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A glance into the anatomy of monotonic maps

In memory of my dearest friend Natashka Gamzina-Kandaurova
May your light guide me to kindness

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Abstract

Given an autohomeomorphism on an ordered topological space or its subspace, we show that it is sometimes possible to introduce a new topology-compatible order on that space so that the same map is monotonic with respect to the new ordering. We note that the existence of such a re-ordering for a given map is equivalent to the map being conjugate (topologically equivalent) to a monotonic map on some homeomorphic ordered space. We observe that the latter cannot always be chosen to be order-isomorphic to the original space. Also, we identify other routes that may lead to similar affirmative statements for other classes of spaces and maps.

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1. Introduction

It is one of classical problems of various areas of topology if a given continuous map on a topological space with perhaps a richer structure has nice properties related to this rich structure. For clarity of exposition, let us agree on terminology. An *autohomeomorphism* on a topological space X is any homeomorphism of X onto itself. An open interval with end points a and b of a linearly ordered set L will be denoted by $(a,b)_L$. If it is clear what ordered set is under consideration, we simply write (a,b). The same concerns other types

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of intervals. Linearly ordered topological spaces are abbreviated as LOTS and their subspaces as GO-spaces. We will mostly be concerned with GO-spaces. It is due to Čech ([4]) that a Hausdorff space X is a GO-space if and only if a family of convex sets with respect to some ordering on X is a basis for the topology of X. Given a GO-space X, an order $\prec$ on X is said to be *GO-compatible* if some collection of $\prec$-convex subsets of $\langle X, \prec\rangle$ is a basis for the topology of X. Note that if X is a LOTS, a GO-compatible order on X need not witness the fact that X is a LOTS. Some of our attention will be directed at topological groups. Recall that G is a *topological group* if it is a topological space with a group operation $\cdot$ such that both $\cdot$ and the operation of taking the inverse (inversion) are continuous with respect to the topology of G. If G is Abelian and $a \in G$, the map $g \mapsto ag$ is called a *shift*. We will be concerned with the following general problem.

Problem 1.1. *Let X be a GO-space and let f be an autohomeomorphism on X. What conditions on X and/or f guarantee that X has a GO-compatible ordering with respect to which f is monotonic?*

Since monotonicity is an order-dependent concept, we will specify with respect to which ordering a map is monotonic. If no clarification is given, the assumed order is the original one and should be clear from the context. Since our discussion will be around Problem 1.1, we will isolate the target property into a definition.

Definition 1.2. An autohomeomorphism f on a GO-space X is potentially monotonic if there exists a GO-compatible order on X with respect to which f is monotonic.

Definition 1.2 is equivalent to the following definition:

Definition 1.3 (Equivalent to 1.2)**.** An autohomeomorphism f on a GO-space X is potentially monotonic if there exists a GO-space Y, a homeomorphism $h : X \to Y$, and a monotonic autohomeomorphism m on Y such that $f = h^{-1} \circ m \circ h$.

To see why these two definitions are equivalent, let f be an autohomeomorphism on a GO-space X. Assume f is potentially monotonic by Definition 1.2. Fix a GO-compatible order $\prec$ on X with respect to which X is monotonic. Put $Y = \langle X, \prec\rangle$, $h = id_X$ (the identity map), and $m = f$. Clearly, $f = id_X^{-1} \circ m \circ id_X$. Hence, f is potentially monotonic with respect to Definition 1.3. We now assume that f is potentially monotonic with respect to Definition 1.3. Fix Y, f, m as in the definition. The order on Y induces an order $\prec$ on X as follows: $a \prec b$ if and only if $h(a) < h(b)$. Since h is a homeomorphism, $\prec$ is compatible with the GO-topology of X. Next, let us show that f is $\prec$-monotonic. We have $a \prec b$ is equivalent to $h(a) < h(b)$. By the choice of m, the latter is equivalent to $m \circ h(a) < m \circ h(b)$. By the definition of $\prec$, the latter is equivalent to $h^{-1} \circ m \circ h(a) \prec h^{-1} \circ m \circ h(b)$. Since $f = h^{-1} \circ m \circ h$, we conclude that $f(a) \prec f(b)$.

