))$ .

(3) Let  $(u_0, v_0), (u_1, v_1) \in X$  such that  $u_0, u_1 \sqsubseteq z$  and  $F_{u_0,z}^L(v_0) = F_{u_1,z}^L(v_1)$ , for some  $z \in D$ . The case that both pairs are contained in  $\text{Tr}(f)$  is obvious. Without restriction let  $(u_0, v_0) \in X \setminus \text{Tr}(f)$ . Then  $u_0 = x$ . If  $(u_1, v_1) \in \text{Tr}(f)$  we obtain with Claim 3.13 that  $u_1 \sqsubseteq x$ , contradicting the fact that  $(u_0, v_0) \in X \setminus \text{Tr}(f)$ . Hence  $(u_1, v_1) \in X \setminus \text{Tr}(f)$ , which means that also  $u_1 = x$ .  $\square$

As a special case we obtain that the dI-domain of stable maps from a dI-domain to a regular dI-domain is regular again [9].

**Theorem 3.14.** *The category of regular dI-domains and stable maps is Cartesian closed.*

#### 4. STRONGLY REGULAR SCOTT DOMAINS

In this section we consider Scott domains which satisfy a strengthened regularity condition.

**Definition 4.1.** A domain  $D$  is strongly regular, if every finite union of basic open sets  $\uparrow\{z\}$  with  $z \in \mathcal{K}(D)$  is regular open.

Note that in the Scott topology the finite unions of basic opens are just the compact open subsets of the domain. By Lemma 2.3 every strongly regular domain is regular. Among the examples given in Section 2 only the domains  $A_\perp$  generated by finite nonempty sets  $A$  are not strongly regular.

The following characterization will turn out to be useful in the remainder of this section. We call a subset  $S$  of  $\mathcal{K}(D)$  *subbasis*, if every compact element of  $D$  is the least upper bound of a finite subset of  $S$ .

**Lemma 4.2.** Let  $S$  be a subbasis of a domain  $D$ . Then the following three statements are equivalent:

- (1)  $D$  is strongly regular.
- (2) For any subset  $X$  of  $S$  and all  $v \in \mathcal{K}(D)$  the next condition holds:

$$(\forall u \in X)u \not\sqsubseteq v \Rightarrow (\exists z \in \mathcal{K}(D))[v \sqsubseteq z \wedge (\forall u \in X)u \not\uparrow z].$$

- (3) For any finite subset  $X$  of  $S$ ,  $\bigcup\{\uparrow\{z\} \mid z \in X\}$  is regular open.

*Proof.* In order to see that (1) implies (2), let  $X$  be a finite subset of  $S$  and  $v \in \mathcal{K}(D)$  such that for all  $u \in X$ ,  $u \not\sqsubseteq v$ . Since  $\bigcup\{\uparrow\{u\} \mid u \in X\}$  is regular open, it follows that  $\uparrow\{v\}$  intersects  $\text{ext}(\bigcup\{\uparrow\{u\} \mid u \in X\})$ . Hence, there is some  $z \in \mathcal{K}(D)$  such that  $v \sqsubseteq z$  and for all  $u \in X$ ,  $u \not\uparrow z$ .

For the proof that (3) follows from (2), let  $X$  be a finite subset of  $S$  and set  $U = \bigcup\{\uparrow\{u\} \mid u \in X\}$ . Moreover, assume that there is some  $v \in \text{int}(\text{cl}(U)) \setminus U$ . Then we have for all  $u \in X$  that  $u \not\sqsubseteq v$ . Because of (2) there exists some  $z \in \mathcal{K}(D)$  such that  $v \sqsubseteq z$  and for all  $u \in X$ ,  $z \not\uparrow u$ . It follows that  $z \in \text{int}(\text{cl}(U))$ . Hence  $z \in \text{cl}(U)$ , which implies that  $\uparrow\{z\}$  intersects  $U$ . Thus, for some  $u \in X$ ,  $u$  and  $z$  have a common upper bound. This contradicts what has been said about  $z$  before.

