This document was last edited on 10 December 2020, 22:05

Efficient evaluation of a continued fraction involved in the calculation of the moments of the sample multivariate coefficient of variation squared

Vicent Giner-Bosch
Centre for Quality and Change Management,
Universitat Politècnica de València, Spain. vigibos@eio.upv.es

Abstract

This document presents a pseudocode routine to calculate the term $\mathcal{C}(a,z)$ described in the work by Giner-Bosch, Tran, Castagliola and Khoo [Qual. Reliab. Eng. Int. 35 (2019), pp. 1515–1541].

Introduction

Aiming at obtaining a reliable, stable way to calculate the mean and the standard deviation of the sample multivariate coefficient of variation (MCV) squared, $\hat{\gamma}^2$, Giner-Bosch et al. (2019) deduced that this was connected to the first and second raw moments of a doubly-noncentral F distribution, and they developed an effective way to compute these raw moments which involves the calculation of the continued fraction denoted by $\mathcal{C}(a,z)$ (see section 4.2.1 of the aforementioned paper for more details). This continued fraction is reproduced in equation (1).

$$\mathcal{C}(a,z) = \cfrac{1}{a + \cfrac{-az}{a+1+\cfrac{z}{a+2+\cfrac{-(a+1)z}{a+3+\cfrac{2z}{a+4+\cfrac{-(a+2)z}{a+5+\cfrac{3z}{a+6+\cdots}}}}}}} \qquad (1)$$

As mentioned by Giner-Bosch et al., this continued fraction is known to converge for any real value of a and z except for $a \in \{-1,-2,\dots\}$.

Keep in mind that, from a mathematical perspective, a continued fraction is an expression that involves an infinite number of calculations (nested fractions). Therefore, when using a continued fraction to calculate something, we are returning an approximation to the actual value that we want to calculate, since the number of steps must be finite. This approximation can be as accurate as needed, just by including enough nested fractions.

It is worth to mention that continued fractions are often regarded as an efficient, powerful way to evaluate functions (Press et al., 2007).

The calculation

Let us express the continued fraction $\mathcal{C}(a,z)$ in a more general way:

$$\mathcal{C}(a,z) = \cfrac{f_0(a,z)}{g_0(a,z) + \cfrac{f_1(a,z)}{g_1(a,z) + \cfrac{f_2(a,z)}{g_2(a,z) + \cfrac{f_3(a,z)}{g_3(a,z) + \cfrac{f_4(a,z)}{g_4(a,z) + \cdots}}}}} \, . \tag{2}$$

In our case, we have:

$$f_k(a,z) = \begin{cases} 1\,, & \text{if } k = 0 \\ \dfrac{k}{2} z\,, & \text{if } k \text{ is even}, k \neq 0 \\ -\left(a + \dfrac{k-1}{2}\right) z\,, & \text{if } k \text{ is odd} \end{cases} \tag{3}$$

and

$$g_k(a,z) = a + k\,, \quad \text{for } k \geq 0 \tag{4}$$

(see equations (A10) to (A12) of Giner-Bosch et al.).

Imagine now that we are interested in calculating $\mathcal{C}(a,z)$. We will use equation (2) to approximate its value using K nested fractions (with $K \geq 1$), which means:

$$\mathcal{C}(a,z) \approx \cfrac{f_0(a,z)}{g_0(a,z) + \cfrac{f_1(a,z)}{g_1(a,z) + \cfrac{\cdots}{\cdots + \cfrac{f_{K-1}(a,z)}{g_{K-1}(a,z) + \cfrac{f_K(a,z)}{g_K(a,z)}}}}} \, . \tag{5}$$

The value of K can be customised by the researcher. The higher the value of K, the more accurate the approximation will be. Giner-Bosch et al. reported that a value of $K = 300$ nested fractions was found to converge with sufficient accuracy.

The routine

The following routine in pseudocode implements equation (5).

Function $C(a, z, K)$

Input:

- Parameters a and z. $a \notin \{-1, -2, \dots\}$
- Number of nested fractions K. $K \in \{2, 4, 6, \dots\}$

Output: An approximation of $C(a, z)$ using K nested fractions.

// Initialisation

$s = 0$

// Calculation of $f_0, \dots, f_K$

$f_0 = 1$

For $k = 2$ **to** K **step** $+2$

$$f_{k-1} = -\left(a + \frac{k}{2} - 1\right) z$$

$$f_k = \frac{k}{2} z$$

End For

// Calculation of $g_0, \dots, g_K$

$g_k = a + k, \quad \text{for } k = 0, \dots, K$

// Calculation of the continued fraction from term $k = K$ backwards to term $k = 0$

$$s = \frac{f_K}{g_K}$$

For $k = K - 1$ **to** 0 **step** -1

$$s = \frac{f_k}{g_k + s}$$

End For

Return s

Discussion and concluding remarks

This way of evaluating $C(a, z)$ is quite obvious and simple. As a matter of fact, it is not recommended by Press et al. (2007), mainly because it requires to know how far we should go (i.e., how many nested fractions we must consider) to obtain a good approximation before starting. Other approaches and techniques to evaluate continued fractions are discussed by Press et al. The interested reader is encouraged to examine them. However, this way of calculating $C(a, z)$ was found to be fast and accurate enough by Giner-Bosch et al., while some other strategies for calculating the moments of $\hat{\gamma}^2$ (based on either continued fractions or numerical series) showed problems of computational stability, due to the particular behaviour of the random variable involved (the doubly-noncentral F).

Acknowledgments

VGB's research work is partially supported by the Ministerio de Ciencia e Innovación of Spain under grant No. PID2019-110442GB-I00.

How to cite this technical note

This pseudocode has been developed by VGB. When using this pseudocode in their research, the reader *should* make refence to it as:

> Giner-Bosch V (2020). Efficient evaluation of a continued fraction involved in the calculation of the moments of the sample multivariate coefficient of variation squared. Technical report DEIOAC-2020-01. Universitat Politècnica de València, Valencia. [http://hdl.handle.net/10251/156814]

Evidently, the paper by Giner-Bosch et al. (2019) should be also properly cited.

References

Giner-Bosch V, Tran KP, Castagliola P, Khoo MBC (2019). An EWMA control chart for the multivariate coefficient of variation. *Quality and Reliability Engineering International*, 35(6):1515–1541. [DOI:10.1002/qre.2459]

Press WH, Teukolsky SA, Vetterling WT, Flannery BP (2007). *Numerical Recipes. The Art of Scientific Computing*. 3rd ed. Cambridge University Press, Cambridge. [ISBN: 978-0-521-88068-8]