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Convexity and boundedness relaxation for fixed point theorems in modular spaces

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Abstract

Although fixed point theorems in modular spaces have remarkably applied to a wide variety of mathematical problems, these theorems strongly depend on some assumptions which often do not hold in practice or can lead to their reformulations as particular problems in normed vector spaces. A recent trend of research has been dedicated to studying the fundamentals of fixed point theorems and relaxing their assumptions with the ambition of pushing the boundaries of fixed point theory in modular spaces further. In this paper, we focus on convexity and boundedness of modulars in fixed point results taken from the literature for contractive correspondence and single-valued mappings. To relax these two assumptions, we seek to identify the ties between modular and b-metric spaces. Afterwards we present an application to a particular form of integral inclusions to support our generalized version of Nadler's theorem in modular spaces.

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1. Introduction

Compared to 1922 when Banach fixed point theorem has been proved [8], certainly, fixed point theory now plays a significant and meaningful role in both the field of Mathematics and many real-life applications due to providing

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a general framework which opens the door to the development of many other approaches. In one approach, fixed point theory in modular spaces has received a lot of attention after being proposed as a generalization of normed spaces [39, 40, 42, 45, 46]. A growing literature on fixed point theorems in Modular spaces deals with rigorous formulations and proofs of many interesting problems which are applicable in a wide variety of settings, including Quantum Mechanics, Machine Learning and etc. Fixed point theory in modular spaces has its root in [27] by using some constructive techniques for single-valued mappings. This work has been widely cited as the inspiration for a variety of fixed point work along with [25, 26]. This line of work was extended by several works in a variety of ways. In one successful approach, in 1969, Nadler proposed the Banach contraction principle for multivalued mappings of in modular spaces [41]. A wide range of extensions was subsequently proposed by various authors, based on different relaxations [1, 18, 19, 31, 50, 51, 53]. Furthermore, authors in [34] focus on a particular case of multivalued mappings in modular spaces with a key property of modulars, additivity. Then, [3] explores the existence of fixed points of a specific type of G-contraction and G-nonexpansive mappings in modular function spaces.

In another approach, in 1993, Czerwik in [15, 16] proposed the first Banach's fixed point theorem for both single and multivalued mapping in b-metric spaces, introduced by Bourbaki and Bakhtin [7, 13]. Then, authors in [30] extended it for some particular types of contractions in the context of b-metric spaces. Along this direction, many researchers studied the extension of various well known fixed point results for various types of contractive mapping in the framework of b-metric spaces [12, 33, 47, 48, 57].

Although fixed point theory is shown to be successful in challenging problems and has contributed significantly to many real-world problems, various fixed point theorems strongly are proved under strong assumptions. In particular, in modular spaces, some of these assumptions can lead to having some induced norms. So, some assumptions that often do not hold in practice or can lead to their reformulations as a particular problem in a normed vector spaces. A recent trend of research has been dedicated to studying the fundamentals of fixed point theorems and relaxing their assumptions with the ambition of pushing the boundaries of fixed point theory in modular spaces further [2, 18, 27].

The aim of the work presented in this paper is to contribute to a deeper understanding of fixed point results in modular spaces and to improve their conditions and assumptions by addressing the open questions and challenges outlined in the literature by identifying the ties between modular spaces and b-metric spaces. In a bird-eyes view, the paper starts with Section 2 which is a brief introduction to modular and b-metric spaces along with the required concepts. Afterwards we describe the relation between these two particular spaces. Section 3 introduces some techniques and ways of improving some current fixed point results. Finally, an application to integral inclusions is provided in Section 5.

2. Background

This section will serve as an introduction to some fundamental concepts of modular and b-metric spaces. A detailed introduction can be found, for example, in the textbooks [4, 5, 7, 32, 40, 52].

2.1. **Modular spaces.** A modular space is a pair (X, ρ) where X is a real linear space and ρ is a real valued functional on X which satisfies the conditions:

(1) $\rho(x) = 0$ if and only if $x = 0$,
(2) $\rho(-x) = \rho(x)$,
(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for any nonnegative real numbers α, β with $\alpha + \beta = 1$.

The functional ρ is called a modular on X. There are many arguably important special instances of well known spaces in which these properties are fulfilled [44, 45, 46, 54]. Interestingly, it is shown that a modular induces a vector space $X_\rho = \{x \in X : \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}$ which is called a modular linear space. Furthermore, Musielak and Orlicz in [39, 45, 46] naturally provide the first definitions of the following key concepts in a modular space (X, ρ):

D1. A sequence x_n in $B \subseteq X$ is said to be ρ-convergent to a point $x \in B$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
D2. A ρ-closed subset $B \subseteq X$ is meant that it contains the limit of all its ρ-convergent sequences.
D3. A sequence x_n in $B \subseteq X$ is said to be ρ-Cauchy if $\rho(x_m - x_n) \to 0$ as $m, n \to \infty$.
D4. A subset B of X is said to be ρ-complete if each ρ-Cauchy sequence in B is ρ-convergent to a point of B.
D5. **ρ-bounded subsets:** A subset $B \subseteq X_\rho$ is called ρ-bounded if $\sup_{x,y \in B} \rho(x-y) < \infty$.
D6. **ρ-compact subsets:** A ρ-closed subset $B \subseteq X$ is called ρ-compact if any sequence $x_n \in B$ has a ρ-convergent subsequence.

For a modular space (X, ρ), the function ω_ρ which is said growth function [17] is defined on $[0, \infty)$ as follows:

$$\omega_\rho(t) = \inf\{\omega : \rho(tx) \leq \omega\rho(x) : x \in X, 0 < \rho(x)\}.$$

It is easy to show that when (X, ρ) satisfies $\omega_\rho(2) < \infty$, then every ρ-convergent sequence in (X, ρ) is ρ-Cauchy. Also, we note that in such case every ρ-compact set is ρ-bounded and ρ-complete [35].