It remains to show that (1) is a consequence of (3). Let  $Y$  be finite subset of  $\mathcal{K}(D)$ . Then for any  $y \in Y$  there is some finite subset  $X_y$  of  $S$  such  $y$  is the least upper bound of  $X_y$ . Thus

$$\bigcup\{\uparrow\{y\} \mid y \in Y\} = \bigcup\{\bigcap\{\uparrow\{u\} \mid u \in X_y\} \mid y \in Y\}.$$

By distributivity it follows that  $\bigcup\{\uparrow\{y\} \mid y \in Y\}$  is a finite intersection over finite unions of basic open sets  $\uparrow\{z\}$  with  $z \in S$ . Since every such finite union is regular open by (3), and the class of regular open sets is closed under finite intersections, we obtain that  $\bigcup\{\uparrow\{y\} \mid y \in Y\}$  is regular open.  $\square$

It follows that a domain is strongly regular, exactly if every compact element that does not extend finitely many given compact elements  $u_i$  can be extended to a compact element which contains additional information, contradictory to the information coded into the  $u_i$ .

We will now consider the dependent product of a continuous family of Scott domains. Let to this end  $D$  be Scott domain and  $F: D \rightarrow \mathbf{SD}^{\text{ep}}$  a continuous functor.



Define  $\Pi_D F$  to be the set of all continuous sections of  $F$ , and order  $\Pi_D F$  pointwise, that is, let

$$f \sqsubseteq_p g \Leftrightarrow (\forall x \in D) f(x) \sqsubseteq_{F(x)} g(x).$$

Then  $\Pi_D F$  with the pointwise order is the *dependent product of  $F$  over  $D$* .

As will follow from the next theorem the sections  $(u \searrow v)$  with  $u \in \mathcal{K}(D)$  and  $v \in \mathcal{K}(F(u))$  form a subbasis of  $\Pi_D F$ . Here  $(u \searrow v)$  is defined by

$$(u \searrow v)(x) = \begin{cases} F_{u,x}^L(v) & \text{if } u \sqsubseteq x, \\ \perp_{F(x)} & \text{otherwise.} \end{cases}$$

The following result has been established in [5, 10].

**Theorem 4.3.** *Let  $D$  be a Scott domain and  $F: D \rightarrow \mathbf{SD}^{\text{ep}}$  a continuous functor. Then  $\Pi_D F$  is a Scott domain. The compact elements are exactly the least upper bounds of finite consistent sets of sections of the form  $(u \searrow v)$ .*

Let  $\mathbf{SRSD}^{\text{ep}}$  be the category of strongly regular Scott domains and embedding/projections, let  $D$  be strongly regular and  $F: D \rightarrow \mathbf{SRSD}^{\text{ep}}$ . We shall show next that  $\Pi_D F$  is also strongly regular. We do this by using an idea of U. Berger, to which the next lemma is due as well.

**Lemma 4.4.** *Let  $D$  be strongly regular and  $u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1} \in \mathcal{K}(D)$ . Moreover, let all  $v_j$  be pairwise distinct. Then there exist  $z_0, \dots, z_{m-1} \in \mathcal{K}(D)$  with the following properties:*

- (1)  $(\forall j < m) v_j \sqsubseteq z_j$ .
- (2)  $(\forall j < m)(\forall i < n)[u_i \not\sqsubseteq v_j \Rightarrow u_i \not\uparrow z_j]$ .
- (3)  $(\forall j_1, j_2 < m)[j_1 \neq j_2 \Rightarrow z_{j_1} \not\uparrow z_{j_2}]$ .

*Proof.* The lemma will be proved by induction on  $m$ . The case  $m = 0$  is trivial. Therefore, let  $m > 0$ . Moreover, without restriction, let  $v_{m-1}$  be minimal among  $v_0, \dots, v_{m-1}$ . Set

$$X = \{y \in \{u_0, \dots, u_{n-1}, v_0, \dots, v_{m-2}\} \mid y \not\sqsubseteq v_{m-1}\}.$$

Since  $D$  is strongly regular, there is some  $z_{m-1} \in \mathcal{K}(D)$  such that  $v_{m-1} \sqsubseteq z_{m-1}$  and  $y \not\uparrow z_{m-1}$ , for all  $y \in X$ . Thus Properties (1) and (2) hold for  $j = m - 1$ . Because  $v_{m-1}$  is minimal and the  $v_j$  are pairwise distinct, we moreover have that  $v_j \not\uparrow z_{m-1}$ , for  $j < m - 1$ .

