

On soft quasi-pseudometric spaces

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ABSTRACT

In this article, we introduce the concept of a soft quasi-pseudometric space. We show that every soft quasi-pseudometric induces a compatible quasi-pseudometric on the collection of all soft points of the absolute soft set whenever the parameter set is finite. We then introduce the concept of soft Isbell convexity and show that a self non-expansive map of a soft quasi-metric space has a nonempty soft Isbell convex fixed point set.

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KEYWORDS: *soft-metric; soft-quasi-pseudometric; soft Isbell convexity.*

1. INTRODUCTION

Soft set theory has several applications in solving practical problems in economics, engineering, social sciences and medical science e.t.c. The study of soft sets was first initiated by Molodtsov [8] in 1999. Since then, many other scholars have taken interest in soft set theory (See [1], [2], [4], [15]). The study of soft metric spaces was initiated by Das and Samanta in [16]. Using the concept of a soft point in a soft set, they introduced a soft metric and some of their basic properties. Thereafter, they investigated some topological structures such as soft open sets, soft closed sets and soft closures of soft sets e.t.c. Furthermore, they investigated the notion of completeness of soft metric

spaces and the Cantor's Intersection Theorem.

Recently, Abbas et al. [5] studied the concept of fixed point theory of soft metric spaces. They showed that a soft metric induces a compatible metric on the collection of all soft points of the absolute soft set, whenever the set of parameters is finite. Thereafter, they used this concept to show that several fixed point theorems for metric spaces can be directly deduced from comparable existing results.

Until recently, most studies in topology has been based on spaces arising from a collection of metrics, which like the Euclidean distance, are symmetric. This was very natural since many problems it was used for were based on the Euclidean topology on the real numbers, which arises from the usual distance on reals numbers. But most topologies are not distance based, they are based on things like "effort" which have many properties of metrics but lack symmetry. A quasi-metric space is an example of a space which lack symmetry. It is also well known that quasi-metric spaces constitute an efficient tool to discuss and solve several problems in topological algebra, approximation theory, theoretical computer science, etc. (see [10]). On the other hand, T -theory is a theory that involves trees, injective envelopes of metric spaces (hyperconvex hull), and all of the areas that are connected with these topics. These topics have been used in the development of mathematical tools for reconstructing phylogenetic trees (see [6]). These are our motivations for generalising soft metric spaces to the asymmetric setting and introducing the concept of hyperconvexity in our new space.

In this article, we introduce soft quasi-pseudometric spaces, a concept that generalise soft metric spaces to the asymmetric setting. We then show that every soft quasi-pseudometric induces a compatible quasi-pseudometric on the collection of all soft points of the absolute soft set whenever the parameter set is finite. We then introduce the concept of hyperconvexity in soft quasi-pseudometric spaces, which we call soft Isbell convexity, and show that a self non-expansive map of a soft quasi-metric space has a nonempty soft Isbell convex fixed point set.

2. PRELIMINARIES

The letters U , E and $P(U)$ will denote the universal set, the set of parameters and the power set of U respectively.

According to [8], if F is a set valued mapping on $A \subset E$ taking values in $P(U)$, then the pair (F, A) is called a soft set over U . We will denote the collection of soft sets over a common universe U by $S(U)$.

A soft set (F, A) over U is said to be a soft point if there is exactly one $\lambda \in A$ such that $F(\lambda) = \{x\}$ and $F(e) = \emptyset$ for all $e \in A \setminus \{\lambda\}$. We shall denote such

a point by (F_λ^x, A) or simply F_λ^x . A soft point F_λ^x is said to belong to a soft set (F, A) , denoted by $F_\lambda^x \tilde{\in} (F, A)$, if $F_\lambda^x(\lambda) = \{x\} \subset F(\lambda)$.

The collection of soft points of (F, A) is denoted by $SP(F, A)$.

A soft set (F, E) is said to be a null soft set, denoted by Φ if for all $e \in A$, $F(e) = \emptyset$. A soft set which is not null is said to be a non-null soft set.

f is a soft mapping from the soft set (F, A) to a soft set (G, B) , denoted by $f : (F, A) \xrightarrow{\sim} (G, B)$, if for each soft point $F_\lambda^x \tilde{\in} (F, A)$ there exists only one soft point $G_\mu^y \tilde{\in} (G, B)$ such that $f(F_\lambda^x) = G_\mu^y$.

Now let \mathbb{R} be the set of real numbers. We denote the collection of all nonempty bounded subsets of \mathbb{R} by $B(\mathbb{R})$.

A soft real set, denoted by (\hat{f}, A) or simply \hat{f} is a mapping $\hat{f} : A \rightarrow B(\mathbb{R})$. If \hat{f} is a single valued mapping on $A \subset E$ taking values in \mathbb{R} , then the pair (\hat{f}, A) or simply \hat{f} is called a soft element of \mathbb{R} or a soft real number. If \hat{f} is a single valued mapping on $A \subset E$ taking values in the set \mathbb{R}^+ of nonnegative real numbers, then the pair (\hat{f}, A) , or simply \hat{f} , is called a nonnegative soft real number. We shall denote the set of nonnegative soft real numbers by $\mathbb{R}(A)^*$. A constant soft real number \bar{c} is a soft real number such that for each $e \in A$, we have $\bar{c}(e) = c$, where c is some real number.