2.2. **b-metric spaces.** Now, we turn our attention to another important and related space in the sense that it can be induced by a modular, namely b-metric spaces. It is shown that a modular induces some well known operators of which we are interested in b-metrics; a b-metric on a nonempty set X is a real function $d : X \times X \to [0, \infty)$ such that for a given real number $s \geq 1$ satisfies the conditions:

(1) $d(x,y)=0$ if and only if $x=y$,
(2) $d(x,y)=d(y,x)$,
(3) $d(x,z)\leq s[d(x,y)+d(y,z)]$, for all $x,y,z\in X$,

the pair (X,d) is called a b-metric space. As stated in [56], it is true that a b-metric space is not necessarily a metric space. With $s=1$, however, a b-metric space is a metric space. Furthermore, many concepts like convergent and Cauchy sequences, complete spaces, closed sets and etc are easily defined in b-metric spaces [11] which are denoted by b- convergent and b- Cauchy sequences, b- complete spaces, b- closed sets. Moreover, it has been shown that a lot of metric fixed point theorems can be extended to b-metric spaces [11, 58], although b-metric, in the general case, is not continuous, $\lim_{n\to\infty} x_n = x$ does not necessarily imply $\lim_{n\to\infty} d(x_n,y)=d(x,y)$ (see [6, 21] for further details).

The notion of b-metric spaces were introduced to reach the generalization of some known fixed point theorems for single valued mappings and correspondences [9, 10, 15, 16].

In the following example, we generalize some examples which are mentioned in [6].

Example 2.1. Suppose that (X,d) is a b-metric space with $s\geq 1$. Then (X,d^r) is a b-metric space, for all $r\in\mathbb{R}^+$. Since from the general form of Holder's inequality [55], for every $x,y,z\in X$ and $r\in\mathbb{R}^+$ with $1+\frac{1}{r}\geq 1$, we get

$$d(x,y)\leq s(d(x,z)+d(z,y))\leq (2s)(d^r(x,z)+d^r(z,y))^{\frac{1}{r}},$$

that is,

$$d^r(x,y)\leq (2s)^r(d^r(x,z)+d^r(z,y)).$$

This implies that, d^r is a b-metric. Since every metric d is a b-metric, then d^r is a b-metric. However, d^r is not necessarily to be a metric. For example, if $d(x,y)=|x-y|$ is the usual Euclidean metric, $d^2(x,y)=|x-y|^2$ is not a metric on $\mathbb{R}$.

We note that a modular ρ induces a b-metric. In fact, for such modular, we can define

$$d(x,y)=\rho(x-y).$$

Then, d is a b-metric with $s=\omega_\rho(2)$ and when (X,ρ) is a ρ-complete space, then (X,d) is a b-complete space. Actually, ρ-Cauchy sequence and ρ-convergent sequence are equivalent to b-Cauchy sequence and b-convergent sequence respectively. We recall that for any subset C of (X,ρ) a correspondence f on a set C, denoted by $f: C\twoheadrightarrow X$ assigns to each $a\in C$ a (nonempty) subset $f(a)$ of X and an element $x\in C$ is said to be a fixed point if $x\in f(x)$. A correspondence f is called continuous if $x_n\to x$ and $y_n\to y$ and $y_n\in f(x_n)$ imply $y\in f(x)$. For a correspondence f we define $d(a,f(b))=\inf\{d(a,y): y\in f(b)\}$ and $dist_\rho(a,f(b))=\inf\{\rho(a-y): y\in f(b)\}$. Also, Hausdorff distance is defined as

$$H_\rho(A,B)=\max\{\sup_{a\in A} dist_\rho(a,B), \sup_{b\in B} dist_\rho(A,b))\}$$

where A and B are subsets of C.

2.3. **Relevant Literature.** Much work has been done on the problem of the fixed point existence for single-valued mappings and in general correspondence in modular spaces [18, 24, 29]. Over the years, multiple authors have analyzed various conditions which suffice to guarantee the existence of fixed points for a board class of functions in modular spaces. Arguably, the following (H1)-(H4) conditions are specified to be some of the most common and popular ones in modular spaces:

(H1) Δ_2-condition: A modular ρ is said to satisfy the Δ_2-condition [45, 46] if $\rho(2x_n) \to 0$, whenever $\sup_n \rho(x_n) \to 0$ as $n \to \infty$.
(H2) Δ_2-type condition: A modular ρ is said to satisfy the Δ_2-type condition [45, 46] if there exists $k > 0$ such that $\rho(2x) \leq k\rho(x)$ for all $x \in X_\rho$.
(H3) $\tilde{s}$-convex modulars: If condition (3) in the modular definition is replaced by $\rho(\alpha x + \beta y) \leq \alpha^{\tilde{s}}\rho(x) + \beta^{\tilde{s}}\rho(y)$ for all $\alpha, \beta \in [0, \infty)$ with $\alpha^{\tilde{s}} + \beta^{\tilde{s}} = 1$ with an $\tilde{s} \in (0, 1]$, the modular ρ is called an $\tilde{s}$-convex modular [22]. In particular, a 1-convex modular is simply called convex.
(H4) Fatou property: A modular ρ has the Fatou property [14] if $\rho(x) \leq \liminf \rho(x_n)$, whenever $x_n \to x$.

