Weak proximal normal structure and coincidence quasi-best proximity points

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ABSTRACT

We introduce the notion of pointwise cyclic-noncyclic relatively non-expansive pairs involving orbits. We study the best proximity point problem for this class of mappings. We also study the same problem for the class of pointwise noncyclic-noncyclic relatively nonexpansive pairs involving orbits. Finally, under the assumption of weak proximal normal structure, we prove a coincidence quasi-best proximity point theorem for pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits. Examples are provided to illustrate the observed results.

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1. INTRODUCTION

Let \( A, B \) be nonempty subsets of Banach space \( X \). A mapping \( T : A \cup B \to A \cup B \) is said to be cyclic provided that \( T(A) \subseteq B \) and \( T(B) \subseteq A \). On the other hand, a mapping \( S : A \cup B \to A \cup B \) is said to be noncyclic if \( S(A) \subseteq A \) and \( S(B) \subseteq B \).
For a cyclic mapping $T : A \cup B \to A \cup B$, a point $p \in A \cup B$ is said to be a best proximity point provided that
\[ d(p, Tp) = \text{dist}(A, B). \]
Furthermore, we say that a pair $(A, B)$ of subsets in a Banach space satisfies a property if each of the sets $A$ and $B$ has that property. Similarly, the pair $(A, B)$ is called convex if both $A$ and $B$ are convex; moreover we write
\[ (A, B) \subseteq (E, F) \iff A \subseteq E, B \subseteq F. \]
In addition, we will use the following notations:
\[ \delta(A, B) = \sup\{\|x - y\| : x \in A, y \in B\}; \]
\[ \delta(x, B) = \sup\{\|x - y\| : y \in B\}. \]
For a nonempty, bounded and convex subset $F$ of a Banach space $X$, we write
\[ r_x(F) = \sup\{\|x - y\| : y \in F\}; \]
\[ r(F) = \inf\{r_x(F) : x \in F\}; \]
\[ F_c = \{x \in F : r_x(F) = r(F)\}. \]
In 2017, M. Gabeleh introduced the notion of a pointwise cyclic relatively nonexpansive mapping involving orbits, and proved a theorem on the existence of best proximity points.

**Definition 1.1** ([11]). Let $(A, B)$ be a nonempty pair of subsets of a Banach space $X$. A mapping $T : A \cup B \to A \cup B$ is said to be pointwise cyclic relatively nonexpansive involving orbits if $T$ is cyclic and for any $(x, y) \in A \times B$, if $\|x - y\| = \text{dist}(A, B)$, then
\[ \|Tx - Ty\| = \text{dist}(A, B), \]
and otherwise, there exists a function $\alpha : A \times B \to [0, 1]$ such that
\[ \|Tx - Ty\| \leq \alpha(x, y)\|x - y\| + (1 - \alpha(x, y)) \min\{\delta_x(O^2(y; \infty)), \delta_y(O^2(x; \infty))\}, \]
where, for any $(x, y) \in A \times B$
\[ \delta_x(O^2(y; \infty)) = \sup_{n \in \mathbb{N}} \|x - T^{2n}y\|, \quad \delta_y(O^2(x; \infty)) = \sup_{n \in \mathbb{N}} \|T^{2n}x - y\|. \]

Note that, if $A = B$, then we say that $T$ is a pointwise nonexpansive mapping involving orbits. In [12], M. Gabeleh, O. Olela Otafudu, and N. Shahzad considered a pair of mappings $T$ and $S$. According to [12], for a nonempty pair of subsets $(A, B)$ in a metric space $(X, d)$, and a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ (that is, $T : A \cup B \to A \cup B$ is cyclic and $S : A \cup B \to A \cup B$ is noncyclic); they called a point $p \in A \cup B$ a coincidence best proximity point for $(T; S)$ if
\[ d(Sp, Tp) = \text{dist}(A, B). \]
Note that if $S = I$, the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for $T$. 

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In 2019, A. Abkar and M. Norouzian introduced the concept of coincidence quasi-best proximity point and proved the existence of such points for quasi-cyclic-noncyclic contraction pairs. We remark that the coincidence quasi-best proximity point theory is more general and includes both the best proximity point theory and the coincidence best proximity point theory.

**Definition 1.2 ([2]).** Let \((A, B)\) be a nonempty pair of subsets of a metric space \((X,d)\) and \(T,S : X \to X\) be a quasi-cyclic-noncyclic pair on \(A \cup B\); that is, \(T(A) \subseteq S(B)\) and \(T(B) \subseteq S(A)\). A point \(p \in A \cup B\) is said to be a coincidence quasi-best proximity point for \((T; S)\) if

\[
d(Sp,Tp) = \text{dist}(S(A), S(B)).
\]

In case that \(S = I\), the point \(p\) reduces to a best proximity point for \(T\).

In this article, we will focus on the coincidence quasi-best proximity point problem for pointwise cyclic-noncyclic and noncyclic-noncyclic relatively non-expansive pairs. To do this, we need to recall some definitions and theorems. We begin with the following definition which is a modification of a concept in [8].

**Definition 1.3.** Let \((A, B)\) be a nonempty pair of subsets of a Banach space \(X\) and \(S : A \cup B \to A \cup B\) be a noncyclic mapping on \(A \cup B\). A convex pair \((S(A), S(B))\) is called a proximal pair if for each \((a_1, b_1) \in A \times B\), there exists \((a_2, b_2) \in A \times B\) such that for each \(i, j \in \{1, 2\}\) with \(i \neq j\) we have

\[
\|Sa_i - Sb_j\| = \text{dist}(S(A), S(B)).
\]

Given \((A, B)\) a pair of nonempty subsets of a Banach space \(X\), the associated proximal pair of \((S(A), S(B))\) is the pair \((S(A)_0, S(B)_0)\) given by

\[
A_0 := \{a \in A : \|Sa - Sb\| = \text{dist}(S(A), S(B))\text{ for some } b \in B\},
\]

\[
B_0 := \{b \in B : \|Sa - Sb\| = \text{dist}(S(A), S(B))\text{ for some } a \in A\}.
\]

In fact, if the pair \((S(A), S(B))\) is nonempty, weakly compact and convex, then its associated pair \((S(A)_0, S(B)_0)\) is also nonempty, weakly compact and convex. Furthermore, we have

\[
\text{dist}(S(A)_0, S(B)_0) = \text{dist}(S(A), S(B)).
\]

The proof of the above statements goes in the same lines as in the case for the pair \((A, B)\); see for instance [21]. Here’s a definition we derive from [8] and we’ve made some changes to meet our needs.