Definition 2.1 ([8]). For two soft real numbers \hat{f}, \hat{g} we say that

- (i) $\hat{f} \tilde{\leq} \hat{g}$ if $\hat{f}(e) \leq \hat{g}(e)$ for all $e \in A$
- (ii) $\hat{f} \tilde{\geq} \hat{g}$ if $\hat{f}(e) \geq \hat{g}(e)$ for all $e \in A$
- (iii) $\hat{f} \tilde{<} \hat{g}$ if $\hat{f}(e) < \hat{g}(e)$ for all $e \in A$
- (iv) $\hat{f} \tilde{>} \hat{g}$ if $\hat{f}(e) > \hat{g}(e)$ for all $e \in A$

Definition 2.2. Let U be a universal set, A be a nonempty subset of parameters and \tilde{U} be the absolute soft set, i.e $F(\lambda) = U$ for all $\lambda \in A$, where $(F, A) = \tilde{U}$. A mapping $d : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ is said to be a soft pseudometric on \tilde{U} if for any $U_\lambda^x, U_\mu^y, U_\lambda^z \in SP(\tilde{U})$ (equivalently $U_\lambda^x, U_\mu^y, U_\lambda^z \tilde{\in} \tilde{U}$), the following hold:

- (i) $d(U_\lambda^x, U_\lambda^x) = \bar{0}$
- (ii) $d(U_\lambda^x, U_\mu^y) = d(U_\mu^y, U_\lambda^x)$
- (iii) $d(U_\lambda^x, U_\lambda^z) \tilde{\leq} d(U_\lambda^x, U_\mu^y) + d(U_\mu^y, U_\lambda^z)$

The soft set \tilde{U} endowed with a soft pseudometric d is called a soft pseudometric space and is denoted by (\tilde{U}, d, A) , or simply by (\tilde{U}, d) if no confusion arises.

Definition 2.3. Let U be a universal set, A be a nonempty subset of parameters and \tilde{U} be the absolute soft set, i.e $F(\lambda) = U$ for all $\lambda \in A$, where $(F, A) = \tilde{U}$. A mapping $d : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ is said to be a soft metric on \tilde{U} if for any $U_\lambda^x, U_\mu^y, U_\lambda^z \in SP(\tilde{U})$ (equivalently $U_\lambda^x, U_\mu^y, U_\lambda^z \tilde{\in} \tilde{U}$), the following hold:

- (i) $d(U_\lambda^x, U_\mu^y) = \bar{0}$ iff $U_\lambda^x = U_\mu^y$
- (ii) $d(U_\lambda^x, U_\mu^y) = d(U_\mu^y, U_\lambda^x)$
- (iii) $d(U_\lambda^x, U_\lambda^z) \lesssim d(U_\lambda^x, U_\mu^y) + d(U_\mu^y, U_\lambda^z)$

The soft set \tilde{U} endowed with a soft metric d is called a soft metric space and is denoted by (\tilde{U}, d, A) , or simply by (\tilde{U}, d) if no confusion arises.

3. SOFT QUASI-PSEUDOMETRIC SPACES

In this section, we introduce the concept of a soft quasi-pseudometric space. We show that the symmetrised soft pseudometric coincides with the soft pseudometric in the sense of [16].

Definition 3.1. Let U be a universal set, A be a nonempty subset of parameters and \tilde{U} be the absolute soft set. A mapping $q : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ is said to be a soft quasi-pseudometric on \tilde{U} if for any $U_\lambda^x, U_\mu^y, U_\lambda^z \in SP(\tilde{U})$ (equivalently $U_\lambda^x, U_\mu^y, U_\lambda^z \tilde{\in} \tilde{U}$), the following hold:

- (i) $q(U_\lambda^x, U_\lambda^x) = \bar{0}$
- (ii) $q(U_\lambda^x, U_\lambda^z) \lesssim q(U_\lambda^x, U_\mu^y) + q(U_\mu^y, U_\lambda^z)$.

We say q is a soft quasi-metric provided that q also satisfies the following condition:

$$q(U_\lambda^x, U_\mu^y) = \bar{0} = q(U_\mu^y, U_\lambda^x) \text{ implies } U_\lambda^x = U_\mu^y.$$

The soft set \tilde{U} endowed with a soft quasi-pseudometric is called a soft quasi-pseudometric space denoted by (\tilde{U}, q, A) or simply by (\tilde{U}, q) if no confusion arises.