Now, by applying the induction hypothesis to  $u_0, \dots, u_{n-1}$  and  $v_0, \dots, v_{m-2}$  we obtain that there are  $z_0, \dots, z_{m-2} \in \mathcal{K}(D)$  such that Properties (1)-(3) hold. Because of our choice of  $z_{m-1}$  (1) and (2) then also hold for  $z_0, \dots, z_{m-1}$ . For Property (3) we only have to show that for all  $j < m - 1$ ,  $z_j \not\uparrow z_{m-1}$ . Assume that  $z_j \uparrow z_{m-1}$ , for some  $j < m - 1$ . Since  $v_j \sqsubseteq z_j$ , it then follows that  $v_j \uparrow z_{m-1}$  as well, which is impossible by our choice of  $z_{m-1}$ .  $\square$

**Theorem 4.5.** *Let  $D$  be a strongly regular Scott domain and  $F: D \rightarrow \mathbf{SRSD}^{\text{ep}}$  a continuous functor. Then  $\Pi_D F$  is strongly regular.*

*Proof.* We verify Condition (2) in Lemma 4.2 for the subbasis

$$S = \{ (u \searrow y) \mid u \in \mathcal{K}(D) \wedge y \in \mathcal{K}(F(u)) \}.$$

Let to this end  $g^{i,a} = (u_i \searrow y_i^a) \in S$  ( $i < n$ ,  $a < b_i$ ) such that  $u_0, \dots, u_{n-1}$  are pairwise distinct, let  $f = \bigsqcup \{ (v_j \searrow z_j) \mid j \in J \} \in \mathcal{K}(\Pi_D F)$ , and assume that  $g^{i,a} \not\sqsubseteq_p f$ , for  $i < n$  and  $a < b_i$ . We have to construct a section  $h \in \mathcal{K}(\Pi_D F)$  with  $f \sqsubseteq_p h$  and  $h \not\uparrow g^{i,a}$ , for  $i < n$  and  $a < b_i$ .

Obviously,  $g^{i,a} \not\sqsubseteq_p f$  if and only if  $y_i^a \not\sqsubseteq f(u_i)$ . Since every domain  $F(u_i)$  is strongly regular, there is some  $\tilde{y}_i \in \mathcal{K}(F(u_i))$ , for each  $i < n$ , such that  $f(u_i) \sqsubseteq \tilde{y}_i$  and  $y_i^a \not\uparrow \tilde{y}_i$ , for all  $a < b_i$ . Because  $D$  is strongly regular too, it follows with the preceding lemma that there are  $\bar{u}_0, \dots, \bar{u}_{n-1} \in \mathcal{K}(D)$  such that the following properties hold:

- (1)  $(\forall i < n) u_i \sqsubseteq \bar{u}_i$ .
- (2)  $(\forall i < n)(\forall j \in J)[v_j \not\sqsubseteq u_i \Rightarrow v_j \not\uparrow \bar{u}_i]$ .
- (3)  $(\forall i_1, i_2 < n)[i_1 \neq i_2 \Rightarrow \bar{u}_{i_1} \not\uparrow \bar{u}_{i_2}]$ .

For  $i < n$  let  $\bar{y}_i = F_{u_i, \bar{u}_i}^L(\tilde{y}_i)$ . We shall show now that the set  $\{ (\bar{u}_i \searrow \bar{y}_i) \mid i < n \} \cup \{ (v_j \searrow z_j) \mid j \in J \}$ , which we denote by  $Y$ , is bounded in  $\Pi_D F$ .

Let hereto  $w \in D$  and assume first that for all  $i < n$ ,  $\bar{u}_i \not\sqsubseteq w$ . Then  $f(w)$  is an upper bound of all  $(u \searrow y)(w)$  with  $(u \searrow y) \in Y$  and  $u \sqsubseteq w$ . Now, let  $i < n$  such that  $\bar{u}_i \sqsubseteq w$ . We want to show that in this case  $F_{\bar{u}_i, w}^L(\bar{y}_i)$  is an upper bound of all  $(u \searrow y)(w)$  with  $(u \searrow y) \in Y$  and  $u \sqsubseteq w$ .

Since  $\bar{u}_i \sqsubseteq w$ , we have that  $u_i \sqsubseteq w$  and for all  $i' < n$  with  $i' \neq i$  that  $\bar{u}_{i'} \not\sqsubseteq w$ , by Property (3). Moreover, we obtain with Property (2) that for  $j \in J$ ,  $v_j \sqsubseteq w$  exactly if  $v_j \sqsubseteq u_i$ . Thus, it follows for  $(u \searrow y) \in Y$  with  $u \sqsubseteq w$  that either  $u = \bar{u}_i$  or  $u = v_j$ , for some  $j \in J$  with  $v_j \sqsubseteq u_i$ . In the first case, we obviously have that  $(u \searrow y)(w) \sqsubseteq F_{\bar{u}_i, w}^L(\bar{y}_i)$ . Let us therefore consider the second case. Since  $F_{u_i, \bar{u}_i}^L$  is continuous, we obtain with Property (2) that  $F_{u_i, \bar{u}_i}^L(f(u_i)) = f(\bar{u}_i)$ . Hence  $f(\bar{u}_i) \sqsubseteq \bar{y}_i$ . It follows for  $j \in J$  with  $v_j \sqsubseteq u_i$  that