One may wonder if the property in Definition 1.2 is equivalent to the property of being topologically equivalent to a monotonic map with respect to the existing order. Recall that homeomorphisms $f, g : X \to X$ are *topologically equivalent* (or conjugate) if there exists a homeomorphism $t : X \to X$ such that $t \circ f = g \circ t$. A supported explanation will be given later in Remark 2.9 that a map can be potentially monotonic but not topologically equivalent to a monotonic map (with respect to the existing order). It is clear, however, from Definition 1.3 that a map topologically equivalent to a monotonic map is potentially monotonic. In our arguments, given a monotonic function f on a GO-space L and an $x \in L$, we will make a frequent use of the set $\{f^n(x) : n \in \mathbb{Z}\}$. In literature, similarly defined sets are often referred to as the orbit of x under f. We will also refer to this set as *the f-orbit of x*. Similarly, the *f-orbit* of a set $A \subset X$ is the collection $\{f^n(A) : n \in \mathbb{Z}\}$. Recall that an indexed family $\{S_i : i \in I\}$ of subsets of a space X is called *discrete* if any point $x \in X$ has a neighborhood U such that $|\{i \in I : S_i \cap U \neq \varnothing\}| \leq 1$. By looking at the behavior of monotonic maps on the reals, we quickly observe that the orbit of each point under such maps exhibits very strong properties. Namely, the following holds.

Proposition 1.4. *Let f be a fixed-point free monotonic autohomeomorphism on a GO-space L and $x \in L$. Then there exists an open neighborhood I of x such that the family $\{f^n(I) : n \in \mathbb{Z}\}$ is discrete.*

Proof. Without loss of generality, we may assume that f is strictly increasing. To avoid repetition, we next isolate a useful statement into a claim:

Claim. $S = \{f^n(\{x\}) : n \in \mathbb{Z}\}$ *is a discrete family of sets for any* $x \in L$.

To prove the claim, we first observe that the elements of S are distinct singletons. Indeed, by strict monotonicity, $f^n(x) \neq f^m(x)$ for distinct integers n and m.Therefore it suffices to show that S has no limit points. Assume the contrary and let y be a limit point for S. By monotonicity, $\lim_{n\to\infty} f^n(x) = y$ or $\lim_{n\to\infty} f^{-n}(x) = y$. By continuity, $f(y) = y$, contradicting the fact that f is fixed-point free. The claim is proved.

Fix $x \in L$. If x is isolated, then $I = \{x\}$ is as desired by Claim. Assume now that x is not isolated. Since f is an increasing homeomorphism, the intervals $(x, f(x))$ and $(f^{-1}(x), x)$ are not empty. Pick and fix $a \in (f^{-1}(x), x)$. Since f is strictly increasing, $f(a) \in (x, f(x))$. Let I be an open neighborhood of x such that the closure of I is a subset of $(a, f(a))$. Let us show that I is as desired. Fix $y \in L$. We need to find an open neighborhood U of y that meets $f^n(I)$ for at most one $n \in \mathbb{Z}$. We have three cases.

- Case ($y \in (f^n(a), f^{n+1}(a))$ for some integer n)**:** By monotonicity of f, the interval $(f^n(a), f^{n+1}(a))$ contains $f^n(I)$ and misses $f^m(I)$ for every other m. Therefore, $U = (f^n(a), f^{n+1}(a))$ is as desired.
- Case ($y = f^n(a)$ for some integer n)**:** Then $(f^{n-1}(a), f^{n+1}(a))$ contains y and meets $f^m(I)$ only for $m = n-1$ and $m = n$. Since the closure of I is in $(a, f(a))$, we conclude that $f^n(a)$ is not in the closure of $f^{n-1}(I)$ or

$f^n(I)$. Hence, there exists an neighborhood U of y that misses $f^m(I)$ for any m.