Remark 3.2. If q is a soft quasi-pseudometric (soft quasi-metric) on \tilde{U} , then $q^t : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ and $q^s : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ defined by $q^t(U_\lambda^x, U_\mu^y) = q(U_\mu^y, U_\lambda^x)$ and $q^s(U_\lambda^x, U_\mu^y) = \max\{q(U_\lambda^x, U_\mu^y), q^t(U_\lambda^x, U_\mu^y)\}$ are also a soft quasi-pseudometric (soft quasi-metric) and soft pseudometric (soft metric) on \tilde{U} respectively. Note that q^s is a soft (pseudometric) metric in the sense of [16]. Furthermore, q^t is called the conjugate of q . Furthermore, we have

$$q(U_\lambda^x, U_\mu^y) \leq q^s(U_\lambda^x, U_\mu^y) \text{ and } q^t(U_\lambda^x, U_\mu^y) \leq q^s(U_\lambda^x, U_\mu^y).$$

Definition 3.3. Let (\tilde{U}, q) be a soft quasi-pseudometric space and \hat{r} be a non-negative soft real number. For any $U_\lambda^x \tilde{\in} \tilde{U}$, we define the open and closed balls with radius \hat{r} and center U_λ^x respectively as follows:

$$B_q(U_\lambda^x, \hat{r}) = \{U_\mu^y \tilde{\in} \tilde{U} : q(U_\lambda^x, U_\mu^y) \tilde{<} \hat{r}\}$$

and

$$C_q(U_\lambda^x, \hat{r}) = \{U_\mu^y \tilde{\in} \tilde{U} : q(U_\lambda^x, U_\mu^y) \tilde{\leq} \hat{r}\}.$$

$SS(B_q(U_\lambda^x, \hat{r}))$ is called the soft open ball with center U_λ^x and radius \hat{r} while $SS(C_q(U_\lambda^x, \hat{r}))$ is called the soft closed ball with center U_λ^x and radius \hat{r} .

Example 3.4. Let $U \subseteq \mathbb{R}$ be a non-empty set and $A \subset \mathbb{R}$ be the non-empty set of parameters. Let \tilde{U} be the absolute soft set, that is, $F(\lambda) = U \forall \lambda \in A$, where $(F, A) = \tilde{U}$. Let \bar{x} denote the soft real number such that $\bar{x}(\lambda) = x$ for all $\lambda \in A$. Furthermore, for constant soft real numbers \bar{x} and \bar{y} , put $\bar{x} \dot{-} \bar{y} = \max\{\bar{x} - \bar{y}, \bar{0}\}$. Then $q : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$, defined by

$$q(U_\lambda^x, U_\mu^y) = \bar{x} \dot{-} \bar{y} + \bar{\lambda} \dot{-} \bar{\mu},$$

is a soft quasi-metric.

Proof. (i) $q(U_\lambda^x, U_\lambda^x) = \bar{x} \dot{-} \bar{x} + \bar{\lambda} \dot{-} \bar{\lambda} = \bar{0}$
 (ii) $q(U_\lambda^x, U_\mu^y) = q(U_\mu^y, U_\lambda^x) = \bar{0} \implies \bar{x} \dot{-} \bar{y} + \bar{\lambda} \dot{-} \bar{\mu} = \bar{0} \implies \bar{x} \dot{-} \bar{y} = \bar{0}$ and $\bar{\lambda} \dot{-} \bar{\mu} = \bar{0} \implies \bar{x} = \bar{y}$ and $\bar{\lambda} = \bar{\mu} \implies U_\lambda^x = U_\mu^y$.
 (iii)

$$\begin{aligned} q(U_\lambda^x, U_\mu^y) &= \bar{x} \dot{-} \bar{y} + \bar{\lambda} \dot{-} \bar{\mu} \\ &= \bar{x} \dot{-} \bar{y} + \bar{z} \dot{-} \bar{z} + \bar{\lambda} \dot{-} \bar{\mu} + \bar{\gamma} \dot{-} \bar{\gamma} \\ &\stackrel{\sim}{\leq} \bar{x} \dot{-} \bar{z} + \bar{z} \dot{-} \bar{y} + \bar{\lambda} \dot{-} \bar{\gamma} + \bar{\gamma} \dot{-} \bar{\mu} \\ &\stackrel{\sim}{\leq} q(U_\lambda^x, U_\gamma^z) + q(U_\gamma^z, U_\mu^y). \end{aligned}$$

Therefore, (\tilde{U}, q, A) is a soft quasi-metric space. □

Remark 3.5. Notice in the example above that $q^t : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ defined by $q^t(U_\lambda^x, U_\mu^y) = q(U_\mu^y, U_\lambda^x)$ is also a soft quasi-metric on \tilde{U} . Furthermore, $q^s : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}(A)^*$ defined by

$$q^s(U_\lambda^x, U_\mu^y) = \max\{q(U_\lambda^x, U_\mu^y), q^t(U_\lambda^x, U_\mu^y)\} = |\bar{x} - \bar{y}| + |\bar{\lambda} - \bar{\mu}|$$

is a soft metric on \tilde{U} in the sense of [16].