$$\begin{aligned} (v_j \searrow z_j)(w) &= F_{v_j, w}^L(z_j) \\ &= F_{\bar{u}_i, w}^L \circ F_{u_i, \bar{u}_i}^L \circ F_{v_j, u_i}^L(z_j) \\ &\sqsubseteq F_{\bar{u}_i, w}^L \circ F_{u_i, \bar{u}_i}^L(f(u_i)) \\ &= F_{\bar{u}_i, w}^L(f(\bar{u}_i)) \\ &\sqsubseteq F_{\bar{u}_i, w}^L(\bar{y}_i). \end{aligned}$$

Set  $h = \bigsqcup Y$ . Then  $f \sqsubseteq_p h$ . Moreover  $h \not\uparrow g^{i,a}$ , for  $i < n$  and  $a < b_i$ , since  $h(\bar{u}_i) = \bar{y}_i = F_{u_i, \bar{u}_i}^L(\tilde{y}_i)$ ,  $g^{i,a}(\bar{u}_i) = F_{u_i, \bar{u}_i}^L(y_i^a)$  and  $\tilde{y}_i \not\uparrow y_i^a$ .  $\square$

As a special case we obtain that the Scott domain of continuous maps between two strongly regular Scott domains is also strongly regular.

**Theorem 4.6.** *The category of strongly regular Scott domains and continuous maps is Cartesian closed.*

## 5. FURTHER CLOSURE PROPERTIES

In this section we study further closure properties of the class of regular and/or strongly regular domains. We start by considering different kinds of sums.

Let  $I$  be an index set with at least two members and  $(D_i)_{i \in I}$  be a family of domains. Set

$$\biguplus_{i \in I} D_i = \bigcup \{ \{i\} \times D_i \mid i \in I \} \cup \{\perp\}$$

and order  $\biguplus_{i \in I} D_i$  by

$$x \sqsubseteq y \Leftrightarrow x = \perp \vee (\exists i \in I)[x = (i, x') \wedge y = (i, y') \wedge x' \sqsubseteq_{D_i} y'].$$

Then  $\biguplus_{i \in I} D_i$  is the *separated sum* of the family  $(D_i)_{i \in I}$  of domains. Obviously, it is a domain again with

$$\mathcal{K}((D_i)_{i \in I}) = \bigcup \{ \{i\} \times \mathcal{K}(D_i) \mid i \in I \} \cup \{\perp\}.$$

**Theorem 5.1.** *Let  $I$  be an index set with at least two members and  $(D_i)_{i \in I}$  be a family of domains. Then the following two statements hold:*

- (1) *If  $D_i$  is regular, for all  $i \in I$ , then  $\biguplus_{i \in I} D_i$  is regular.*
- (2) *If  $I$  is at least infinite and for all  $i \in I$ ,  $D_i$  is strongly regular, then  $\biguplus_{i \in I} D_i$  is strongly regular.*

In order to see that the separated sum of a finite family of strongly regular domains need not be strongly regular again, consider the separated sum of two one-point domains.

Next, define

$$\bigoplus_{i \in I} D_i = \bigcup \{ \{i\} \times (D_i \setminus \{\perp_{D_i}\}) \mid i \in I \} \cup \{\perp\}$$

and order  $\bigoplus_{i \in I} D_i$  as in the case of the separated sum. Then  $\bigoplus_{i \in I} D_i$  is the *coalesced sum* of the family  $(D_i)_{i \in I}$ . It is also a domain with

$$\mathcal{K}(\bigoplus_{i \in I} D_i) = \bigcup \{ \{i\} \times \mathcal{K}(D_i \setminus \{\perp_{D_i}\}) \mid i \in I \} \cup \{\perp\}.$$

As is well known, the coalesced sum is the categorical coproduct in the category of domains with strict continuous maps [13].