Some excellent overviews of (H1)-(H4) conditions are provided in [27, 22, 54]. It is shown that a modular ρ implies that

$$\|x\|_\rho = \inf\{a > 0 : \rho(\frac{x}{a}) \leq 1\},$$

defines an F-norm on X_ρ. Specifically, if ρ is convex, $\|\cdot\|_\rho$ is a norm and it is frequently called the Luxemburg norm [23]. Note that a modular space determined by a function modular ρ will be called a modular function space and will be denoted by L_ρ. Then, it is not difficult to show that $\|\cdot\|_\rho$ is an F-norm induced by ρ. More importantly, $(L_\rho, \|\cdot\|_\rho)$ is a complete space.

Being able to define such norm in a real vector space can lead to a smooth proof for many fixed point theorems in very specific modular spaces. For instance, an earlier Work on this topic goes back to Theorem 2-2 of [27] which was proposed in the early 1990s:

Theorem 2.2 ([27]). *Let ρ be a function modular satisfying the Δ_2-condition and let B be a $\|\cdot\|_\rho$-closed subset of L_ρ. Let $T : B \to B$ be a single-valued mapping such that $\rho(T(f) - T(g)) \leq k\rho(f - g)$ where $f, g \in B$ and $k \in (0, 1)$. Then T has a fixed point if $\sup_n(2T^n(f_0)) < 1$.*

Since then, there has been significant work on extending and improving this result further in many ways.

Ait Taleb and Hanebaly present some example illustrating that the following result (Theorem I-1 of [2]) tends to be more applicable than Theorem 2.2:

Theorem 2.3 ([2]). *Suppose that X_ρ is a ρ-complete modular space where ρ is an $\tilde{s}$-convex modular satisfying the Δ_2-condition and has the Fatou property.*

Moreover, assume that B is a ρ-closed subset of X_ρ and $T : B \to B$ is a single-valued mapping such that there are $c, k \in \mathbb{R}^+$ that $c > \max\{1, k\}$ and $\rho(c(T(x) - T(y))) \leq k^{\tilde{s}}\rho(x - y)$ where $x, y \in B$. Then T has a fixed point.

However, it should be stressed that Theorem 2.2 is not generalized by Theorem 2.3. As mentioned in [2], we can ask what if it is mentioned that they are unable to prove whether the conclusion of Theorem 2.3 is true if we have $c = 1$ and $0 < k < 1$. We note that our main result, Theorem 3.3, replied to this open question, also it generalized Theorem 2.2, when $w_\rho(2) < \infty$.

Various extensions were subsequently proposed by various authors, based on different relaxations that require: A recent extension of Theorem 2.2 to the cases of correspondence maps appeared in 2006 in Theorem 3-1 of [18] as:

Theorem 2.4 ([18])**.** *Let ρ be a convex function modular satisfying the Δ_2-type condition, B a nonempty ρ-bounded ρ-closed subset of L_ρ, and $f : B \twoheadrightarrow B$ a ρ-closed valued correspondence such that there exists a constant $k \in [0, 1)$ that*

$$H_\rho(f(f_1), f(f_2)) \leq k\rho(f_1 - f_2),$$

where $f_1, f_2 \in B$. Then f has a fixed point.

Additionally, in 2009, this result is improved to Theorem 2-1 of [34]:

Theorem 2.5 ([34])**.** *Let ρ be a convex modular satisfying Δ_2-type condition and $B \subset L_\rho$ be a nonempty ρ-closed ρ-bounded subset of the modular space L_ρ. Then any closed valued correspondence $f : B \twoheadrightarrow B$ such that for $f_1, f_2 \in B$ and $f_3 \in f(f_1)$, there is $f_4 \in f(f_2)$ such that $\rho(f_3 - f_4) \leq k\rho(f_2 - f_1)$, where $k \in (0, 1)$, has a fixed point.*

In both Theorems 2.4 and 2.5, it is assumed that the correspondence defined on a ρ-bounded subset of a modular space (X, ρ) with convex modular. In [35] Theorem 2-5, the correspondence has ρ-compact set values.

Theorem 2.6 ([35])**.** *Let B be a ρ-bounded subset of ρ-complete space (X, ρ). Let $f : B \twoheadrightarrow B$ be a correspondence with ρ-compact values that for each $x, y \in C$ and $z \in f(x)$, there exists $w \in f(y)$ such that*

$$\rho(z - w) \leq k\rho(x - y),$$

where $2kw_\rho(2)^2 < 1$. Then f has a fixed point.

Our main result, Theorem 3.3, is definitely a generalization of Theorems 2.4, 2.5, 2.6.

In the following sections, we provide certain conditions under which we can guarantee the existence of fixed points for myriad mappings and some strong assumptions such as the convexity of modulars and the ρ-boundedness of the domain of a correspondence are relaxed which can lead to making our theorems much stronger and more applicable.

3. Main results

In this section, we focus on the case of the ρ-complete modular space X and consider $f : B \twoheadrightarrow B$ is a correspondence with ρ-closed valued where B is a ρ-closed subset of X. Further, we assume $\omega_\rho(2) < \infty$. Also, to ease the notation let us now denote $X = (X, \rho)$.

To prove our main results, we make use of the following lemma suggested by the reviewer.

Lemma 3.1 ([36]). *Let (X, d) be a b-metric space with $s \geq 1$ and $\{x_n\}$ be a b-convergent sequence in X with $\lim_{n\to\infty} x_n = x$. Then for all $y \in X$,*

$$s^{-1}d(x, y) \leq \lim_{n\to\infty} \inf d(x_n, y) \leq \lim_{n\to\infty} \sup d(x_n, y) \leq sd(x, y).$$

We also use the following lemma taken from the literature to obtain our main results.