Proposition 3.6. Let (\tilde{U}, q) be a soft quasi-pseudometric space and $U_\lambda^x \tilde{\in} \tilde{U}$. Then we have the following:

- (i) $B_{q^s}(U_\lambda^x, \hat{r}) \subseteq B_q(U_\lambda^x, \hat{r})$
- (ii) $C_{q^s}(U_\lambda^x, \hat{r}) \subseteq C_q(U_\lambda^x, \hat{r})$

Definition 3.7. A soft subset (Y, A) in a soft quasi-pseudometric space (\tilde{U}, q, A) is said to be $\tau(q)$ -soft open if for any soft point U_λ^x of (Y, A) , there exists a positive soft real number \hat{r} such that

$$U_\lambda^x \in B_q(U_\lambda^x, \hat{r}) \subset SP(Y, A).$$

Remark 3.8. The collection $\tau(q)$ of all $\tau(q)$ -soft open sets in a soft quasi-pseudometric space (\tilde{U}, q) form a $\tau(q)$ -soft topology on \tilde{U} . Similarly, the collection $\tau(q^t)$ of all $\tau(q^t)$ -soft open sets in a soft quasi-pseudometric space (\tilde{U}, q) form a $\tau(q^t)$ -soft topology on \tilde{U} . Furthermore, the collection $\tau(q^s)$ of all $\tau(q^s)$ -soft open sets in a soft quasi-pseudometric space (\tilde{U}, q) form a $\tau(q^s)$ -soft topology on \tilde{U} . The soft topology $\tau(q^s)$ is finer than the soft topologies $\tau(q)$ and $\tau(q^t)$. Finally the triple $(X, \tau(q), \tau(q^t))$ is a soft bitopological space.

Definition 3.9. Let (\tilde{U}, q) be a soft quasi-pseudometric space. A sequence $(U_{\lambda_n}^{x_n})_{n \in \mathbb{N}}$ of soft points in \tilde{U} is said to be $\tau(q)$ -convergent in (\tilde{U}, q) if there is a soft point $U_\mu^y \in \tilde{U}$ such that

$$q(U_{\lambda_n}^{x_n}, U_\mu^y) \rightarrow \bar{0} \text{ as } n \rightarrow \infty.$$

Definition 3.10. Let (\tilde{U}, q) be a soft quasi-pseudometric space. A sequence $\{U_{\lambda_i}^{x_i}\}_{i \in \mathbb{N}}$ of soft points in \tilde{U} is said to be $\tau(q^t)$ -convergent in (\tilde{U}, q) if there is a soft point $U_\mu^y \in \tilde{U}$ such that

$$q^t(U_{\lambda_n}^{x_n}, U_\mu^y) = q(U_\mu^y, U_{\lambda_n}^{x_n}) \rightarrow \bar{0} \text{ as } n \rightarrow \infty.$$

Proposition 3.11. Let $(U_{\lambda_n}^{x_n})_{n \in \mathbb{N}}$ be a sequence in a soft quasi-pseudometric space (\tilde{U}, q) . Then

- (i) if $(U_{\lambda_n}^{x_n})_{n \in \mathbb{N}}$ is $\tau(q)$ -convergent to U_λ^x and $\tau(q^t)$ -convergent to U_μ^y , then $q(U_\lambda^x, U_\mu^y) = \bar{0}$.
- (ii) if $(U_{\lambda_n}^{x_n})_{n \in \mathbb{N}}$ is $\tau(q)$ -convergent to U_λ^x and $q(U_\mu^y, U_\lambda^x) = \bar{0}$, then $(U_{\lambda_n}^{x_n})_{n \in \mathbb{N}}$ is $\tau(q)$ -convergent to U_μ^y .

Proof.

- (i) By letting $n \rightarrow \infty$ in the inequality $q(U_\lambda^x, U_\mu^y) \lesssim q(U_\lambda^x, U_{\lambda_n}^{x_n}) + q(U_{\lambda_n}^{x_n}, U_\mu^y)$, we get $q(U_\lambda^x, U_\mu^y) = \bar{0}$.
- (ii) Follows from the relation $q(U_\mu^y, U_{\lambda_n}^{x_n}) \lesssim q(U_\mu^y, U_\lambda^x) + q(U_\lambda^x, U_{\lambda_n}^{x_n}) \rightarrow \bar{0}$ as $n \rightarrow \infty$.

□

Definition 3.12. A sequence $(U_{\lambda_n}^{x_n})_{n \in \mathbb{N}}$ of soft points in a soft metric space (\tilde{U}, d) is said to be Cauchy in (\tilde{U}, d) if for each $\hat{\epsilon} \gtrsim \bar{0}$, there exists an $m \in \mathbb{N}$ such that $d(U_{\lambda_i}^{x_i}, U_{\lambda_j}^{x_j}) \lesssim \hat{\epsilon}$ for all $i, j \geq m$.

Definition 3.13. A soft metric space (\tilde{U}, d) is said to be complete if every Cauchy sequence in (\tilde{U}, d) converges to some soft point of \tilde{U} .

Definition 3.14. A soft quasi-metric space (\tilde{U}, q) is said to be bicomplete provided that (\tilde{U}, q^s) is a complete soft metric space.

4. THE COMPATIBLE QUASI-PSEUDOMETRIC

In [5], Abbas et al. introduced the concept of a compatible soft metric and used this concept to prove some fixed point theorems. In this section, we introduce the concept of a compatible soft quasi-pseudometric metric whose symmetrised (pseudo) metric coincides with the compatible metric in the sense of [5].