**Definition 1.4.** Let \((K_1, K_2)\) be a nonempty pair of subsets of a Banach space \(X\) and \(S : K_1 \cup K_2 \to K_1 \cup K_2\) be a noncyclic mapping on \(K_1 \cup K_2\). We say that a convex pair \((S(K_1), S(K_2))\) has proximal normal structure (PNS) if for any closed, bounded, convex and proximal pair \((S(H_1), S(H_2)) \subseteq (S(K_1), S(K_2))\) which

\[
\text{dist}(S(H_1), S(H_2)) = \text{dist}(S(K_1), S(K_2)), \quad \delta(S(H_1), S(H_2)) > \text{dist}(S(H_1), S(H_2)),
\]

where \(\delta(S(H_1), S(H_2))\) denote the Hausdorff distance between \(S(H_1)\) and \(S(H_2)\).
there exists \((x, y) \in H_1 \times H_2\) such that
\[
\delta(Sx, S(H_2)) < \delta(S(H_1), S(H_2)), \quad \delta(Sy, S(H_1)) < \delta(S(H_1), S(H_2)).
\]

Note that the pair \((K, K)\) has proximal normal structure if and only if \(K\) has normal structure in the sense of Brodskii and Milman (see [4] and [20]).

**Theorem 1.5** ([8]). *Every bounded, closed and convex pair in a uniformly convex Banach space \(X\) has proximal normal structure.*

The following definition is a modification of what already appeared in [11].

**Definition 1.6.** Let \((K_1, K_2)\) be a nonempty pair of subsets of a Banach space \(X\) and \(S : K_1 \cup K_2 \to K_1 \cup K_2\) be a noncyclic mapping on \(K_1 \cup K_2\). We say that a convex pair \((S(K_1), S(K_2))\) has weak proximal normal structure (WPNS) if for each nonempty, weakly compact and convex proximal pair \((S(H_1), S(H_2))\) \(\subseteq (S(K_1), S(K_2))\) for which

\[
\text{dist}(S(H_1), S(H_2)) = \text{dist}(S(K_1), S(K_2)), \quad \delta(S(H_1), S(H_2)) > \text{dist}(S(H_1), S(H_2)),
\]

there exists \((x, y) \in H_1 \times H_2\) such that
\[
\delta(Sx, S(H_2)) < \delta(S(H_1), S(H_2)), \quad \delta(Sy, S(H_1)) < \delta(S(H_1), S(H_2)).
\]

In this article, we intend to generalize some results of [8] and [11]. Our results have the following advantages: First, we introduce the class of the pointwise cyclic-noncyclic and noncyclic-noncyclic relatively nonexpansive mappings involving orbits, that in particular, includes the class of pointwise cyclic-noncyclic and noncyclic-noncyclic relatively nonexpansive mappings involving orbits. Second, we consider a pair of mappings while the previous articles are concerned with one single mapping, and finally, we study the coincidence quasi-best proximity point problem, which in particular, includes the best proximity point problem as a special case.

### 2. Cyclic-noncyclic pairs

We begin this section by introducing the new concept of a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits.

**Definition 2.1.** Assume that \((A, B)\) is a nonempty pair of subsets of a Banach space \(X\) and \(T, S : A \cup B \to A \cup B\) are two mappings. A pair \((T; S)\) is said to be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits if \((T; S)\) is a cyclic-noncyclic pair and for any \((x, y) \in A \times B\), if \(\|x - y\| = \text{dist}(S(A), S(B))\), then
\[
\|Tx - Ty\| = \text{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \text{dist}(S(A), S(B))
\]

and otherwise, there exists a function \(\alpha : A \times B \to [0, 1]\) such that
\[
\|Tx - Ty\| \leq \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[O^2(y; \infty)], \delta_y[O^2(x; \infty)]\},
\]

where, for any \((x, y) \in A \times B\)
\[
\delta_x[O^2(y; \infty)] = \sup_{n \in \mathbb{N}} \|x - T^{2n}y\|, \quad \delta_y[O^2(x; \infty)] = \sup_{n \in \mathbb{N}} \|T^{2n}x - y\|.
\]
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We note that if $S = I$, then the class of pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits reduces to the class of pointwise cyclic relatively nonexpansive mappings involving orbits introduced in [11].

**Definition 2.2 ([20]).** We say that a Banach space $X$ has the property (C) if every bounded decreasing sequence of nonempty, closed and convex subsets of $X$ have a nonempty intersection.

For $C \subseteq X$, we denote the diameter of $C$ by $\delta(C)$. A point $x \in C$ is a diametral point of $C$ provided that $\sup\{\|x - y\| : y \in C\} = \delta(C)$. A convex set $K \subseteq X$ is said to have normal structure if for each bounded convex subset $H$ of $K$ which contains at least two points, there is some point $x \in H$ which is not a diametral point of $H$.

**Lemma 2.3 ([20]).** Assume that $X$ is a Banach space with the property (C), then $F_c$ is nonempty, closed and convex.

**Lemma 2.4 ([20]).** Assume that $F$ is a closed and convex subset of a Banach space $X$ which contains at least two points. If $F$ has normal structure, then $\delta(F_c) < \delta(F)$.

**Theorem 2.5.** Assume that $K$ is a nonempty, bounded, closed and convex subset of a Banach space $X$ with property (C). Suppose that $K$ has normal structure. Let $(T, S)$ be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits on $K$. Then there exists a point $p \in K$ such that $\|Tp - Sp\| = 0$.