Lemma 3.2 ([38]). *A sequence $\{x_n\}$ in a b-metric space (X, d) is a b-Cauchy sequence if there exists $k \in [0, 1)$ such that*

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),$$

for every $n \in \mathbb{N}$.

Now we can state one of our main results which is an equivalent of Nadler's theorem in [41] in a modular space. We would like to highlight that while the convexity of ρ is required for both theorems 2.4 and 2.5, we show that it can be removed.

Theorem 3.3. *Consider $k \in [0, 1)$ and for every $y \in B$, there exists $w \in f(y)$ such that $\rho(z - w) \leq k\rho(x - y)$ for every $x \in B$ and $z \in f(x)$. Then f has a fixed point.*

Proof. Take $x_0 \in B$ and $x_1 \in f(x_0)$. We know from our assumption that for every $n \geq 1$ there exists $x_{n+1} \in B$ such that $x_{n+1} \in f(x_n)$ and

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}),$$

where $d(x, y) = \rho(x - y)$ is the b-metric induced by the modular ρ. Note that by Lemma 3.2, $\{x_n\}$ is a b-Cauchy sequence in the ρ-complete space B. Which means that there exists $x \in B$ such that $x_n \to x$ as $n \to \infty$.

On the other hand, from our assumption, it is true that for every $x_n \in f(x_{n-1})$, there exists $y_n \in f(x)$ such that

$$\rho(y_n - x_n) \leq k\rho(x - x_{n-1}).$$

It implies that $\lim \rho(y_n - x_n) = 0$, and as a result we have $\lim d(x_n, y_n) = 0$. By Lemma 3.1, $s^{-1} \lim d(y_n, x) \leq \lim d(y_n, x_n)$. Therefore, $\lim d(y_n, x) = 0$ which means $\lim \rho(y_n - x) = 0$. Since $y_n \in f(x)$ and $f(x)$ is ρ-closed, it concludes our proof. □

Example 3.4. Define the modular $\rho : \ell_\infty(\mathbb{R}) \to \mathbb{R}$ as follows:

$$\rho((x_n)) = \sup_{n\in\mathbb{N}} |x_n|,$$

for every $(x_n) \in \ell_\infty(\mathbb{R})$ and the correspondence $f : \ell_\infty(\mathbb{R}) \twoheadrightarrow \ell_\infty(\mathbb{R})$ by

$$f((x_n)) = \{(\frac{1}{i}, \frac{x_1}{2}, \frac{x_2}{2}, \dots) : i \in \mathbb{N}\}.$$

So $(\ell_\infty(\mathbb{R}), \rho)$ is a ρ-complete space and the correspondence f satisfies the contraction of Theorem 3.3. Indeed, for $(x_n), (y_n) \in \ell_\infty(\mathbb{R})$, and

$$(z_n) = (\frac{1}{i}, \frac{x_1}{2}, \frac{x_2}{2}, \dots) \in f((x_n)),$$

there is $(w_n) = (\frac{1}{i}, \frac{y_1}{2}, \frac{y_2}{2}, \dots) \in f((y_n))$ such that

$$\rho((z_n) - (w_n)) = \sup_n |\frac{x_n - y_n}{2}| \leq \frac{1}{2} \sup_n |x_n - y_n| = \frac{1}{2}\rho((x_n) - (y_n))$$

Thus all assumptions of Theorem 3.3 are fulfilled and f has many fixed points such as $\{(\frac{1}{i}, \frac{1}{2i}, \frac{1}{4i}, \dots) : i \geq 2\}$.

The following result shows that Theorem 3.3 can be even further generalized:

Theorem 3.5. *Consider for every $x,\ y \in B$ and $z \in f(x)$, there exists $w \in f(y)$ such that*

$$\rho(z - w) \leq k \max\{\rho(x - y), \alpha\rho(x - z), \alpha\rho(y - w), \frac{\beta}{2}(\rho(x - w) + \rho(y - z))\},$$

where $\alpha, \beta \in [0, 1]$, and $k \in [0, 1)$. Then f has a fixed point if one of the following assumptions satisfies:

i: *f is continuous.*
ii: *ρ is continuous i.e. $\lim \rho(x_n) = \rho(x)$ as $x_n \to x$.*
iii: *$k\beta\omega_\rho(2) < 1$.*

Proof. Let us define a sequence $\{x_n\}$ with $x_0 \in B$, $x_1 \in f(x_0)$ and $x_{n+1} \in f(x_n)$ such that

$$\begin{aligned} \rho(x_{n+1} - x_n) \quad &\leq \quad k \max\{\rho(x_n - x_{n-1}), \alpha\rho(x_n - x_{n-1}), \\ &\qquad \alpha\rho(x_{n+1} - x_n), \frac{\beta}{2}\rho(x_{n-1} - x_{n+1})\}, \end{aligned} \tag{3.1}$$

for every $n \geq 1$. As becomes clear by equation (3.1), the right hand side of this equation is not $\alpha\rho(x_n - x_{n+1})$ or $\alpha\rho(x_n - x_{n-1})$. Now it is easy to see that

$$\rho(x_{n+1} - x_n) \leq k \max\{\rho(x_n - x_{n-1}), \frac{\beta\omega_\rho(2)}{2}(\rho(x_{n-1} - x_n) + \rho(x_n - x_{n+1}))\}, \tag{3.2}$$

Now we distinguish the two cases whether the right hand side of equation (3.2) is $\rho(x_n - x_{n-1})$ or

$$\frac{\beta\omega_\rho(2)}{2}(\rho(x_{n-1} - x_n) + \rho(x_n - x_{n+1})).$$

If the former is the case, then using equation (3.2), we have $\rho(x_{n+1} - x_n) \leq k\rho(x_n - x_{n-1})$. The latter leads to

$$\rho(x_{n+1} - x_n) \leq \frac{k\beta\omega_\rho(2)}{2 - k\beta\omega_\rho(2)}\rho(x_n - x_{n-1}).$$

Now, we are ready to derive a new upper bound for equation (3.2) as:

$$\rho(x_{n+1} - x_n) \leq \max\{k, \frac{k\beta\omega_\rho(2)}{2 - k\beta\omega_\rho(2)}\}\rho(x_n - x_{n-1}).$$

Thus it follows from Lemma 3.2 and ρ-completeness of B that there exists $x \in B$ such that $x_n \to x$ as $n \to \infty$.