Theorem 4.1. Let (\tilde{U}, q, A) be a soft quasi-pseudometric with A a finite set. Define a function $M_q : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}^+$ as

$$M_q(U_\lambda^x, U_\mu^y) = \max_{\eta \in A} q(U_\lambda^x, U_\mu^y)(\eta)$$

for all $U_\lambda^x, U_\mu^y \in SP(\tilde{U})$. Then the following holds:

- (i) M_q is a quasi-pseudometric on $SP(\tilde{U})$
- (ii) M_q is a quasi-metric on $SP(\tilde{U})$ if and only if q is a soft quasi-metric on \tilde{U}
- (iii) (\tilde{U}, q, A) is bicomplete if and only if $(SP(\tilde{U}), M_q)$ is bicomplete.

Proof. Let $U_\lambda^x, U_\mu^y, U_\gamma^z \in SP(\tilde{U})$. Then we have

- (i) We first show that M_q satisfies the conditions of a quasi-pseudometric.
 - (i) $M_q(U_\lambda^x, U_\lambda^x) = 0$ by condition (i) of Definition 3.1.
 - (ii) $M_q(U_\lambda^x, U_\gamma^z) + M_q(U_\gamma^z, U_\mu^y)$ by condition (ii) of Definition 3.1. This is because

$$\begin{aligned} M_q(U_\lambda^x, U_\mu^y) &= \max_{\eta \in A} q(U_\lambda^x, U_\mu^y) \\ &\leq \max_{\eta \in A} q(U_\lambda^x, U_\gamma^z) + \max_{\eta \in A} q(U_\gamma^z, U_\mu^y) \\ &= M_q(U_\lambda^x, U_\gamma^z) + M_q(U_\gamma^z, U_\mu^y). \end{aligned}$$

Therefore, (\tilde{U}, M_q) is a quasi-pseudometric space.

- (ii) If q is a quasi-metric on \tilde{U} , then

$$M_q(U_\lambda^x, U_\mu^y) = M_q(U_\mu^y, U_\lambda^x) = 0 \implies U_\lambda^x = U_\mu^y.$$

- (iii) Suppose (\tilde{U}, q, A) is bicomplete. Then (\tilde{U}, q^s, A) is complete. Then by [5, Theorem 1], $(SP(\tilde{U}), (M_q)^s)$ is complete and so $(SP(\tilde{U}), M_q)$ is bicomplete. Conversely, suppose $(SP(\tilde{U}), M_q)$ is bicomplete. Then $(SP(\tilde{U}), (M_q)^s)$ is complete. Thus by [5, Theorem 1], (\tilde{U}, q^s, A) is complete. Therefore, (\tilde{U}, q, A) is bicomplete.

□

Proposition 4.2. *Let (\tilde{U}, d, A) be a soft quasi-pseudometric space. Then the function $(M_q)^t : SP(\tilde{U}) \times SP(\tilde{U}) \rightarrow \mathbb{R}^+$ defined by $(M_q)^t(U_\lambda^x, U_\mu^y) = M_q(U_\mu^y, U_\lambda^x)$ is also a quasi-pseudometric on $SP(\tilde{U})$. Moreover, $(M_q)^t = M_{q^t}$.*

Proof. One can easily check that (M_q) satisfies the axioms of a quasi-pseudometric. We only show that $(M_q)^t = M_{q^t}$. Observe that

$$(M_q)^t(U_\lambda^x, U_\mu^y) = M_q(U_\mu^y, U_\lambda^x) = \max_{\eta \in A} q(U_\mu^y, U_\lambda^x)(\eta) = \max_{\eta \in A} q^t(U_\lambda^x, U_\mu^y)(\eta) = M_{q^t}$$

□

Proposition 4.3. *Let (\tilde{U}, q, A) be a soft quasi-pseudometric space. Then for any soft point U_λ^x of \tilde{U} the following holds:*

- (i) $B_q(U_\lambda^x, \bar{r}) = B_{M_q}(U_\lambda^x, r)$ and $B_{q^t}(U_\lambda^x, \bar{r}) = B_{M_{q^t}}(U_\lambda^x, r)$
- (ii) $C_q(U_\lambda^x, \bar{r}) = C_{M_q}(U_\lambda^x, r)$ and $C_{q^t}(U_\lambda^x, \bar{r}) = C_{M_{q^t}}(U_\lambda^x, r)$

Proof. We show that $B_q(U_\lambda^x, \bar{r}) \subseteq B_{M_q}(U_\lambda^x, r)$, the rest follows the same arguments. Suppose $U_\mu^y \in B_q(U_\lambda^x, \bar{r})$. Then $q(U_\lambda^x, U_\mu^y) \lesssim \bar{r}$. This implies that $q(U_\lambda^x, U_\mu^y)(\eta) < \bar{r}(\eta)$ for all $\eta \in A$. Thus $\max_{\eta \in A} q(U_\lambda^x, U_\mu^y)(\eta) < r$. Therefore, $M_q(U_\lambda^x, U_\mu^y) < r$ and so $B_q(U_\lambda^x, \bar{r}) \subseteq B_{M_q}(U_\lambda^x, r)$. Conversely, suppose $U_\mu^y \in B_{M_q}(U_\lambda^x, r)$, then $M_q(U_\lambda^x, U_\mu^y) < r$ this implies that $\max_{\eta \in A} q(U_\lambda^x, U_\mu^y)(\eta) < r$.