**Proof.** Suppose $\Gamma$ denotes the collection of all nonempty, closed and convex subsets of $K$ such that $(T, S)$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits on $K$. By Zorn’s Lemma, $\Gamma$ has a minimal member, say $F$. We complete the proof by verifying that $F$ consists of a single point. Assume that $x \in F_c$. In this case, for any $y \in F_c$ we have

$$\|Sx - y\| \leq \sup\{\|z - y\| : z \in F\} = r_y(F) = r(F),$$

therefore,

$$\sup\{\|Sx - y\| : x \in F_c\} \leq r(F).$$

Then,

$$r_{Sz}(F) = \sup\{\|Sx - y\| : y \in F\} \leq \sup\{\|Sx - y\| : x \in F_c, y \in F\} \leq \sup\{r(F), y \in F\} = r(F).$$
Then, for any $x \in F_c$ we have $r_{Sx}(F) = r(F)$; that is, $S : F_c \to F_c$. Moreover, for any $x, y \in F_c$ we have $\|Sx - Sy\| \leq r(F)$. On the other hand, for any $x, y \in F_c$,
\[
\delta_x[O^2(y; \infty)] = \sup_{n \in \mathbb{N}} \|x - T^{2n}y\|
\leq \sup\{\|x - z\| : z \in F\}
= r_x(F) = r(F).
\]
Similarly, for any $x, y \in F_c$ we have $\delta_y[O^2(x; \infty)] \leq r(F)$. In particular, for each $x, y \in F_c$,
\[
\|Tx - Ty\| \leq \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[O^2(y; \infty)], \delta_y[O^2(x; \infty)]\}
\leq \alpha(x, y)r(F) + (1 - \alpha(x, y)) r(F)
= r(F);
\]
that is, $r_{Tx}(F) = r(F)$. Then, $T : F_c \to F_c$. By Lemma 2.3, we have $F_c \in \Gamma$. If $\delta(F) > 0$, then by Lemma 2.4, $F_c$ is properly contained in $F$ which contradicts the minimality of $F$. Hence $\delta(F) = 0$ and $F$ consists of a single point; this is, there exists a point $p \in K$ such that $Tp = p$ and $Sp = p$. So, there exists a $p \in K$ such that $\|Tp - p\| = 0$. \hfill \Box

**Theorem 2.6.** Assume that $(A, B)$ is a nonempty pair of subsets in a Banach space $X$ with PNS. Let $T, S : A \cup B \to A \cup B$ be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ is a weakly compact and convex pair of subsets in $X$. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have
\[
\|Tp - Sp\| = \text{dist}(S(A), S(B)).
\]

**Proof.** The result follows from Theorem 2.5 if $\text{dist}(S(A), S(B)) = 0$, so we assume that $\text{dist}(S(A), S(B)) > 0$. Let $(S(A_0), S(B_0))$ be the associated proximal pair of $(S(A), S(B))$. We have already observed that $S(A_0)$ and $S(B_0)$ are nonempty, weakly compact and convex, moreover
\[
\text{dist}(S(A_0), S(B_0)) = \text{dist}(S(A), S(B)).
\]
Assume that $x \in A_0$, then there exists $y \in B_0$ such that $\|Sx - Sy\| = \text{dist}(S(A), S(B))$. On the other hand, $(T; S)$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits. Thus,
\[
\|T(Sx) - T(Sy)\| = \text{dist}(S(A), S(B)), \quad \|S(Sx) - S(Sy)\| = \text{dist}(S(A), S(B)).
\]
This implies that
\[
\|S(Sx) - S(Sy)\| = \text{dist}(S(A_0), S(B_0)),
\]
and
\[
\|T(Sx) - T(Sy)\| = \text{dist}(S(A_0), S(B_0)).
\]
Therefore, we have
\[
T(Sx) \in S(B_0), \quad T(Sy) \in S(A_0);
\]
\[
\|T(Sx) - T(Sy)\| = \text{dist}(S(A_0), S(B_0)).
\]
that is,
\[ T(S(A_0^a)) \subseteq S(B_0^a), \quad T(S(B_0^a)) \subseteq S(A_0^a). \]
Similarly,
\[ S(S(A_0^a)) \subseteq S(A_0^a), \quad S(S(B_0^a)) \subseteq S(B_0^a). \]
So, for each \( x \in A_0^a \) and \( y \in B_0^a \) we have
\[ \|T(Sx) - T(Sy)\| = \text{dist}(S(A_0^a), S(B_0^a)), \]
and
\[ \|S(Sx) - S(Sy)\| = \text{dist}(S(A_0^a), S(B_0^a)). \]
Clearly \((S(A_0^a), S(B_0^a))\) also has proximal normal structure. Now, assume that \( \Omega \) denotes the collection of all nonempty subsets \( S(F) \) of \( S(A_0^a) \cup S(B_0^a) \) for which \( S(F) \cap S(A_0^a) \) and \( S(F) \cap S(B_0^a) \) are nonempty, closed, convex, and such that
\[ T(S(F) \cap S(A_0^a)) \subseteq S(F) \cap S(B_0^a), \quad T(S(F) \cap S(B_0^a)) \subseteq S(F) \cap S(A_0^a), \]
and
\[ S(S(F) \cap S(A_0^a)) \subseteq S(F) \cap S(A_0^a), \quad S(S(F) \cap S(B_0^a)) \subseteq S(F) \cap S(B_0^a), \]
and so
\[ \text{dist}(S(F) \cap S(A_0^a), S(F) \cap S(B_0^a)) = \text{dist}(S(A), S(B)). \]
Since, \( S(A_0^a) \cup S(B_0^a) \in \Omega \) and \( \Omega \) is nonempty, we may assume that \( \{S(F_\alpha)\}_{\alpha \in \Omega} \) is a decreasing chain in \( \Omega \) such that \( S(F_0) = \cap_{\alpha \in \Omega} S(F_\alpha) \). Then \( S(F_0) \cap S(A_0^a) = \cap_{\alpha \in \Omega} (S(F_\alpha) \cap S(A_0^a)) \), so \( S(F_0) \cap S(A_0^a) \) is nonempty, closed and convex. Similarly, \( S(F_0) \cap S(B_0^a) \) is nonempty, closed and convex. Also,
\[ T(S(F_0) \cap S(A_0^a)) \subseteq S(F_0) \cap S(B_0^a), \quad T(S(F_0) \cap S(B_0^a)) \subseteq S(F_0) \cap S(A_0^a) \]
and
\[ S(S(F_0) \cap S(A_0^a)) \subseteq S(F_0) \cap S(A_0^a), \quad S(S(F_0) \cap S(B_0^a)) \subseteq S(F_0) \cap S(B_0^a). \]
To show that \( S(F_0) \in \Omega \) we only need to verify that
\[ \text{dist}(S(F_0) \cap S(A_0^a), S(F_0) \cap S(B_0^a)) = \text{dist}(S(A), S(B)). \]
Note that for each \( \alpha \in J \) it is possible to select \( Sx_\alpha \in S(F_\alpha) \cap S(A_0^a) \), \( Sy_\alpha \in S(F_\alpha) \cap S(B_0^a) \) such that
\[ \|Sx_\alpha - Sy_\alpha\| = \text{dist}(S(A), S(B)). \]
It is also possible to choose convergent subnets \( \{Sx_{\alpha'}\} \) and \( \{Sy_{\alpha'}\} \) (with the same indices), say
\[ \lim_{\alpha'} Sx_{\alpha'} = Sx, \quad \lim_{\alpha'} Sy_{\alpha'} = Sy. \]
Then clearly $Sx \in S(F_0) \cap S(A^+_0)$ and $Sy \in S(F_0) \cap S(B^+_0)$. By weak lower semicontinuity of the norm, we have $\|Sx - Sy\| \leq \text{dist}(S(A), S(B))$; hence,