Case ($y \notin [f^n(a), f^{n+1}(a)]$ for any integer n): If y is not in the closure of $\bigcup_n [f^n(a), f^{n+1}(a)]$, then some neighborhood of y misses $[f^n(a), f^{n+1}(a)]$ for any n. Assume y is in the closure of $\bigcup_n [f^n(a), f^{n+1}(a)]$. By the case's condition, y must be a limit point for $\{f^n(\{a\}) : n \in \mathbb{Z}\}$, which is impossible by Claim.

Since we exhausted all cases, the proof is complete. □

We will next isolate the necessary condition identified in Proposition 1.4 into a property.

Definition 1.5. Let $f : X \to X$ be a map and $A \subset X$. The f-orbit of A is strongly discrete if there exists an open neighborhood U of A such that the family $\{f^n(U) : n \in \mathbb{Z}\}$ is discrete. The f-orbit of $x \in X$ is strongly discrete if the f-orbit of $\{x\}$ is strongly discrete.

In this note we will present partial results addressing Problem 1.1. At the end of our study we will identify a few questions that may have a good chance for an affirmative resolution.

In notation and terminology we will follow [3]. In particular, if $\prec$ is an order on L and $A, B \subset L$, by $A \prec B$ we denote the fact that $a \prec b$ for any $a \in A$ and $b \in B$.

2. Study

One may wonder if our introduction of the concepts of strongly discrete orbits is really necessary. Can we use the requirement of being "period-point free" instead? The next example shows that a periodic-point free autohomeomorphism even on a nice space need not have strongly discrete orbits.

Example 2.1. There exist a periodic-point free autohomeomorphism f of the space of rationals $\mathbb{Q}$ and a point $q \in \mathbb{Q}$ such that the f-orbit of q is not strongly discrete.

Proof. Example [1, Example 2.5] provides a construction of a fixed point autohomeomorphism f on the rationals that satisfies the hypothesis of Lemma [1, Lemma 2.4]. For convenience, the cited hypotheses is copied next:

Hypothesis of Lemma [1, Lemma 2.4]: *"Suppose $f : \mathbb{Q} \to \mathbb{Q}$ is not an identity map and $p \in \mathbb{Q}$ satisfy the following property:*

(*) $\forall n > 0 \exists m > 0$ such that $f^{m+1}((p-1/n, p+1/n)_{\mathbb{Q}})$ meets $f^{-m}((p-1/n, p+1/n)_{\mathbb{Q}})$."

Clearly an f that satisfies the above hypothesis fails having a strong f-orbit at p. □

For our next affirmative result we need a technical statement that incorporates our general strategy for showing that a map is potentially monotonic.

Lemma 2.2. *Let L be a GO-space and $f : L \to L$ an autohomeomorphism. Suppose that $\mathcal{O}$ is a collection of clopen subsets of L with the following properties:*

(1) *The f-orbit of each $O \in \mathcal{O}$ is strongly discrete.*
(2) $f^n(O) \cap f^m(O') = \varnothing$ *for distinct $O, O' \in \mathcal{O}$ and $n, m \in \mathbb{Z}$.*
(3) $\{f^n(O) : n \in \mathbb{Z}, O \in \mathcal{O}\}$ *is a cover of L .*

Then there exists a GO-compatible order $\prec$ on L with respect to which f is strictly increasing.

Proof. By $<$ we denote some ordering with respect to which L is a generalized ordered space. Enumerate elements of $\mathcal{O}$ as $\{O_\alpha : \alpha < |\mathcal{O}|\}$. We will define $\prec$ in three stages.

Stage 1: For each $O \in \mathcal{O}$ and $n \in \omega \setminus \{0\}$, define $\prec$ on $f^n(O)$ and $f^{-n}(O)$ recursively as follows:

Step 0: Put $\prec|_O = <|_O$.