**Theorem 5.2.** *Let  $I$  be an index set with at least two members and  $(D_i)_{i \in I}$  be a family of domains. Then the following two statements hold:*

- (1) *If  $D_i$  is regular, for all  $i \in I$ , then  $\bigoplus_{i \in I} D_i$  is regular.*
- (2) *If  $D_i$  is strongly regular, for all  $i \in I$ , then  $\bigoplus_{i \in I} D_i$  is strongly regular.*

Let  $\mathbf{DOM}^{\text{ep}}$  be the category of domains and embedding/projections. For a domain  $D$  and a continuous functor  $F: D \rightarrow \mathbf{DOM}^{\text{ep}}$  set

$$\Sigma_D F = \{ (y, z) \mid y \in D \wedge z \in F(y) \}$$

and order  $\Sigma_D F$  by

$$(u, v) \sqsubseteq (y, z) \Leftrightarrow u \sqsubseteq_D y \wedge F_{u,y}^L(v) \sqsubseteq_{F(y)} z.$$

Then  $\Sigma_D F$  is the *dependent sum of  $F$  over  $D$* . As follows from [5, 10],  $\Sigma_D F$  is a domain again with

$$\mathcal{K}(\Sigma_D F) = \{ (u, v) \mid u \in \mathcal{K}(D) \wedge v \in \mathcal{K}(F(u)) \}.$$

**Theorem 5.3.** *Let  $D$  be domain and  $F: D \rightarrow \mathbf{DOM}^{\text{ep}}$  a continuous functor. Then the following two statements hold:*

- (1) *If  $D$  and, for every  $x \in D$ ,  $F(x)$  are regular, then  $\Sigma_D F$  is regular.*
- (2) *If  $D$  and, for every  $x \in D$ ,  $F(x)$  are strongly regular, then  $\Sigma_D F$  is strongly regular.*

*Proof.* We only show Statement (2). Let to this end  $X$  be a finite subset of  $\mathcal{K}(\Sigma_D F)$  and  $v \in \mathcal{K}(\Sigma_D F)$  such that  $u \not\sqsubseteq v$ , for all  $u \in X$ . Then it follows for  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  that either  $u_1 \not\sqsubseteq v_1$ , or  $u_1 \sqsubseteq v_1$  and  $F_{u_1, v_1}^L(u_2) \not\sqsubseteq v_2$ . Set  $X_1 = \{ u \in X \mid u_1 \not\sqsubseteq v_1 \}$  and  $X_2 = X \setminus X_1$ . Since  $D$  is strongly regular, there is some  $\hat{u} \in \mathcal{K}(D)$  such that  $v_1 \sqsubseteq \hat{u}$ , but  $u_1 \not\uparrow \hat{u}$ , for all  $u \in X_1$ . Let  $\hat{v} = F_{v_1, \hat{u}}^L(v_2)$ . Then  $(\hat{u}, \hat{v}) \in \mathcal{K}(\Sigma_D F)$ . Moreover, we have for  $u \in X_2$  that  $u_1 \sqsubseteq \hat{u}$  and  $F_{u_1, \hat{u}}^L(u_2) \not\sqsubseteq \hat{v}$ .

By the strong regularity of  $F(\hat{u})$  there is now some  $\bar{v} \in \mathcal{K}(F(\hat{u}))$  such that  $\hat{v} \sqsubseteq \bar{v}$  and  $F_{u_1, \hat{u}}^L(u_2) \not\uparrow \bar{v}$ , for all  $u \in X_2$ . Then  $(\hat{u}, \bar{v}) \in \mathcal{K}(\Sigma_D F)$ . Moreover  $v \sqsubseteq (\hat{u}, \bar{v})$ , since  $v_1 \sqsubseteq \hat{u}$  and  $F_{v_1, \hat{u}}^L(v_2) = \hat{v} \sqsubseteq \bar{v}$ . Finally, we have that  $(\hat{u}, \bar{v}) \not\uparrow u$ , for all  $u \in X$ .

In order to see this, assume that for some  $u \in X$  there is some  $x \in \Sigma_D F$  with  $u \sqsubseteq x$  and  $(\hat{u}, \bar{v}) \sqsubseteq x$ . So  $u_1, \hat{u} \sqsubseteq x_1$  and  $F_{u_1, x_1}^L(u_2), F_{\hat{u}, x_1}^L(\bar{v}) \sqsubseteq x_2$ . By our choice of  $\hat{u}$  it follows that  $u_1 \in X_2$ , that is  $u_1 \sqsubseteq \hat{u}$ . Hence, we have that  $F_{u_1, \hat{u}}^L(u_2), \bar{v} \sqsubseteq F_{\hat{u}, x_1}^R(x_2)$ . This contradicts our choice of  $\bar{v}$ .  $\square$

As is well known, in the case of a constant functor, say  $F(x) = E$ , the dependent sum of  $F$  over  $D$  is the product of the domains  $D$  and  $E$ . Thus, it follows that the product of two regular domains is regular and that of two strongly regular domains is strongly regular. This can easily be generalized to products of arbitrary families of domains. Note that in this case the compact elements are those maps from the index set into the respective domains which almost always have the least element of the respective domain as value. Moreover, the result holds for the smash product of two domains.