$$\text{dist}(S(A), S(B)) \leq \text{dist}(S(F_0) \cap S(A^+_0), S(F_0) \cap S(B^+_0)) \leq \|Sx - Sy\| \leq \text{dist}(S(A), S(B)).$$

Therefore,

$$\text{dist}(S(F_0) \cap S(A^+_0), S(F_0) \cap S(B^+_0)) = \text{dist}(S(A), S(B)).$$

Since, every chain in $\Omega$ is bounded below by a member of $\Omega$, Zorn’s Lemma implies that $\Omega$ has a minimal element, say $S(K)$. Assume that $S(K_1) = S(K) \cap S(A^+_0)$ and $S(K_2) = S(K) \cap S(B^+_0)$. Observe that if

$$\delta(S(K_1), S(K_2)) = \text{dist}(S(K_1), S(K_2)),$$

then for any $x \in S(K_1)$, we have

$$\|Tx - Sx\| = \text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B)).$$

Similarly, for any $y \in S(K_2)$, we have

$$\|Ty - Sy\| = \text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B)).$$

Now, we assume that

$$\delta(S(K_1), S(K_2)) > \text{dist}(S(K_1), S(K_2)).$$

We complete the proof by showing that this leads to a contradiction. Since $S(K)$ is minimal, it follows that $(S(K_1), S(K_2))$ is a proximal pair in $(S(A^+_0), S(B^+_0))$. By the PNS property of $X$, there exist $(x_1, y_1) \in K_1 \times K_2$ and $\beta \in (0, 1)$ such that

$$\delta(Sx_1, S(K_2)) \leq \beta \delta(S(K_1), S(K_2)) \quad \text{and} \quad \delta(Sy_1, S(K_1)) \leq \beta \delta(S(K_1), S(K_2)).$$

Since, $(S(K_1), S(K_2))$ is a proximal pair, there exists $(x_2, y_2) \in K_1 \times K_2$ such that for each distinct $i, j \in \{1, 2\}$, we have

$$\|Sx_i - Sy_j\| = \text{dist}(S(K_1), S(K_2)).$$

So, for each $u \in S(K_2)$ we have

$$\|\frac{Sx_1 + Sx_2}{2} - u\| \leq \|\frac{Sx_1 - u}{2}\| + \|\frac{Sx_2 - u}{2}\| \leq \frac{\beta \delta(S(K_1), S(K_2))}{2} + \frac{\delta(S(K_1), S(K_2))}{2} = \alpha \delta(S(K_1), S(K_2)),$$

where $\alpha = \frac{1 + \beta}{2} \in (0, 1)$. Assume that $Sw_1 = \frac{(Sx_1 + Sx_2)}{2}$ and $Sw_2 = \frac{(Sy_1 + Sy_2)}{2}$. Then