Assumption: Assume that $\prec$ is defined on $f^k(O)$ and on $f^{-k}(O)$ for all $k = 0, 1, ..., n-1$.

Step n: If $x, y \in f^n(O)$, put $x \prec y$ if and only if $f^{-1}(x) \prec f^{-1}(y)$. This is well defined since $f^{-1}(x), f^{-1}(y)$ are in $f^{n-1}(O)$ and $\prec$ is defined on $f^{n-1}(O)$ by assumption. Similarly, if $x, y \in f^{-n}(O)$, put $x \prec y$ if and only if $f(x) \prec f(y)$.

Stage 2: For any $O \in \mathcal{O}$ and any $n, m \in \mathbb{Z}$ such that $n < m$, put $f^n(O) \prec f^m(O)$.

Stage 3: For any $\alpha < \beta < |\mathcal{O}|$ and any $n, m \in \mathbb{Z}$, put $f^n(O_\alpha) \prec f^m(O_\beta)$.

The next two claims show that $\prec$ is as desired.

Claim 1. $\prec$ *is compatible with the GO-topology of L.*

Proof of Claim. To prove the claim, for each $O \in \mathcal{O}$, let $\mathcal{T}_O$ be the collection of all $<$-convex open subsets of L that are subsets of O. Since $<$ coincides with $\prec$ one every $O \in \mathcal{O}$, we conclude that every element in $\mathcal{T}_O$ is $\prec$-convex. By the constructions at Stage 1, $f^n(O)$ is $\prec$-convex. Since f is an autohomeomorphism, the collection $\{f^n(I) : I \in \mathcal{T}_O, O \in \mathcal{O}\}$ is a basis for the topology of L and consists of open $\prec$-convex sets. The claim is proved.

Claim 2. f *is increasing with respect to* $\prec$.

Proof of Claim. Pick distinct x and y. If $x, y \in \bigcup_n f^n(O)$ for some $O \in \mathcal{O}$, then apply Stages 1 and 2. Otherwise, apply Stage 3. □

Remark to Lemma 2.2. *Note that if $\langle L, < \rangle$ is a LOTS and each O in the argument of the lemma has both extremities or each O has neither extremity, then $\langle L, \prec \rangle$ is a LOTS too.*

The converse of Lemma 2.2 for fixed-point free autohomeomorphisms on zero-dimensional GO-spaces holds too (Lemma 2.4). To prove the converse, we need the following quite technical statement. Recall that given a continuous self-map $f : X \to X$, a closed set $A \subset X$ is an *f-color* if $A \cap f(A) = \varnothing$. For a review of major results on colors of continuous maps, we refer the reader to [6].

Proposition 2.3. *Let L be a zero-dimensional GO-space, $f : L \to L$ a fixed-point free monotonic homeomorphism, and $x \in L$. Then there exists a maximal convex clopen set $I \subset L$ containing x such that the following hold:*

(1) $\bigcup_{n\in\mathbb{Z}} f^n(I)$ *is clopen and convex.*
(2) $f^n(I) \cap f^m(I) = \varnothing$ *for any distinct integers n and m.*
(3) $f^n(I)$ *is a maximal clopen convex f-color for any $n \in \mathbb{Z}$.*

Proof. We may assume that f is strictly increasing. Let dL be the largest ordered compactification of L. That is, if A and B are clopen subsets of L with the properties that $A < B$ and $A \cup B = L$, then $cl_{dL}(A)$ and $cl_{dL}(B)$ are clopen as well. Therefore, dL is zero-dimensional too, and f continuously extends to $\tilde{f} : dL \to dL$. The map f is an autohomeomorphism too and is increasing but not necessarily strictly.