Therefore, $q(U_\lambda^x, U_\mu^y)(\eta) < r(\eta)$ for all $\eta \in A$. Hence $q(U_\lambda^x, U_\mu^y) \lesssim \bar{r}$ and so $B_{M_q}(U_\lambda^x, r) \subseteq B_q(U_\lambda^x, \bar{r})$ □

5. SOFT ISBELL CONVEXITY

In this section, we extend the concept of Isbell convexity, introduced in [13] to soft quasi-pseudometric spaces.

Definition 5.1. A soft quasi-pseudometric space (\tilde{U}, q) is said to be soft Isbell convex provided that for each family $(U_{\lambda_i}^{x_i})_{i \in I}$ of soft points of \tilde{U} and families $(\bar{r}_i)_{i \in I}$ and $(\bar{s}_i)_{i \in I}$ of constant non-negative soft real numbers satisfying $q(U_{\lambda_i}^{x_i}, U_{\lambda_j}^{x_j}) \lesssim \bar{r}_i + \bar{s}_j$ whenever $i, j \in I$, the following holds:

$$\bigcap_{i \in I} SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)) \neq \Phi.$$

or equivalently

$$\bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) \neq \emptyset.$$

Lemma 5.2. *Suppose (\tilde{U}, q, A) , where A is finite, is a soft quasi-pseudometric space. Then (\tilde{U}, q, A) is soft Isbell convex if and only if $(SP(\tilde{U}), M_q)$ is Isbell convex.*

Proof. Suppose (\tilde{U}, q, A) is soft Isbell convex. Let $(U_{\lambda_i}^{x_i})_{i \in I}$ be a family of soft points of \tilde{U} and $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative real numbers satisfying $M_q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j}) \leq r_i + s_j$ whenever $i, j \in I$. Then

$$\max_{\eta \in A} q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j})(\eta) \leq r_i + s_j$$

whenever $i, j \in I$. Thus

$$q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j})(\eta) \leq r_i + s_j$$

for all $\eta \in A$ and $i, j \in I$. Thus $q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j}) \lesssim \bar{r}_i + \bar{s}_j$ whenever $i, j \in I$. Since (\tilde{U}, q, A) is soft Isbell convex, we have

$$\bigcap_{i \in I} \tilde{SS}(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} \tilde{SS}(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)) \neq \Phi.$$

or equivalently

$$\bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) \neq \emptyset.$$

Then by Proposition 4.3, we have

$$\bigcap_{i \in I} C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i) = \bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) \neq \emptyset.$$

Therefore, $(SP(\tilde{U}), M_q)$ is Isbell convex.

Conversely, suppose $(SP(\tilde{U}), M_q)$ is Isbell convex. Let $(U_{\lambda_i}^{x_i})_{i \in I}$ be a family of soft points of \tilde{U} and $(\bar{r}_i)_{i \in I}$ and $(\bar{s}_i)_{i \in I}$ be families of non-negative soft real numbers satisfying $q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j}) \lesssim \bar{r}_i + \bar{s}_j$ whenever $i, j \in I$. Then $q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j})(\eta) \leq (\bar{r}_i + \bar{s}_j)(\eta)$ for all $\eta \in A$ and $i, j \in I$. Thus

$$\max_{\eta} q(U_{\lambda_i}^{x_j}, U_{\lambda_j}^{x_j})(\eta) \leq r_i + s_j$$

whenever $i, j \in I$. This implies that $M_q(U_{\lambda_i}^{x_j}, U_{\lambda_i}^{x_j}) \leq r_i + s_j$ whenever $i, j \in I$.

By Isbell convexity of $(SP(\tilde{U}), M_q)$, we have

$$\bigcap_{i \in I} C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i) \neq \emptyset.$$

By Proposition 4.3, we have

$$\bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) = \bigcap_{i \in I} C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i) \neq \emptyset.$$

Hence

$$\bigcap_{i \in I} \tilde{SS}(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} \tilde{SS}(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)) \neq \Phi.$$

Therefore, (\tilde{U}, q, A) is soft Isbell convex. □

Definition 5.3. Let (\tilde{U}, q, A) be a soft quasi-pseudometric space. A family of soft double balls $[(SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)))_{i \in I}, (SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)))_{i \in I}]$, where \bar{r}_i and \bar{s}_i are non-negative constant soft real numbers and $U_{\lambda_i}^{x_i}$ is a soft point of \tilde{U} whenever $i \in I$, is said to have a mixed binary intersection property if for all indices $i, j \in I$,

$$SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} SS(C_{q^t}(U_{\lambda_j}^{x_j}, \bar{s}_j)) \neq \Phi$$

or equivalently,

$$C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_j}^{x_j}, \bar{s}_j) \neq \emptyset.$$

Definition 5.4. A soft quasi-pseudometric space (\tilde{U}, q, A) is said to be soft Isbell complete if for every family of soft double balls

$$[(SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)))_{i \in I}, (SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)))_{i \in I}],$$

where \bar{r}_i and \bar{s}_i are non-negative constant soft real numbers and $U_{\lambda_i}^{x_i}$ is a soft point of \tilde{U} whenever $i \in I$, having a mixed binary intersection property satisfy

$$\bigcap_{i \in I} SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)) \neq \Phi$$

or equivalently

$$\bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) \neq \emptyset.$$

Lemma 5.5. A soft quasi-pseudometric space (\tilde{U}, q, A) , where A is finite, is soft Isbell complete if and only if $(SP(\tilde{U}), M_q)$ is an Isbell complete quasi-pseudometric space.