The proof is obviously complete under assumption (i). Now assume (ii) holds. We know that for x_n, there exists $y_n \in f(x)$ such that

$$\begin{aligned} \rho(x_n - y_n) \quad &\leq \quad k \max\{\rho(x_{n-1} - x), \alpha\rho(x_{n-1} - x_n), \\ &\qquad \alpha\rho(y_n - x), \frac{\beta}{2}(\rho(x_{n-1} - y_n) + \rho(x - x_n))\}. \end{aligned} \tag{3.3}$$

By looking at the definition of the ρ-convergent sequences, it becomes clear that $\lim\limits_{x_n \to x} \rho(x_n - x) = 0$ and $\lim\limits_{x_n \to x} \rho(x_n - x_{n-1}) = 0$. Now (ii) leads to $\lim\limits_{x_n \to x} \rho(x_n - y_n) = \rho(x - y_n)$. Using this and equation (3.3), it is easy to derive that

$$\begin{aligned} \lim_{x_n \to x} \rho(x_n - y_n) \quad &\leq \quad k \max\{\lim_{x_n \to x} \alpha\rho(y_n - x_n), \lim_{x_n \to x} \frac{\beta\rho(y_n - x_n)}{2}\} \\ &\leq \quad k \lim_{x_n \to x} \rho(y_n - x_n), \end{aligned}$$

which yields $\lim\limits_{x_n \to x} \rho(x_n - y_n) = 0$. Now from the facts that $f(x)$ is ρ-closed and

$$\rho(y_n - x) \leq \omega_\rho(2)(\rho(x_n - x) + \rho(y_n - x_n)),$$

we have $\lim_{n \to \infty} y_n = x \in f(x)$.

Finally, if assumption (iii) is satisfied, it is possible to repeat the proof presented for assumption (ii) to get equation (3.3). Therefore, it follows that

$$\begin{aligned} \rho(x_n - y_n) \quad &\leq \quad k \max\{\rho(x_{n-1} - x), \alpha\rho(x_n - x_{n-1}), \alpha\rho(y_n - x), \\ &\qquad \frac{\beta}{2}(\rho(x_{n-1} - y_n) + \rho(x - x_n))\}, \\ &\leq \quad k \max\{\alpha\omega_\rho(2)\rho(y_n - x_n), \frac{\beta}{2}(\rho(x_{n-1} - y_n) + \rho(x - x_n))\}, \\ &\leq \quad k \max\{\alpha\omega_\rho(2)\rho(y_n - x_n), \\ &\qquad \frac{\beta}{2}[\omega_\rho(2)(\rho(x_n - x_{n-1}) + \rho(x_n - y_n)) + \rho(x - x_n)]\}, \\ &\leq \quad \max\{k\alpha\omega_\rho(2), \frac{k\beta\omega_\rho(2)}{2}\}\rho(x_n - y_n). \end{aligned}$$

This implies that $\lim_{n \to \infty} \rho(x_n - y_n) = 0$, by Lemma 3.1, $\lim \rho(y_n - x) = 0$ and then $x \in f(x)$. □

Example 3.6. For a function $p : \mathbb{N} \to [1, \infty)$, define the vector space

$$\ell_{p(\cdot)} = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \Sigma_{n=1}^{\infty} |\lambda x_n|^{p(n)} < \infty, \text{ for some } \lambda > 0\},$$

and the modular $\rho : \ell_{p(\cdot)} \to \mathbb{R}$ by

$$\rho((x_n)) = \Sigma_{n=1}^{\infty} |x_n|^{p(n)},$$

for every $(x_n) \in \ell_{p(\cdot)}$. Now define the corespondence $f : \ell_{p(\cdot)} \twoheadrightarrow \ell_{p(\cdot)}$ by

$$f((x_n)) = \{(\frac{sinx_1}{2}, \frac{x_2}{3}, \cdots, \frac{x_n}{n+1}, \ldots), (\frac{cosx_1}{2}, \frac{x_2}{3}, \cdots, \frac{x_n}{n+1}, \ldots)\},$$

for every $(x_n) \in \ell_{p(\cdot)}$. For $(x_n), (y_n) \in \ell_{p(\cdot)}$ and $(z_n) = (\frac{sinx_1}{2}, \frac{x_2}{3}, \cdots, \frac{x_n}{n+1}, \ldots)$, there is $(w_n) = \frac{siny_1}{2}, \frac{y_2}{3}, \cdots, \frac{y_n}{n+1}, \ldots)$ such that

$$\begin{aligned} \rho((z_n) - (w_n)) &= |\frac{sinx_1 - siny_1}{2}|^{p(1)} + \Sigma_{n=2}^{\infty} |\frac{x_n - y_n}{n+1}|^{p(n)}, \\ &\leq \frac{1}{2} \max\{\rho((x_n) - (y_n)), \rho((x_n) - (z_n)), \rho((y_n) - (w_n)), \\ &\quad \frac{1}{2}(\rho((x_n) - (w_n)) + \rho((y_n) - (z_n)))\}. \end{aligned}$$