Since dL is a zero-dimensional compact LOTS, any neighborhood of x contains $a \in dL$ such that a is the right member of a gap and $a \leq_{dL} x$. Since $\tilde{f}$ does not fix x we can select such an $a \in dL$ with an additional property that $\tilde{f}(a) >_{dL} x$. Put $I = [a, \tilde{f}(a))_L$, where the right-hand side is an abbreviation for $[a, \tilde{f}(a))_{dL} \cap L$. Let us show that I is as desired. By monotonicity, $\{f^n(I) : n \in \mathbb{Z}\} = \{..., [\tilde{f}^{-1}(a), a)_L, [a, \tilde{f}(a))_L, [\tilde{f}(a), \tilde{f}^2(a))_L, ...\}$. Enlarging any interval in this sequence would make that interval meet its image. Therefore, (3) is met. Visual inspection of the sequence is a convincing evidence that the union $\bigcup_n f^n(I)$ is convex. The union is also open as the union of open sets. Since f is fixed-point free, $f^n(I)$'s form a discrete collection, and hence, the union is closed. By our choice, $f^{n+1}(I) = [\tilde{f}^{n+1}(a), \tilde{f}^{n+2}(a))_L$, which guarantees that (2) is met. □

Lemma 2.4. *Let L be a zero-dimensional GO-space and let $f : L \to L$ be a fixed-point free monotonic autohomeomorphism. Then there exists a collection $\mathcal{O}$ of convex clopen subsets of L with the following properties:*

(1) *The f-orbit of each $O \in \mathcal{O}$ is strongly discrete.*
(2) $f^n(O) \cap f^m(O') = \varnothing$ *for distinct $O, O' \in \mathcal{O}$ and $n, m \in \mathbb{Z}$.*
(3) $\{f^n(O) : n \in \mathbb{Z}, O \in \mathcal{O}\}$ *is a cover of L .*

Proof. Without loss of generality, we may assume that f is strictly increasing. We will construct $\mathcal{O} = \{O_\alpha\}_\alpha$ recursively. Assume that O_β is constructed for each $\beta < \alpha$ and the following properties hold:

P1: $\bigcup_{n\in\mathbb{Z}} f^n(O_\beta)$ is clopen and convex.
P2: $f^n(O_\beta) \cap f^m(O_\beta) = \varnothing$ for any distinct integers n and m.
P3: $f^n(O_\beta)$ is a maximal clopen convex f-color for any $n \in \mathbb{Z}$.

Note that P1 and P2 imply the following:

P4: The f-orbit of O_β is strongly discrete.

Construction of O_α: Let $L_\alpha = L \setminus \bigcup\{f^n(O_\beta) : \beta < \alpha, n \in \mathbb{Z}\}$. If L_α is empty, then the recursive construction is complete and $\mathcal{O} = \{O_\beta : \beta < \alpha\}$. Otherwise, we have $f(L_\alpha) = f^{-1}(L_\alpha) = L_\alpha$. Let us show that L_α is clopen in L. Firstly, it is closed as the complement of the union of open sets. To show that it is open, fix $x \in L_\alpha$. Let I be as in Proposition 2.3 for given x, f, L. If x is a limit point for $L \setminus L_\alpha$, then it must contain some $f^n(O_\beta)$ for $\beta < \alpha$ and $n \in \mathbb{Z}$, which contradicts property P3. Hence, I is an open neighborhood of x contained in L_α. Since properties (1)-(3) of I in the conclusion of Proposition 2.3 coincide with the properties P1-P3, we can put $O_\alpha = I$.

The family $\mathcal{O} = \{O_\alpha\}_\alpha$ is as desired by construction. □

Lemmas 2.2 and 2.4 form the following criterion.

Theorem 2.5. *Let f be a fixed-point free autohomeomorphism on a zero-dimensional GO-space X. Then f is potentially monotonic if and only if there exists a collection $\mathcal{O}$ of convex clopen subsets of L with the following properties:*

(1) *The f-orbit of each $O \in \mathcal{O}$ is strongly discrete.*
(2) *$f^n(O) \cap f^m(O') = \varnothing$ for distinct $O, O' \in \mathcal{O}$ and $n, m \in \mathbb{Z}$.*
(3) *$\{f^n(O) : n \in \mathbb{Z}, O \in \mathcal{O}\}$ is a cover of L .*

We next put one part (Lemma 2.2) of the above criterion to a good use.