$$\delta(Sw_1, S(K_2)) \leq \alpha \delta(S(K_1), S(K_2)) \quad \text{and} \quad \delta(Sw_2, S(K_1)) \leq \alpha \delta(S(K_1), S(K_2)).$$
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Since,
\[
\text{dist}(S(K_1), S(K_2)) \leq \|Sw_1 - Sw_2\|
\]
\[
= \left\| \frac{Sx_1 + Sx_2}{2} - \frac{(Sy_1 + Sy_2)}{2} \right\|
\]
\[
\leq \frac{1}{2} \left( \|Sx_1 - Sy_2\| + \|Sx_2 - Sy_1\| \right)
\]
\[
= \text{dist}(S(K_1), S(K_2)),
\]
we have \(\|Sw_1 - Sw_2\| = \text{dist}(S(K_1), S(K_2))\). Put
\[
S(L_1) = \{Sx \in S(K_1) : \delta(Sx, S(K_2)) \leq \alpha \delta(S(K_1), S(K_2))\},
\]
\[
S(L_2) = \{Sy \in S(K_2) : \delta(Sy, S(K_1)) \leq \alpha \delta(S(K_1), S(K_2))\}.
\]
Then for each \(i = 1, 2\), \(S(L_i)\) is a nonempty, closed and convex subset of \(S(K_i)\) and since \(Sw_1 \in S(L_1)\) and \(Sw_2 \in S(L_2)\), we have
\[
\text{dist}(S(K_1), S(K_2)) \leq \text{dist}(S(L_1), S(L_2)) \leq \|Sw_1 - Sw_2\| = \text{dist}(S(K_1), S(K_2)).
\]
Therefore,
\[
\text{dist}(S(L_1), S(L_2)) = \text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B)).
\]
Now, assume that \(Sx \in S(L_1)\) and \(Sy \in S(K_2)\). Then \(Sx \in S(A^*_0)\) and \(Sy \in S(B^*_0)\); that is, \(x \in A^*_0\) and \(y \in B^*_0\). Thus,
\[
\|T(Sx) - T(Sy)\| = \text{dist}(S(A), S(B)) \leq \delta(Sx, S(K_2)) \leq \alpha \delta(S(K_1), S(K_2)).
\]
So, \(T(Sy) \in B(T(Sx); \alpha \delta(S(K_1), S(K_2))) \cap S(K_1);\) that is,
\[
T(S(K_2)) \subseteq B(T(Sx); \alpha \delta(S(K_1), S(K_2))) \cap S(K_1) =: S(K'_1).
\]
Clearly \(S(K'_1)\) is closed and convex. Also, if \(Sy \in S(K_2)\) satisfies \(\|Sx - Sy\| = \text{dist}(S(A), S(B))\), then
\[
\|T(Sx) - T(Sy)\| = \text{dist}(S(K_1), S(K_2)).
\]
Since, \(T(Sy) \in S(K'_1)\), we conclude that \(\text{dist}(S(K'_1), S(K_2)) = \text{dist}(S(A), S(B))\).
Therefore, \(S(K'_1) \cup S(K_2) \in \Omega\) and by the minimality of \(S(K)\) we must have \(S(K'_1) = S(K_1)\). Hence,
\[
S(K_1) \subseteq B(T(Sx); \alpha \delta(S(K_1), S(K_2))); \quad \text{that is,} \quad \delta(T(Sx), S(K_1)) \leq \alpha \delta(S(K_1), S(K_2))
\]
and since \(Sx \in S(L_1)\) was arbitrary, we obtain \(T(S(L_1)) \subseteq S(L_2)\). Similarly, \(T(S(L_2)) \subseteq S(L_1)\) and \(S(S(L_1)) \subseteq S(L_2)\). Thus, \(S(L_1) \cup S(L_2) \in \Omega\), but \(\delta(S(L_1), S(L_2)) \leq \alpha \delta(S(K_1), S(K_2))\), contradicting the minimality of \(S(K)\).

**Corollary 2.7.** Assume that \((A, B)\) is a nonempty pair of subsets in a uniformly convex Banach space \(X\). Let \(T, S : A \cup B \to A \cup B\) be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and such that \(T(A) \subseteq S(B)\) and \(T(B) \subseteq S(A)\). Suppose that \((S(A), S(B))\) is a bounded,
closed and convex pair of subsets in $X$. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have

$$\|Tp - Sp\| = \text{dist}(S(A), S(B)).$$

3. **Noncyclic-Noncyclic Pairs**

In this section we study the case in which both mappings are noncyclic. Indeed, we first introduce a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits, and proceed to study its best proximity points.

**Definition 3.1.** Assume that $(A, B)$ is a nonempty pair of subsets of a Banach space $X$ and $T, S : A \cup B \to A \cup B$ are two mappings. A pair $(T, S)$ is said to be a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits if $(T, S)$ is a noncyclic-noncyclic pair and for any $(x, y) \in A \times B$, if $\|x - y\| = \text{dist}(S(A), S(B))$, then

$$\|Tx - Ty\| = \text{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \text{dist}(S(A), S(B))$$

and otherwise, there exists a function $\alpha : A \times B \to [0, 1]$ such that

$$\|Tx - Ty\| \leq \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[O(y; \infty)], \delta_y[O(x; \infty)]\},$$

where, for any $(x, y) \in A \times B$

$$\delta_x[O(y; \infty)] = \sup_{n \in \mathbb{N}} \|x - T^n y\|, \quad \delta_y[O(x; \infty)] = \sup_{n \in \mathbb{N}} \|T^n x - y\|.$$

**Theorem 3.2.** Assume that $(A, B)$ is a nonempty pair of subsets in a strictly convex Banach space $X$ with PNS, and $T, S : A \cup B \to A \cup B$ is a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits such that $T(A) \subseteq S(A)$ and $T(B) \subseteq S(B)$. Suppose that $(S(A), S(B))$ is a weakly compact and convex pair of subsets in $X$. Then, there exists $x_0 \in A$ and $y_0 \in B$ such that

$$Tx_0 = x_0, \quad Ty_0 = y_0$$

and

$$\|x_0 - y_0\| = \text{dist}(S(A), S(B)).$$

**Proof.** Suppose that $(S(A_0^\ast), S(B_0^\ast))$ is the associated proximal pair of $(S(A), S(B))$, and choose $x \in A_0^\ast$. Then there exists $y \in B_0^\ast$ such that $\|Sx - Sy\| = \text{dist}(S(A), S(B))$, and furthermore

$$\|T(Sx) - S(Ty)\| = \text{dist}(S(A), S(B)) = \text{dist}(S(A_0^\ast), S(B_0^\ast)).$$