Otherwise $(z_n) = (\frac{cosx_1}{2}, \frac{x_2}{3}, \cdots, \frac{x_n}{n+1}, \ldots)$, there is $(w_n) = \frac{cosy_1}{2}, \frac{y_2}{3}, \cdots, \frac{y_n}{n+1}, \ldots)$ such that

$$\begin{aligned} \rho((z_n) - (w_n)) &= |\frac{cosx_1 - cosy_1}{2}|^{p(1)} + \Sigma_{n=2}^{\infty} |\frac{x_n - y_n}{n+1}|^{p(n)}, \\ &\leq \frac{1}{2} \max\{\rho((x_n) - (y_n)), \rho((x_n) - (z_n)), \rho((y_n) - (w_n)), \\ &\quad \frac{1}{2}(\rho((x_n) - (w_n)) + \rho((y_n) - (z_n)))\}. \end{aligned}$$

Thus all assumptions of Theorem 3.5 are fulfilled and f has a fixed point (0).

From now on, for the sake of clearness of notation, we consider ψ to be a continuous and nondecreasing self-map on $[0, \infty)$ such that $\psi(t) = 0$ if and only if $t = 0$. This notation has been taken from the literature [28] and it was shown that, under mild assumptions, fixed point exists for many families.

Theorem 3.7. *Let $k \in [0, 1)$, $\alpha \geq 0$ and for every $x, y \in B$ we have*

$$H_{x,y} = \frac{dist_\rho(x, f(y)) + dist_\rho(y, f(x))}{2w_\rho(2)}.$$

Furthermore, suppose that for every $y \in B$ there is $w \in f(y)$ such that

$$\psi(\frac{1}{k}\rho(z - w)) \leq \psi(S(x, y)) + \alpha\psi(I(x, y)),$$

for every $x \in B$ and $z \in f(x)$ where

$$S(x, y) = \max\{\rho(x - y), dist_\rho(x, f(x)), dist_\rho(y, f(y)) \frac{1 + dist_\rho(x, f(x))}{1 + \rho(x - y)}, H_{x,y}\},$$

and

$$I(x,y)=\min\{dist_\rho(x,f(x))+dist_\rho(y,f(y)),dist_\rho(x,f(y)),dist_\rho(y,f(x))\}.$$

Then f has a fixed point, provided that it satisfies one of the following condition:

i: *f is continuous.*
ii: *ρ is continuous.*
iii: $kw_\rho(2)<1$.

Proof. Let $x_0\in X$ and $x_1\in f(x_0)$. By assumption, there exists $x_2\in f(x_1)$ such that

$$\psi(\frac{1}{k}\rho(x_2-x_1))\leq\psi(S(x_1,x_0))+\alpha\psi(I(x_1,x_0)).$$

Thus, one can define a sequence of $\{x_n\}$ in B such that

$$\psi(\frac{1}{k}\rho(x_n-x_{n+1}))\leq\psi(S(x_{n-1},x_n))+\alpha\psi(I(x_{n-1},x_n)),\tag{3.4}$$

where $x_n\in f(x_{n-1})$. On the other hand, taking into account that $\lim_{n\to\infty}I(x_{n-1},x_n)=0$, we have

$$\begin{aligned}S(x_{n-1},x_n)&=\max\{\rho(x_{n-1}-x_n),dist_\rho(x_{n-1},f(x_{n-1})),\\&\quad dist_\rho(x_n,f(x_n))\frac{1+dist_\rho(x_{n-1},f(x_{n-1}))}{1+\rho(x_{n-1}-x_n)},H_{x_{n-1},x_n}\},\\&\leq\max\{\rho(x_{n-1}-x_n),\rho(x_n-x_{n+1}),\\&\quad\frac{\rho(x_{n-1}-x_{n+1})+\rho(x_n-x_n)}{2w_\rho(2)}\},\\&=\max\{\rho(x_{n-1}-x_n),\rho(x_n-x_{n+1})\}.\end{aligned}$$

The right side of this inequality can be either $\rho(x_{n-1}-x_n)$ or $\rho(x_n-x_{n+1})$. However, the latter follows that

$$\psi(\frac{1}{k}\rho(x_n-x_{n+1}))\leq\psi(\rho(x_n-x_{n+1}))+\alpha\psi(0),$$

which gives a contradiction

$$\rho(x_n-x_{n+1})\leq k\rho(x_n-x_{n+1}),$$

based on the fact that ψ is nondecreasing. Therefore, we have

$$\psi(\frac{1}{k}\rho(x_n-x_{n+1}))\leq\psi(\rho(x_{n-1}-x_n))+\alpha\psi(0)=\psi(\rho(x_{n-1}-x_n)).$$

It leads to $\rho(x_n-x_{n+1})\leq k\rho(x_{n-1}-x_n)$ for every $n\in\mathbb{N}$. Lemma 3.2 implies that there exists $x\in B$ such that $x_n\to x$. Now, our goal is to prove $x\in f(x)$.