Proof. Suppose (\tilde{U}, q, A) is soft Isbell complete. Let

$$[(C_{M_q}(U_{\lambda_i}^{x_i}, r_i))_{i \in I}, (C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i))_{i \in I}],$$

where r_i and s_i are non-negative real numbers and $U_{\lambda_i}^{x_i}$ is a soft point of \tilde{U} whenever $i \in I$, have a mixed binary intersection property. Then

$$C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_j}^{x_j}, s_j) \neq \emptyset.$$

By Proposition 4.3,

$$C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_j}^{x_j}, \bar{s}_j) \neq \emptyset.$$

Whenever $i, j \in I$. Hence the family of soft double balls

$$[(SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)))_{i \in I}, (SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)))_{i \in I}],$$

where \bar{r}_i and \bar{s}_i are non-negative constant soft real numbers and $U_{\lambda_i}^{x_i}$ is a soft point of \tilde{U} whenever $i \in I$, have a mixed binary intersection property. By Isbell

completeness of (\tilde{U}, q, A) , we have

$$\bigcap_{i \in I} SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)) \neq \Phi$$

or equivalently

$$\bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) \neq \emptyset.$$

Therefore by Proposition 4.3,

$$\bigcap_{i \in I} C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i) = \bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) \neq \emptyset.$$

Conversely, suppose $(SP(\tilde{U}), q, A)$ is Isbell complete. Let

$$[(SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)))_{i \in I}, (SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)))_{i \in I}],$$

where \bar{r}_i and \bar{s}_i are non-negative soft real numbers and $U_{\lambda_i}^{x_i}$ is a soft point of \tilde{U} whenever $i \in I$, be a family of soft double balls having a mixed binary intersection property. Then

$$SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} SS(C_{q^t}(U_{\lambda_j}^{x_j}, \bar{s}_j)) \neq \Phi$$

or equivalently,

$$C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_j}^{x_j}, \bar{s}_j) \neq \emptyset$$

whenever $i, j \in I$. By Proposition 4.3, we have

$$C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_j}^{x_j}, s_j) \neq \emptyset$$

whenever $i, j \in I$. Since $(SP(\tilde{U}), q, A)$ is Isbell complete, it follows that

$$\bigcap_{i \in I} C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i) \neq \emptyset.$$

By Proposition 4.3, we have

$$\bigcap_{i \in I} C_q(U_{\lambda_i}^{x_i}, \bar{r}_i) \cap C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i) = \bigcap_{i \in I} C_{M_q}(U_{\lambda_i}^{x_i}, r_i) \cap C_{M_{q^t}}(U_{\lambda_i}^{x_i}, s_i) \neq \emptyset.$$

Therefore,

$$\bigcap_{i \in I} SS(C_q(U_{\lambda_i}^{x_i}, \bar{r}_i)) \tilde{\cap} SS(C_{q^t}(U_{\lambda_i}^{x_i}, \bar{s}_i)) \neq \Phi.$$

Hence, (\tilde{U}, q, A) is soft Isbell complete. □

Definition 5.6. A soft quasi-pseudometric space (\tilde{U}, q, A) is said to be soft metrically convex if for any soft points U_{λ}^x and U_{μ}^y of \tilde{U} and non-negative constant soft real numbers \bar{r} and \bar{s} , such that $q(U_{\lambda}^x, U_{\mu}^y) \lesssim \bar{r} + \bar{s}$ there exists a

soft point U_λ^z of \tilde{U} such that $q(U_\lambda^x, U_\lambda^z) \leq \tilde{r}$ and $q(U_\lambda^z, U_\mu^y) \leq \tilde{s}$ or equivalently $C_q(U_\lambda^x, \tilde{r}) \cap C_{q^t}(U_\mu^y, \tilde{s}) \neq \emptyset$.

Remark 5.7. Notice that $C_q(U_\lambda^x, \tilde{r}) \cap C_{q^t}(U_\mu^y, \tilde{s}) \neq \emptyset$ is equivalent to $SS(C_q(U_\lambda^x, \tilde{r})) \cap SS(C_{q^t}(U_\mu^y, \tilde{s})) \neq \emptyset$.

Lemma 5.8. *A soft quasi-pseudometric space (\tilde{U}, q, A) , where A is finite, is soft metrically convex if and if $(SP(\tilde{U}), M_q)$ is metrically convex.*

Proof. Suppose (\tilde{U}, q, A) is soft metrically convex. Let U_λ^x and U_μ^y be soft points of \tilde{U} and r and s are non-negative real numbers such that $M_q(U_\lambda^x, U_\mu^y) \leq r + s$. Then $\max_{\eta \in A} q(U_\lambda^x, U_\mu^y)(\eta) \leq r + s$. Thus $q(U_\lambda^x, U_\mu^y)(\eta) \leq r + s$ for all $\eta \in A$. Therefore, $q(U_\lambda^x, U_\mu^y) \leq \tilde{r} + \tilde{s}$. By soft metric convexity of (\tilde{U}, q, A) , we have $C_q(U_\lambda^x, \tilde{r}) \cap C_{q^t}(U_\mu^y, \tilde{s}) \neq \emptyset$. By Proposition 4.3, we have $C_{M_q}(U_\lambda^x, r) \cap C_{M_{q^t}}(U_\mu^y, s) \neq \emptyset$. Therefore, $(SP(\tilde{U}), M_q)$ is metrically convex.