Theorem 2.6. *Let X be a zero-dimensional subspace of the reals and let $f : X \to X$ be an autohomeomorphism with strongly discrete orbits at all points. Then f is potentially monotonic.*

Proof. To prove the statement, we will construct a collection $\mathcal{O}$ as in the hypothesis of Lemma 2.2. Let $x \in X$ be an arbitrary point. Let U be an open neighborhood of x that witnesses the fact that x has a strongly discrete f-orbit. Let V be a clopen neighborhood of x that is a subset of U. Clearly, V witnesses the property too. Since V is its own neighborhood, V has a strongly discrete f-orbit as well. Therefore, we can fix a countable cover $\mathcal{F} = \{F_n : n \in \omega\}$ of X so that each F_i is clopen and has strongly discrete f-orbit.

Step 0*:* Put $O_0 = F_0$.

Assumption: Assume that O_k is defined for $k < n$, clopen, and has strongly discrete f-orbit. In addition, assume that $\bigcup_{m\in\mathbb{Z}} f^m(O_i)$ misses $\bigcup_{m\in\mathbb{Z}} f^m(O_j)$, whenever $i \neq j$ and $i, j < n$.

Step n: Let i_n be the smallest index such that F_{i_n} is not covered by $\{f^m(O_i) : i < n, m \in \mathbb{Z}\}$. Put $O_n = F_{i_n} \setminus \cup\{f^m(O_i) : i < n, m \in \mathbb{Z}\}$.

Construction is complete. The collection $\mathcal{O} = \{O_n : n \in \omega\}$ has properties (1) and (2) in the hypothesys of Lemma 2.2 by construction. To show (3), that is, the equality $X = \cup\{f^m(O_i) : i \in \omega, m \in \mathbb{Z}\}$, fix any $x \in X$. Since $\mathcal{F}$ is a cover of X, there exists n such that $x \in F_n$. If x is not in $f^m(O_i)$ for some $i < n$ and $m \in \omega$, then F_n is the first element in $\mathcal{F}$ that meets the construction requirements at step n. Therefore, $x \in O_n$. □

Corollary 2.7. *Every periodic-point free bijection on $\mathbb{Z}$ is potentially monotonic.*

In contrast with Corollary 2.7, we next observe that not every periodic-point free bijection on $\mathbb{Z}$ is topologically equivalent to a monotonic map.

Example 2.8. There exists a periodic-point free bijection on $\mathbb{Z}$ that is not topologically equivalent to a monotonic map.

Proof. First observe that every monotonic bijection on $\mathbb{Z}$ is a shift. Therefore, any bijection on $\mathbb{Z}$ that is topologically equivalent to a monotonic map is also topologically equivalent to a shift. It is observed in [2, Example 1.2] that if a bijection f on $\mathbb{Z}$ has infinitely many points with mutually disjoint orbits, then such a map is not topologically equivalent to a shift. Thus, any such fixed-point free map is an example of a potentially monotonic map on $\mathbb{Z}$ that is not topologically equivalent to a monotonic map. □

Remark 2.9. Corollary 2.7 and Example 2.8 imply that the property of being potentially monotonic does not imply the property of being topologically equivalent to a monotonic map (with respect to the existing order).

We can strengthen Corollary 2.7 as follows.

Theorem 2.10. *Let f be a periodic-point free bijection on $\mathbb{Z}$. Then there exist an ordering $\prec$ and a binary operation $\oplus$ on Z such that $\mathbb{Z}' = \langle \mathbb{Z}, \oplus, \prec \rangle$ is a discrete ordered topological group and f is a shift in $\mathbb{Z}'$.*

Proof. Let M be a minimal subset of $\mathbb{Z}$ with respect to the property that the f-orbit of M covers $\mathbb{Z}$.