Obviously, x is a fixed point of f if (i) holds. By considering assumption (ii), it becomes obvious that $x\in f(x)$. For $x_n\in f(x_{n-1})$, there is $q_n\in f(x)$ such that

$$\varphi(\frac{1}{k}\rho(q_n-x_n))\leq\varphi(S(x_{n-1},x)+l\varphi(I(x_{n-1},x)).$$

We have

$$\begin{aligned} S(x_{n-1},x) &= \max\{\rho(x_{n-1}-x), dist_\rho(x_{n-1},f(x_{n-1})),\\ &\quad dist_\rho(x,f(x))\frac{1+dist_\rho(x_{n-1},f(x_{n-1}))}{1+\rho(x_{n-1}-x)},\\ &\quad \frac{dist_\rho(x_{n-1},f(x))+dist_\rho(x,f(x_{n-1}))}{2w_\rho(2)}\},\\ &\le \max\{\rho(x_{n-1}-x),\rho(x_{n-1}-x_n),\\ &\quad dist_\rho(x,f(x))\frac{1+\rho(x_{n-1}-x_n)}{1+\rho(x_{n-1}-x)},\\ &\quad \frac{w_\rho(2)(dist_\rho(x,f(x))+\rho(x-x_{n-1}))+\rho(x-x_n)}{2w_\rho(2)}\}. \end{aligned}$$

This implies that $\lim S(x_{n-1},x) \le dist_\rho(x,f(x))$. Therefore

$$\lim \varphi(\frac{1}{k}\rho(q_n-x_n)) \le \varphi(dist_\rho(x,f(x))) \le \varphi(\lim \rho(x-q_n)).$$

So $\lim \rho(q_n-x_n) \le k \lim \rho(x-q_n)$. Since ρ is continuous and $x_n \to x$, $\lim \rho(q_n-x) \le k\rho(x-q_n)$. Thus $\lim \rho(q_n-x)=0$. This implies that, since $f(x)$ is closed and $q_n \in f(x)$, we have $x \in f(x)$. If condition (iii) is provided, since we have

$$\begin{aligned} \lim \varphi(\frac{1}{k}\rho(q_n-x_n)) &\le \varphi(dist_\rho(x,f(x))),\\ &\le \varphi(\rho(x-q_n)),\\ &\le \varphi(w_\rho(2)(\rho(x-x_n)+\rho(x_n-q_n))). \end{aligned}$$

Therefore $\lim \rho(x_n-q_n) \le kw_\rho(2) \lim \rho(x_n-q_n)$. Now, $kw_\rho(2) < 1$ implies $\lim \rho(x_n-q_n)=0$. Therefore, by Lemma 3.1, $q_n \to x$. Thus $x \in f(x)$. ☐

4. Addressing some open questions and challenges

A series of research articles addressing challenges and hitherto open questions in the context of fixed point theory in modular spaces has been presented [2, 49]. Our aim is to contribute to a deeper understanding of fixed point theorems in modular spaces and to extend them to further general cases. A way of doing this is to address open problems taken from the literature. For instance, Radenović et. al. in [49] considered the following open problem

If $T : B \to B$ is a single valued mapping such that

$$\rho(T(x)-T(y)) \le k\max\{\rho(x-y),\rho(x-T(x)),\rho(y-T(y)),\rho(x-T(y)),\rho(y-T(x))\},$$

for every $x,y \in B$ where $B \subseteq X$ and $k \in \mathbb{R}$, then under what constraints does T have a fixed point? and can answer this question under the constraints that $T : B \to B$ is a single valued mapping and $k \in (0, \frac{1}{w_\rho(2)(1+w_\rho(2))})$. However, there is no answer to this question in the case of multi-valued T or $k \ge \frac{1}{w_\rho(2)(1+w_\rho(2))}$.

The open questions that have arisen address open questions outlined in [49] existence of a fixed point for some certain maps:

Existing of a fixed point for T has been successfully shown if $k \in (0, \frac{1}{w_\rho(2)(1+w_\rho(2))})$ (given by Radenovic et. al. in [49]). Interestingly, this question can be reformulated for correspondence, and we prove it with $k \in [0, \frac{1}{2w_\rho(2)}]$ in the next theorem.

Theorem 4.1. *Let f be a correspondence that for each x, $y \in B$ and $z \in f(x)$, there exists $w \in f(y)$ such that*

$$\rho(z-w) \leq k\max\{\rho(x-y), \rho(x-z), \rho(y-w), \rho(x-w), \rho(y-z)\},$$

where $2kw_\rho(2) < 1$. Then f has a fixed point.

Proof. We first find $x_1 \in f(x_0)$ for an arbitrary $x_0 \in B$. By assumption, for every $n \geq 1$ there exists $x_{n+1} \in f(x_n)$ such that
(4.1)
$$\rho(x_{n+1}-x_n) \leq k\max\{\rho(x_n-x_{n-1}), \rho(x_{n+1}-x_n), \rho(x_{n-1}-x_{n+1}), \rho(x_n-x_n)\},$$
It follow that

$$\begin{aligned}\rho(x_{n+1}-x_n) &\leq k\max\{\rho(x_n-x_{n-1}), \rho(x_{n+1}-x_n),\\ &\quad w_\rho(2)(\rho(x_{n-1}-x_n)+\rho(x_n-x_{n+1}))\},\\ &\leq kw_\rho(2)(\rho(x_{n-1}-x_n)+\rho(x_n-x_{n+1})\end{aligned}$$

which implies that

$$\rho(x_{n+1}-x_n) \leq k'\rho(x_{n-1}-x_n).$$

where $k' = \frac{kw_\rho(2)}{1-kw_\rho(2)}$. Note that k' is never larger than one Since $k < \frac{1}{2w_\rho(2)}$. In addition, from the fact that B is a ρ-complete set and x_n is a ρ-Cauchy sequence by Lemma 3.2, there exists $x \in B$ such that $x_n \to x$.