Thus, $T : S(A_0^\ast) \to S(A_0^\ast)$ and similarly, $T : S(B_0^\ast) \to S(B_0^\ast)$. Now let $\Omega$ denote the collection of nonempty subsets $S(F)$ of $S(A_0^\ast) \cup S(B_0^\ast)$ for which $S(F) \cap S(A_0^\ast)$ and $S(F) \cap S(B_0^\ast)$ are nonempty, closed and convex,

$$T(S(F) \cap S(A_0^\ast)) \subseteq S(F) \cap S(A_0^\ast), \quad T(S(F) \cap S(B_0^\ast)) \subseteq S(F) \cap S(B_0^\ast),$$

$$S(S(F) \cap S(A_0^\ast)) \subseteq S(F) \cap S(A_0^\ast), \quad S(S(F) \cap S(B_0^\ast)) \subseteq S(F) \cap S(B_0^\ast).$$
Weak proximal normal structure

and

\[ \text{dist}(S(F) \cap S(A^n_0), S(F) \cap S(B^n_0)) = \text{dist}(S(A), S(B)). \]

Since, \( S(A^n_0) \cup S(B^n_0) \in \Omega \), \( \Omega \) is nonempty. We proceed as in the proof of Theorem 2.6 to show that \( \Omega \) has a minimal element \( S(K) \). Assume that \( S(K_1) = S(K) \cap S(A^n_0) \), and \( S(K_2) = S(K) \cap S(B^n_0) \). First, assume that one of the sets is a singleton, say \( S(K_1) = \{ x \} \). Then \( Tx = x \) and if \( y \) is the unique point of \( S(K_2) \) for which \( \| x - y \| = \text{dist}(S(K_1), S(K_2)) \), it must be the case that \( Ty = y \). Since, \( \| y - x \| = \text{dist}(S(A), S(B)) \), we are finished. So, we may assume that \( S(K_1) \) and \( S(K_2) \) have positive diameter and because the space is strictly convex, this in turn implies that

\[ \delta(S(K_1), S(K_2)) > \text{dist}(S(K_1), S(K_2)). \]

We shall see that this leads to a contradiction. Since \( (S(A^n_0), S(B^n_0)) \) has proximal normal structure, we may define \( S(L_1) \) and \( S(L_2) \) as in the proof of Theorem 2.6. Choose \( Sx \in S(L_1) \). For any \( Sy \in S(K_2) \), we have \( Sx \in S(A^n_0) \) and \( Sy \in S(B^n_0) \); that is, \( x \in A^n_0 \) and \( y \in B^n_0 \). Thus, \( \| Sx - Sy \| = \text{dist}(S(A), S(B)) \) and so,

\[ \| T(Sx) - T(Sy) \| = \text{dist}(S(A), S(B)) \leq \delta(Sx, S(K_2)) \leq \alpha \delta(S(K_1), S(K_2)). \]

This implies that

\[ T(Sy) \in B(T(Sx); \alpha \delta(S(K_1), S(K_2))) \cap S(K_2), \]

thus,

\[ T(S(K_2)) \subseteq B(T(Sx); \alpha \delta(S(K_1), S(K_2))) \cap S(K_2). \]

It follows from the minimality of \( S(K) \) that \( S(K_2) \subseteq B(T(Sx); \alpha \delta(S(K_1), S(K_2))) \) and this in turn implies that

\[ \delta(T(Sx), S(K_2)) \leq \alpha \delta(S(K_1), S(K_2)). \]

Therefore, \( T(Sx) \in S(L_1) \); in fact \( T(S(L_1)) \subseteq S(L_1) \). Similarly, \( T(S(L_2)) \subseteq S(L_2) \), \( S(S(L_1)) \subseteq S(L_1) \) and \( S(S(L_2)) \subseteq S(L_2) \). Since, \( S(L_1) \) and \( S(L_2) \) are, respectively, nonempty, closed and convex subsets of \( S(K_1) \) and \( S(K_2) \) and since for \( \alpha < 1 \) we have

\[ \delta(S(L_1), S(L_2)) \leq \alpha \delta(S(K_1), S(K_2)), \]

which contradicts the minimality of \( S(K) \). \[ \square \]

**Corollary 3.3.** Assume that \( (A, B) \) is a nonempty pair of subsets in a uniformly convex Banach space \( X \) and \( T, S : A \cup B \to A \cup B \) is a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits such that \( T(A) \subseteq S(A) \) and \( T(B) \subseteq S(B) \). Suppose that \( (S(A), S(B)) \) is a bounded, closed and convex pair of subsets in \( X \). Then, there exists \( x_0 \in A \) and \( y_0 \in B \) such that

\[ Tx_0 = x_0, \quad Ty_0 = y_0 \]

and

\[ \| x_0 - y_0 \| = \text{dist}(S(A), S(B)). \]
4. WPNS AND CYCLIC-NONCYCLIC PAIRS

In this section, and under weak proximal normal structure, we discuss the coincidence quasi-best proximity point problem for pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits.

**Lemma 4.1.** Assume that \((A, B)\) is a nonempty pair of subsets in a Banach space \(X\), and \(T, S : A \cup B \to A \cup B\) is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits such that \(T(A) \subseteq S(B)\) and \(T(B) \subseteq S(A)\). Suppose that \((S(A), S(B))\) is a weakly compact and convex pair of subsets in \(X\). Then, there exists \((S(K_1), S(K_2)) \subseteq (S(A_1^0), S(B_1^0)) \subseteq (S(A), S(B))\) which is minimal with respect to being nonempty, closed, convex and \(T\) and \(S\)-invariant pair of subsets of \((S(A), S(B))\), such that

\[
\text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B)).
\]

Moreover, the pair \((S(K_1), S(K_2))\) is proximal.

**Proof.** The proof essentially goes in the same lines as in the proof of Theorem 2.6. We omit the details. \(\square\)

**Theorem 4.2.** Assume that \((A, B)\) is a nonempty pair of subsets in a Banach space \(X\) with WPNS, and \(T, S : A \cup B \to A \cup B\) is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits such that \(T(A) \subseteq S(B)\) and \(T(B) \subseteq S(A)\). Suppose that \((S(A), S(B))\) is a weakly compact and convex pair of subsets in \(X\). Then \((T; S)\) has a coincidence quasi-best proximity point.