If $|M| = n$, enumerate the elements of M by $\mathbb{Z}_n$. Clearly, $\mathbb{Z}_n \times_l \mathbb{Z}$ is an ordered discrete topological group with the component-wise addition. Define a bijection $h : \mathbb{Z} \to \mathbb{Z}_n \times_l \mathbb{Z}$ by letting $g(f^k(n_i)) = (i, k)$. Since any element of $\mathbb{Z}$ is in the f-orbit of exactly one element of M, the correspondence is well-defined and is a bijection. Since g is a homeomorphism, we will next abuse notation and will identify $f^k(x_i)$ with (i, k). Let us apply f to (i, k). We have $f(f^k(x_i)) = f^{k+1}(x_i)$, and the latter is identified with $(i, k + 1)$. Therefore, f is a shift by $(0, 1)$ in $\mathbb{Z}'$.

If M is infinite, enumerate its elements by integers as $M = \{n_i : i \in \mathbb{Z}\}$. Define $h : \mathbb{Z} \to \mathbb{Z} \times_l \mathbb{Z}$ by letting $g(f^k(n_i)) = (i, k)$. Argument similar to the $\mathbb{Z}_n$ case shows that the ordering on $\mathbb{Z}$ induced by h is as desired. □

Note that the above statement does not hold for continuous periodic-point free bijections on the rationals. Indeed, as shown in [1, Example 2.5] there exists a continuous periodic-point free bijection on $\mathbb{Q}$ with a point with non-strongly discrete fiber. The mentioned example [1, Example 2.5] is constructed to satisfy the hypothesis of [1, Lemma 2.4], which is a stronger case of not having discrete fibers. Nonetheless, the following takes place.

Theorem 2.11. *A fixed-point free autohomeomorphism $f : \mathbb{Q} \to \mathbb{Q}$ is potentially monotonic if and only if f is topologically equivalent to a shift.*

Proof. (⇒) Since f is potentially monotonic, there exists a collection $\mathcal{O}$ as in the conclusion of Lemma 2.4. The argument of Theorem 2.3 in [1] shows that f with such a collection is topologically equivalent to a non-trivial shift.

(⇐) It is proved in [2, Theorem 2.8] that a periodic-point free homeomorphism h on $\mathbb{Q}$ is topologically equivalent to a shift if and only if one can introduce a group operation $\oplus$ on $\mathbb{Q}$ compatible with the topology of $\mathbb{Q}$ so that the topological group $\langle \mathbb{Q}, \oplus \rangle$ is continuously isomorphic to $\mathbb{Q}$ and h is a shift with respect to new operation. Clearly such an $\langle \mathbb{Q}, \oplus \rangle$ is an ordered topological group, and hence, any shift is monotonic. Therefore, f is potentially monotonic. □

Recall that given a continuous selfmap $f : X \to X$, *the chromatic number of* f is the least number of f-colors needed to cover X.

Theorem 2.12. *Let f be a fixed-point free autohomeomorphism on a zero-dimensional GO-space L. If f is potentially monotonic, then the chromatic number of f is* 2*.*

Proof. (⇒) Since the chromatic number of f is a purely topological property not attached to an order, we may assume that f is strictly monotonic. Let $\mathcal{O}$ be as in the conclusion of 2.4 for the given f and L. Put $A = \cup\{f^n(O) : n \textit{ is an even integer}, O \in \mathcal{O}\}$ and $B = \cup\{f^n(O) : n \textit{ is an odd integer}, n \in \mathcal{O}\}$. Clearly, $\{A, B\}$ is cover of L by colors. □

Theorem 2.12 and Remark 2.9 prompt the following question.

Problem 2.13. *Let f be a periodic point free homeomorphism on a zero-dimensional GO-space L. Let the chromatic number of f be 2. Is f potentially monotonic?*

Theorem 2.6 prompts the following question.

Problem 2.14. *Let X be a GO-space and let $f : X \to X$ be an autohomeomorphism with strongly discrete orbits at all points. Is f potentially monotonic? What if X is hereditarily paracompact?*

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