On the other hand, for every x_n, there exists $y_n \in f(x)$ such that

$$\begin{aligned}\rho(x_n-y_n) &\leq k\max\{\rho(x_{n-1}-x), \rho(x_{n-1}-x_n), \rho(y_n-x), \rho(y_n-x_{n-1}),\\ &\quad \rho(x_n-x)\},\\ &\leq k\max\{\rho(x_{n-1}-x), w_\rho(2)(\rho(y_n-x_n)+\rho(x_n-x)),\\ &\quad \rho(x_{n-1}-x_n), w_\rho(2)(\rho(y_n-x_n)+\rho(x_n-x_{n-1})), \rho(x_n-x)\}.\end{aligned}$$

Hence, it becomes obvious that $\lim\rho(x_n-y_n)=0$ by Lemma 3.1, $x \in f(x)$. □

5. Application to integral inclusions

As outlined in the introduction, a modular fixed point theorem can be used for providing sufficient (but not necessary) conditions for finding a real continuous function u defined on $[a,b]$ such that

$$u(t) \in v(t) + \gamma\int_a^b G(t,s)g(s,u(s))ds, \quad t \in [a,b], \tag{5.1}$$

where γ is a constant, $g : [a,b]\times\mathbb{R} \twoheadrightarrow [a,b]$ is lower semicontinuous, $G : [a,b]\times[a,b] \to [0,\infty)$ and $v : [a,b] \to \mathbb{R}$ are given continuous functions.

For simplicity we introduce the following shorthand notations. We use $X = C[a,b]$ to denote all real continuous functions defined on $[a,b]$, $g_u : [a,b] \twoheadrightarrow [a,b]$ where $g_u(s) = g(s, u(s))$ and a modular ρ defined on X as

$$\rho(u) = \max_{a \leq t \leq b} \mid u(t) \mid^2 .$$

It is not difficult to prove that (X, ρ) is a ρ-complete modular space. Now the aforementioned integral inclusion problem (5.1) can be reformulated as u is a solution of problem (5.1) if and only if it is a fixed point of $f : X \twoheadrightarrow X$ defined as

$$f(u) = \{x \in X : \ x(t) \in v(t) + \gamma \int_a^b G(t,s) g(s, u(s)) ds, \ t \in [a,b]\}.$$

Now we show under the following mild assumptions:

i: for all $x, y \in X$ and $w_x(t) \in g_x(t)$, there exists $h_y(t) \in g_y(t)$ such that

$$|w_x(t) - h_y(t)|^2 \leq \frac{1}{2s} \mid x(t) - y(t) \mid^2, \ t \in [a,b],$$

ii: $\max\limits_{a \leq t \leq b} \int_a^b G^2(t,z) dz \leq \frac{1}{b-a}$,

iii: $\mid \gamma \mid \leq 1$,

the correspondence f has a unique fixed point. So, we assume $x, y \in X$ and $w \in f(x)$ by definition, we have

$$w(t) \in v(t) + \gamma \int_a^b G(t,s) g(s, x(s)) ds = v(t) + \gamma \int_a^b G(t,s) g_x(s) ds.$$

By Michael's selection theorem (see Theorem 1 in [38]), it follows that there exists a continuous single valued mapping $w_x(s) \in g_x(s)$ that $w(t) = v(t) + \gamma \int_a^b G(t,s) w_x(s) ds$. According to assumption (i), for $w_x(s) \in g_x(s)$, there is an $h_y(s) \in g_y(s)$ such that

$$|w_x(s) - h_y(s)|^2 \leq \frac{1}{2s} \mid x(s) - y(s) \mid^2,$$

for all $s \in [a,b]$. We define

$$h(t) = v(t) + \gamma \int_a^b G(t,s) h_y(s) ds$$

which means that

$$h(t) \in v(t) + \gamma \int_a^b G(t,s) g_y(s) ds.$$

Therefore $h \in f(y)$. Using the Cauchy-Schwarz inequality and conditions (i-iii), we have

$$\begin{aligned}
\rho(w-h) &= \max_{a\leq t\leq b} |w(t)-h(t)|^2, \\
&= \max_{a\leq t\leq b} \mid v(t)+\gamma\int_a^b G(t,s)w_x(s)ds-(v(t)+\gamma\int_a^b G(t,s)h_y(s)ds)\mid^2, \\
&= \mid\gamma\mid^2 \max_{a\leq t\leq b} \mid \int_a^b G(t,s)(w_x(s)-h_y(s))ds \mid^2, \\
&\leq \mid\gamma\mid^2 \max_{a\leq t\leq b} \big\{ \int_a^b G^2(t,s)ds \int_a^b \mid w_x(s)-h_y(s)\mid^2 ds \big\}, \\
&= \mid\gamma\mid^2 \big\{ \max_{a\leq t\leq b} \int_a^b G^2(t,s)ds \big\}.\big\{ \int_a^b \mid w_x(s)-h_x(s)\mid^2 ds \big\}, \\
&\leq \frac{\mid\gamma\mid^2}{b-a} \big\{ \frac{1}{2s} \int_a^b \mid x(s)-y(s)\mid^2 ds \big\}, \\
&\leq \frac{\mid\gamma\mid^2}{2s(b-a)} \int_a^b \max_{a\leq s\leq b} \mid x(s)-y(s)\mid^2 ds, \\
&= \frac{\mid\gamma\mid^2}{2s} \max_{a\leq s\leq b} \mid x(s)-y(s)\mid^2, \\
&= \frac{1}{2s}\rho(x-y).
\end{aligned}$$

Theorem 3.3 implies that f has a unique fixed point $u \in X$, that is, the integral inclusion (5.1) has a solution which belongs to $C[a,b]$.

6. Conclusion

Our main results show that strong assumptions such as convexity and boundedness of modulars in fixed point results for contractive correspondence and single-valued mappings can be relaxed by making use of some ties between modular and b-metric spaces. Our approach in this work includes a unifying view on fixed point results to yield some assumptions which are more likely to hold in practice and reformulations as particular normed vector space problems are no longer required. In particular, a generalized version of Nadler's theorem along with an application in modular spaces is presented.

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