In the remainder of this section we consider closure under the inverse limit construction. An  $\omega$ -chain in  $\mathbf{DOM}^{\text{ep}}$  is a diagram of the form  $\Theta = D_0 \xrightarrow{e_0} D_1 \xrightarrow{e_1} \dots$ . Let  $e_{mn} = e_{n-1} \circ \dots \circ e_m$ , for  $m < n$ , and  $e_{mm} = \text{id}_{D_m}$ . Set

$$D_\Theta = \{ x \in \prod_{m \in \omega} D_m \mid (\forall m \in \omega) x_m = e_m^R(x_{m+1}) \}$$

and order  $D_\Theta$  componentwise, that is

$$x \sqsubseteq y \Leftrightarrow (\forall m \in \omega) x_m \sqsubseteq_{D_m} y_m.$$

Then  $D_\Theta$  is the *inverse limit* of the  $\omega$ -cochain  $\Theta^R = (D_m, e_m^R)_{m \in \omega}$ . As is well known,  $D_\Theta$  is a domain with

$$\mathcal{K}(D_\Theta) = \{ u \in D_\Theta \mid (\exists m \in \omega) u_m \in \mathcal{K}(D_m) \wedge (\forall n \geq m) u_{n+1} = e_n^L(u_n) \}.$$

It is a colimit of the  $\omega$ -chain  $\Theta$  in  $\mathbf{DOM}^{\text{ep}}$ .

**Proposition 5.4.** *Let  $(D_n, e_n)_{n \in \omega}$  be an  $\omega$ -chain in  $\mathbf{DOM}^{\text{ep}}$ . Then the following two statements hold:*

- (1) *If  $D_n$  is regular, for all  $n \in \omega$ , then  $D_\Theta$  is regular.*
- (2) *If  $D_n$  is strongly regular, for all  $n \in \omega$ , then  $D_\Theta$  is strongly regular.*

*Proof.* We only show Statement (2). Let to this end  $X$  be a finite subset of  $\mathcal{K}(D_\Theta)$  and  $v \in \mathcal{K}(D_\Theta)$  such that  $u \not\sqsubseteq v$ , for all  $u \in X$ . For  $u \in \mathcal{K}(D_\Theta)$  there is some smallest  $m_u \in \omega$  such that  $u_{m_u} \in \mathcal{K}(D_{m_u})$  and  $u = \text{in}_{m_u}(u_{m_u})$ , where  $\text{in}_{m_u}: D_{m_u} \rightarrow D_\Theta$  defined by

$$\text{in}_{m_u}(z) = \begin{cases} e_{m_u n}^L(z) & \text{if } m_u \leq n, \\ e_{nm_u}^R(z) & \text{otherwise,} \end{cases}$$

is the canonical embedding of  $D_{m_u}$  into  $D_\Theta$ . Let  $j$  be the maximal such  $m_u$ , for  $u \in X \cup \{v\}$ . Moreover, let  $X_j = \text{pr}_j(X)$ , where  $\text{pr}_j: D_\Theta \rightarrow D_j$  is the projection onto the  $j$ th component. Then  $X_j$  is a subset of  $\mathcal{K}(D_j)$  and for all  $w \in X_j$  we have that  $w \not\sqsubseteq v_j$ . Since  $D_j$  is strongly regular, there is some  $z \in \mathcal{K}(D_j)$  such that  $v_j \sqsubseteq z$  and  $w \not\uparrow z$ , for all  $w \in X_j$ . Set  $\bar{z} = \text{in}_j(z)$ . Then  $\bar{z} \in \mathcal{K}(D_\Theta)$ . Furthermore  $v \sqsubseteq \bar{z}$ . Now, assume that for some  $u \in X$ ,  $\bar{z} \uparrow u$ . Then  $\bar{z}_j \uparrow u_j$ . But since  $\bar{z}_j = z$  and  $u_j \in X_j$ , this is impossible by our choice of  $z$ .  $\square$

It follows that every  $\omega$ -chain in the categories of regular and/or of strongly regular domains and embedding/projections has a colimit.