**Proof.** By Lemma 4.1, assume that \((S(K_1), S(K_2))\) is a minimal, weakly compact, convex and proximal pair which is \(T\) and \(S\)-invariant, and such that \(\text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B))\). Notice that

\[
\overline{\text{co}}(T(S(K_1))) \subseteq S(K_2)
\]

and so,

\[
T(\overline{\text{co}}(T(S(K_1)))) \subseteq T(S(K_2)) \subseteq \overline{\text{co}}(T(S(K_2))).
\]

Similarly,

\[
T(\overline{\text{co}}(T(S(K_2)))) \subseteq \overline{\text{co}}(T(S(K_1)));
\]

that is, \(T\) is cyclic on \(\overline{\text{co}}(T(S(K_1))) \cup \overline{\text{co}}(T(S(K_2)))\).

On other hand, \(S\) is noncyclic on \(\overline{\text{co}}(S(S(K_1))) \cup \overline{\text{co}}(S(S(K_2)))\). The minimality of \((S(K_1), S(K_2))\) implies that

\[
\overline{\text{co}}(T(S(K_1))) = S(K_2) \quad \text{and} \quad \overline{\text{co}}(T(S(K_2))) = S(K_1).
\]

Besides,

\[
\overline{\text{co}}(S(S(K_1))) = S(K_1) \quad \text{and} \quad \overline{\text{co}}(S(S(K_2))) = S(K_2).
\]

We note that if \(\delta(S(K_1), S(K_2)) = \text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B))\), then every point of \(S(K_1) \cup S(K_2)\) is a coincidence quasi-best proximity point.
of \((T; S)\) and we are finished. Otherwise, since \((S(A), S(B))\) has WPNS, there exists a point \((x_1, y_1) \in K_1 \times K_2\) and \(c \in (0, 1)\), so that
\[
\delta(Sx_1, S(K_2)) \leq c \delta(S(K_1), S(K_2)), \quad \delta(Sy_1, S(K_1)) \leq c \delta(S(K_1), S(K_2)).
\]
Since \((S(K_1), S(K_2))\) is a proximal pair, there exists \((x_2, y_2) \in K_1 \times K_2\) such that
\[
\|Sx_1 - Sy_2\| = \|Sx_2 - Sy_1\| = \text{dist}(S(A), S(B)).
\]
Put \(Su := \frac{Sx_1 + Sx_2}{2}\) and \(Sv := \frac{Sy_1 + Sy_2}{2}\). Then, \((Su, Sv) \in (K_1)^2 \times S(K_2)\) and
\[
\|Su - Sv\| = \text{dist}(S(K_1), S(K_2)).
\]
Moreover, for each \(z \in K_2\), we have
\[
\|Su - Sz\| = \|\frac{Sx_1 + Sx_2}{2} - Sz\|
\leq \frac{1}{2} \|\|Sx_1 - Sz\| + \|Sx_2 - Sz\||
\leq \frac{c + 1}{2} \delta(S(K_1), S(K_2)).
\]
Now, if \(r := \frac{c + 1}{2}\), then \(r \in (0, 1)\) and \(\delta(Su, (S(K_2)) \leq r \delta(S(K_1), S(K_2))\).
Similarly, we can see that \(\delta(Sv, (S(K_1)) \leq r \delta(S(K_1), S(K_2))\).
Assume that \(S(L_1) = \{Sx \in S(K_1) : \delta(Sx, S(K_2)) \leq r \delta(S(K_1), S(K_2))\}\),
\(S(L_2) = \{Sy \in S(K_2) : \delta(Sy, S(K_1)) \leq r \delta(S(K_1), S(K_2))\}\).
Thus, \((Su, Sv) \in S(L_1) \times S(L_2)\) and so, \(\text{dist}(S(L_1), S(L_2)) = \text{dist}(S(K_1), S(K_2))\).
Moreover, \((S(L_1), S(L_2))\) is a weakly compact and convex pair in \(X\). We show that \(T\) is cyclic on \(S(L_1) \cup S(L_2)\). Suppose \(Sx \in S(L_1)\) and \(Sy \in S(L_2)\). Then, similar to proof of Theorem 2.6, \(Sx \in S(A_0^0)\) and \(Sy \in S(B_0^0)\); that is, \(x \in A_0^0\) and \(y \in B_0^0\). Thus,
\[
\|T(Sx) - T(Sy)\| = \text{dist}(S(A), S(B)) \leq \delta(Sx, S(K_2)) \leq r \delta(S(K_1), S(K_2)).
\]
So, \(T(Sy) \in B(T(Sx); r \delta(S(K_1), S(K_2)))\); that is,
\[
T(Sy) \in B(T(Sx); r \delta(S(K_1), S(K_2)))
\]
and
\[
S(K_1) = \text{conv}T(S(K_2)) \subseteq B(T(Sx); r \delta(S(K_1), S(K_2))).
\]
Therefore, \(\delta(T(Sx), S(K_1)) \leq r \delta(S(K_1), S(K_2))\); that is, \(T(Sx) \in S(L_1)\).
Thus, \(T(S(L_1)) \subseteq S(L_2)\). Similarly, \(T(S(L_2)) \subseteq S(L_1)\) and \(S(S(L_2)) \subseteq S(L_2)\). Hence, \(T\) is cyclic and \(S\) is noncyclic on \(S(L_1) \cup S(L_2)\).
The minimality of \((S(K_1), S(K_2))\) now implies that
\[
S(L_1) = S(K_1) \quad \text{and} \quad S(L_2) = S(K_2).
\]
Now, we have
\[
\delta(S(K_1), S(K_2)) = \sup_{x \in K_1} \delta(Sx, S(K_2)) \leq r \delta(S(K_1), S(K_2)),
\]
which is a contradiction. \(\square\)
5. Examples

We clarify the above results with some examples.