Conversely, suppose $(SP(\tilde{U}), M_q)$ is metrically convex. Let U_λ^x and U_μ^y be soft points of \tilde{U} and \tilde{r} and \tilde{s} are non-negative soft real numbers such that $q(U_\lambda^x, U_\mu^y) \leq \tilde{r} + \tilde{s}$. Then $q(U_\lambda^x, U_\mu^y) \leq r(\eta) + s(\eta)$ for all $\eta \in A$. Then $\max_{\eta \in A} q(U_\lambda^x, U_\mu^y)(\eta) \leq r + s$ and by soft metric convexity of $(SP(\tilde{U}), M_q)$, we have $C_{M_q}(U_\lambda^x, r) \cap C_{M_{q^t}}(U_\mu^y, s) \neq \emptyset$. Therefore, (\tilde{U}, q, A) is soft metrically convex. \square

Lemma 5.9. *A soft quasi-pseudometric space (\tilde{U}, q, A) , where A is finite, is soft Isbell convex if and if (\tilde{U}, q, A) is soft-Isbell complete and soft metrically convex.*

Proof. Suppose (\tilde{U}, q, A) is soft Isbell convex. Then by Lemma 5.2, $(SP(\tilde{U}), M_q)$ is Isbell convex. By [13, Lemma 3.1.1], $(SP(\tilde{U}), M_q)$ is Isbell complete and metrically convex. By Lemma 5.5 and Lemma 5.8, (\tilde{U}, q, A) is soft metrically convex and soft Isbell complete.

Conversely, suppose (\tilde{U}, q, A) is soft metrically convex and soft Isbell complete. Then by Lemma 5.5 and Lemma 5.8, $(SP(\tilde{U}), M_q)$ is Isbell complete and metrically convex. Therefore, by [13, Lemma 3.1.1] $(SP(\tilde{U}), M_q)$ is Isbell convex. Therefore, by Lemma 5.2, (\tilde{U}, q, A) is soft metrically convex. \square

Proposition 5.10. *Suppose (\tilde{U}, q, A) is a soft Isbell convex soft quasi-metric space. Then (\tilde{U}, q, A) is bicomplete.*

Proof. Since (\tilde{U}, q, A) is soft Isbell convex. Then $(SP(\tilde{U}), M_q)$ is Isbell convex. By [13, Corollary 3.1.3] $(SP(\tilde{U}), M_q)$ is bicomplete. This implies that (\tilde{U}, q, A) is bicomplete by Theorem 4.1. \square

Definition 5.11. A soft quasi-pseudometric space (\tilde{U}, q, A) is said to be bounded if for each soft points U_λ^x and U_μ^y of \tilde{U} , there exists a positive soft real number \hat{k} such that $q(U_\lambda^x, U_\mu^y) \lesssim \hat{k}$.

Remark 5.12. Notice that if (\tilde{U}, q, A) is a soft quasi-pseudometric space, where A is finite, then boundedness of (\tilde{U}, q, A) implies boundedness of $(SP(\tilde{U}), M_q)$.

Theorem 5.13. *If (\tilde{U}, q, A) , where A is finite, is a bounded Isbell convex soft quasi-metric space and $T : (\tilde{U}, q, A) \rightarrow (\tilde{U}, q, A)$ is a non-expansive map, then the fixed point set $Fix(T)$ of T in (\tilde{U}, q, A) is nonempty and soft Isbell convex.*

Proof. Since (\tilde{U}, q, A) is soft Isbell convex, then $(SP(\tilde{U}), M_q)$ is Isbell convex. Also, since (\tilde{U}, q, A) is bounded, then $(SP(\tilde{U}), M_q)$ is bounded by Remark 5.12. Furthermore, since $f : (\tilde{U}, q, A) \rightarrow (\tilde{U}, q, A)$ satisfies

$$q(f(U_\lambda^x), f(U_\mu^y)) \lesssim q(U_\lambda^x, U_\mu^y)$$

for all $U_\lambda^x, U_\mu^y \in SP(\tilde{U})$. Then we have

$$\begin{aligned} M_q(f(U_\lambda^x), f(U_\mu^y)) &= \max_{\eta \in A} q(f(U_\lambda^x), f(U_\mu^y))(\eta) \\ &\leq \max_{\eta \in A} q(U_\lambda^x, U_\mu^y)(\eta) \\ &= M_q(U_\lambda^x, U_\mu^y) \end{aligned}$$

Therefore, $f : (SP(\tilde{U}), M_q) \rightarrow (SP(\tilde{U}), M_q)$ is a non-expansive map and by [11, Theorem 3.3] $Fix(f)$ is nonempty and Isbell convex. By Lemma 5.2, $Fix(f)$ is soft Isbell convex. \square

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