**Theorem 5.5.** *The categories of regular domains and of strongly regular domains, both with embedding/projections as morphisms, are  $\omega$ -cocomplete.*

As is well known, both the separated and the coalesced sum of a family of Scott domains is a Scott domain, for any Scott domain  $D$  and any continuous functor  $F: D \rightarrow \mathbf{SD}^{\text{ep}}$  the dependent sum of  $F$  over  $D$  is a Scott domain, and for any  $\omega$ -chain  $\Theta$  in  $\mathbf{SD}^{\text{ep}}$ ,  $D_\Theta$  is a Scott domain. Thus, we have the following consequences.

**Theorem 5.6.**

- (1) *For every family  $(D_i)_{i \in \omega}$  of strongly regular Scott domains the separated sum  $\biguplus_{i \in \omega} D_i$  is a strongly regular Scott domain.*
- (2) *For every index set  $I \subseteq \omega$  and every family  $(D_i)_{i \in I}$  of strongly regular Scott domains the coalesced sum  $\bigoplus_{i \in I} D_i$  is a strongly regular Scott domain.*
- (3) *For every strongly regular Scott domain  $D$  and every continuous functor  $F: D \rightarrow \mathbf{SRSD}^{\text{ep}}$  the dependent sum  $\Sigma_D F$  of  $F$  over  $D$  is a strongly regular Scott domain.*
- (4) *For every  $\omega$ -chain  $\Theta$  in  $\mathbf{SRSD}^{\text{ep}}$  the inverse limit  $D_\Theta$  of the  $\omega$ -cochain  $\Theta^R$  is a strongly regular Scott domain. That is, the category  $\mathbf{SRSD}^{\text{ep}}$  is  $\omega$ -cocomplete.*

For regular dI-domains we have an analogous result.

**Theorem 5.7.**

- (1) For every index set  $I \subseteq \omega$  and every family  $(D_i)_{i \in I}$  of regular dI-domains the separated sum  $\bigsqcup_{i \in I} D_i$  as well as the coalesced sum  $\bigoplus_{i \in I} D_i$  are regular dI-domains.
- (2) For every regular dI-domain  $D$  and every stable functor  $F : D \rightarrow \mathbf{RDI}^{\text{rep}}$  the dependent sum  $\Sigma_D F$  of  $F$  over  $D$  is a regular dI-domain.
- (3) For every  $\omega$ -chain  $\Theta$  in  $\mathbf{RDI}^{\text{rep}}$  the inverse limit  $D_\Theta$  of the  $\omega$ -cochain  $\Theta^R$  is a regular dI-domain. That is, the category  $\mathbf{RDI}^{\text{rep}}$  is  $\omega$ -co-complete.

*Proof.* Statement (1) is obvious. For Statement (2) we only have to show that  $\Sigma_D F$  is a dI-domain. We know already that it is a Scott domain. Axiom I is clearly satisfied.

For the verification of Axiom d let  $(x, u), (y, v), (z, w) \in \Sigma_D F$  such that  $(y, v) \uparrow (z, w)$ . Note that

$$(y, v) \sqcup (z, w) = (y \sqcup z, F_{y, y \sqcup z}^L(v) \sqcup F_{z, y \sqcup z}^L(w))$$

and

$$(y, v) \sqcap (z, w) = (y \sqcap z, F_{y \sqcap z, y}^R(v) \sqcap F_{y \sqcap z, z}^R(w)).$$

Since Axiom d holds in  $D$  we only have to show that

$$\begin{aligned} & F_{x \sqcap (y \sqcup z), x}^R(u) \sqcap F_{x \sqcap (y \sqcup z), y \sqcup z}^R(F_{y, y \sqcup z}^L(v) \sqcup F_{z, y \sqcup z}^L(w)) \\ (5.5) \quad &= F_{x \sqcap y, x \sqcap (y \sqcup z)}^L(F_{x \sqcap y, x}^R(u) \sqcap F_{x \sqcap y, y}^R(v)) \sqcup \\ & F_{x \sqcap z, x \sqcap (y \sqcup z)}^L(F_{x \sqcap z, x}^R(u) \sqcap F_{x \sqcap z, z}^R(w)). \end{aligned}$$

Denote the left-hand side of this equation by  $W$ . Then we have that

$$\begin{aligned} (5.6) \quad W &= W \sqcap F_{x \sqcap (y \sqcup z), y \sqcup z}^R(F_{y, y \sqcup z}^L(v) \sqcup F_{z, y \sqcup z}^L(w)) \\ &= W \sqcap (F_{x \sqcap (y \sqcup z), y \sqcup z}^R \circ F_{y, y \sqcup z}^L(v) \sqcup F_{x \sqcap (y \sqcup z), y \sqcup z}^R \circ F_{z, y \sqcup z}^L(w)) \\ &= (W \sqcap F_{x \sqcap (y \sqcup z), y \sqcup z}^R \circ F_{y, y \sqcup z}^L(v)) \sqcup (W \sqcap F_{x \sqcap (y \sqcup z), y \sqcup z}^R \circ F_{z, y \sqcup z}^L(w)). \end{aligned}$$