Example 5.1. Let \( A = [-4, 0] \) and \( B = [0, 4] \) be subsets of the uniformly convex Banach space \((\mathbb{R}, |.|)\). For any \( x \in A \cup B \) we define
\[
Tx = -\frac{1}{4}x, \quad Sx = \frac{1}{2}x.
\]
Then,
\[
T(A) = [0, 1] \subseteq [0, 2] = S(B), \quad T(B) = [-1, 0] \subseteq [-2, 0] = S(A).
\]
Moreover, for any \( (x, y) \in A \times B \), we define
\[
\alpha(x, y) = \begin{cases} 
0, & \text{if } x = y \\
1, & \text{if } x \neq y.
\end{cases}
\]
If \( (x, y) \in A \times B \) such that \( \|x - y\| = \text{dist}(S(A), S(B)) = 0 \), then \( x = y \) and
\[
\|Tx - Ty\| = \text{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \text{dist}(S(A), S(B)).
\]
Otherwise,
\[
\|Tx - Ty\| = \|y - \frac{1}{4}x\| = \frac{1}{2}\|y - \frac{1}{2}x\|
\]
\[
= \frac{1}{2}\|Sy - Sx\| = \frac{1}{2}\|Sx - Sy\|
\]
\[
\leq \|Sx - Sy\|
\]
\[
= \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[O^2(y; \infty)], \delta_y[O^2(x; \infty)]\}.
\]
Thus, \((T, S)\) is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and by Corollary 2.7, there exists \((x, y) \in A \times B\) such that
\[
\|Tx - Sx\| = \text{dist}(S(A), S(B)), \quad \|Ty - Sy\| = \text{dist}(S(A), S(B)).
\]

Example 5.2. Let \( A = [-4, -1] \) and \( B = [1, 4] \) be subsets in \((\mathbb{R}, |.|)\). Let \( K_1 = [-4, -2], K_2 = [2, 4] \) and
\[
Sx = \begin{cases} 
-\sqrt{-x} - 2, & \text{if } x \in A \setminus K_1 \\
\sqrt{x} + 2, & \text{if } x \in B \setminus K_2 \\
-3, & \text{if } x \in K_1 \\
3, & \text{if } x \in K_2.
\end{cases}
\]
Therefore, \( S \) is a noncyclic mapping. Moreover,
\[
S(A) = [-4, -3] \subseteq A, \quad S(B) = [3, 4] \subseteq B.
\]
So, \((S(A), S(B))\) is a closed, convex and bounded pair and we have
\[
\text{dist}(S(A), S(B)) = 6.
\]
Suppose that
\[
T_x = \begin{cases} \sqrt{x} + 2, & \text{if } x \in A \setminus K_1 \\ -\sqrt{x} - 2, & \text{if } x \in B \setminus K_2 \\ 3, & \text{if } x \in K_1 \\ -3, & \text{if } x \in K_2. \end{cases}
\]

Therefore, \(T\) is a cyclic mapping. Besides,
\[
T(A) = [3, 4] = S(B) \subseteq B, \quad T(B) = [-4, -3] = S(A) \subseteq A.
\]

Moreover, we suppose that for any \((x, y) \in A \times B\),
\[
\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in (A \setminus K_1) \times (B \setminus K_2) \\ 0, & \text{otherwise}. \end{cases}
\]

If \(\|x - y\| = \text{dist}(S(A), S(B))\), then \((x, y) \in K_1 \times K_2\) and we have
\[
\|Sx - Sy\| = \| -3 - 3\| = 6 = \text{dist}(S(A), S(B))
\]
and
\[
\|Tx - Ty\| = \|3 - (-3)\| = 6 = \text{dist}(S(A), S(B)).
\]

Otherwise, for any \((x, y) \in (A \setminus K_1) \times (B \setminus K_2)\), we have
\[
\|Tx - Ty\| = \|\sqrt{-x} + 2 - (-\sqrt{y} - 2)\|
= \|\sqrt{-x} + \sqrt{y} + 4\|
= \|\sqrt{y} + 2 - (-\sqrt{-x} - 2)\|
= \|Sy - Sx\| = \|Sx - Sy\|
\leq \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[O(y; \infty)], \delta_y[O(x; \infty)]\}.
\]

Thus, \((T; S)\) is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and by Corollary 2.7, there exists \((x, y) \in A \times B\) such that
\[
\|Tx - Sx\| = \text{dist}(S(A), S(B)), \quad \|Ty - Sy\| = \text{dist}(S(A), S(B)).
\]

In fact, for any \((x, y) \in K_1 \times K_2\), we have
\[
\|Tx - Sx\| = 6 = \text{dist}(S(A), S(B)), \quad \|Ty - Sy\| = 6 = \text{dist}(S(A), S(B)).
\]

We clarify the above result with an example.

**Example 5.3.** Assume that \(A = [-4, 0]\) and \(B = [0, 4]\) are subsets of \((\mathbb{R}, |.|)\). For any \(x \in A \cup B\), we set
\[
Tx = \frac{1}{4}x, \quad Sx = \frac{1}{2}x.
\]

Then,
\[
T(A) = [-1, 0] \subseteq [-2, 0] = S(A), \quad T(B) = [0, 1] \subseteq [0, 2] = S(B).
\]

Moreover, we suppose that for any \((x, y) \in A \times B\),
\[
\alpha(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}
\]
If \((x, y) \in A \times B\) such that \(\|x - y\| = \text{dist}(S(A), S(B)) = 0\), then \(x = y\) and
\[\|Tx - Ty\| = \text{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \text{dist}(S(A), S(B)).\]

Otherwise,
\[
\|Tx - Ty\| = \frac{1}{4} x - \frac{1}{4} y = \frac{1}{2} \|x - y\|
\leq \|Sx - Sy\|
= \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta(x, \infty), \delta(y, \infty)\}.
\]

Thus, \((T, S)\) is a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits, and by Corollary 3.3, there exists \((x_0, y_0) \in A \times B\) such that
\[\|x_0 - y_0\| = \text{dist}(S(A), S(B)).\]
In fact, for \(x_0 = 0\) and \(y_0 = 0\), we have \(Tx_0 = x_0, Ty_0 = y_0\) and
\[\|x_0 - y_0\| = \text{dist}(S(A), S(B)).\]

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