Here we used that Axiom d holds in  $F(x \sqcap (y \sqcup z))$ . Next we transform the first operand in the last equation. It is

$$\begin{aligned} (5.7) \quad & W \sqcap F_{x \sqcap (y \sqcup z), y \sqcup z}^R \circ F_{y, y \sqcup z}^L(v) \\ &= F_{x \sqcap (y \sqcup z), x}^R(u) \sqcap F_{x \sqcap (y \sqcup z), y \sqcup z}^R \circ F_{y, y \sqcup z}^L(v) \\ &= F_{x \sqcap (y \sqcup z), x}^R(u) \sqcap F_{x \sqcap y, x \sqcap (y \sqcup z)}^L \circ F_{x \sqcap y, y}^R(v) \\ (5.8) \quad &= F_{x \sqcap y, x \sqcap (y \sqcup z)}^L \circ F_{x \sqcap y, x \sqcap (y \sqcup z)}^R(F_{x \sqcap (y \sqcup z), x}^R(u) \\ & \sqcap F_{x \sqcap y, x \sqcap (y \sqcup z)}^L \circ F_{x \sqcap y, y}^R(v)) \\ (5.9) \quad &= F_{x \sqcap y, x \sqcap (y \sqcup z)}^L(F_{x \sqcap y, x}^R(u) \sqcap F_{x \sqcap y, y}^R(v)), \end{aligned}$$

where Equation (5.7) holds since  $(F(x \sqcap y), F_{x \sqcap y, y}, F_{x \sqcap y, x \sqcap (y \sqcup z)})$  is a pullback of  $(F_{y, y \sqcup z}, F_{x \sqcap (y \sqcup z), y \sqcup z})$ . For Equation (5.8) note that

$$F_{x \sqcap (y \sqcup z), x}^R(u) \sqcap F_{x \sqcap y, x \sqcap (y \sqcup z)}^L \circ F_{x \sqcap y, y}^R(v) \sqsubseteq F_{x \sqcap y, x \sqcap (y \sqcup z)}^L \circ F_{x \sqcap y, y}^R(v)$$

and the morphisms are rigid embedding/projections. By the monotonicity of the maps the term in Equation (5.8) is less or equal to the term in Equation (5.9), which in its turn is less or equal to the term in Equation (5.7).

The term in Equation (5.9) coincides with the first operand in the right-hand side of Equation (5.5). A similar transformation shows that the second operand in Equation (5.6) is equal to the second operand in the right-hand side of Equation (5.5). This proves that Axiom d holds.

In the case of Statement (3) too, it only has to be shown that Axioms d and I hold in  $D_\Theta$ . Axiom d is obvious as the order is defined componentwise. For the verification of Axiom I let  $v \in \mathcal{K}(D_\Theta)$  and  $u \in D_\Theta$  with  $u \sqsubseteq v$ . Then there is some  $m \in \omega$  and some  $\bar{v} \in \mathcal{K}(D_m)$  such that  $v = \text{in}_m(\bar{v})$ . Since  $(\text{in}_m, \text{pr}_m)$  is a rigid embedding/projection, we obtain that there is some  $\bar{u} \in D_\Theta$  with  $u = \text{in}_m(\bar{u})$ . It follows that  $\bar{u} \sqsubseteq_{D_m} \bar{v}$ , which shows that  $\downarrow\{u\}$  is finite.  $\square$

## 6. CONCLUSION

In this paper two subclasses of domains have been introduced and order-theoretic as well as topological characterizations have been given. The domains are such that if a compact element  $u$  is strictly larger than a compact element  $v$  in the information ordering of the domain, then a witness for this can be found, that is a compact element  $z$  which extends  $v$  and contains information that is inconsistent with the information contained in  $u$ .

Both subclasses enjoy important closure properties. In the special cases of regular dI-domains and strongly regular Scott domains they are such that models of the untyped lambda calculus can be constructed (*cf. e.g.* [3]). Moreover, models of typed lambda calculi allowing disjoint unions, dependent products as well as dependent sums, and recursive definitions on both the type and the term level can be given.

Note that effective versions of the domains can be defined so that similar closure properties hold.

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