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Convexity and freezing sets in digital topology

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Abstract

We continue the study of freezing sets in digital topology, introduced in [4]. We show how to find a minimal freezing set for a "thick" convex disk X in the digital plane $\mathbb{Z}^2$. We give examples showing the significance of the assumption that X is convex.

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1. Introduction

We often use a digital image as a mathematical model of an object or a set of objects "pictured" by the image. Methods inspired by classical topology are used to determine whether a digital image has properties analogous to the topological properties of a "real world" object represented by the image. The literature now contains considerable success in adapting to digital topology notions from classical topology such as *connectedness, continuous function, homotopy, fundamental group, homology, automorphism group*, etc.

The fixed point properties (including "approximate fixed point" properties) of a digital image are in some ways similar and in other ways very different from those of the Euclidean object modeled by the image. This claim is borne out in such papers as [8, 6, 7].

A large number of papers have been written to study fixed point properties of digital images considered as *digital metric spaces* [12]. Most of assertions of these papers have been shown [9, 2, 3, 5] to be incorrect or trivial; this leads

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to the conclusion that the digital metric space is not an idea worth developing further.

Knowledge of the fixed point set $\text{Fix}(f)$ of a continuous self-map on a nontrivial topological space X rarely tells us much about $f|_{X\setminus\text{Fix}(f)}$. By contrast, it was shown in [9, 4] that knowledge of the fixed point set $\text{Fix}(f)$ of a digitally continuous self-map on a nontrivial digital image (X,κ) may tell us a great deal about $f|_{X\setminus\text{Fix}(f)}$. Indeed, if A is a subset of X that is a "freezing set" and $A \subset \text{Fix}(f)$, then f is constrained to be the identity function id_X.

Some results concerning freezing sets were presented in [4]. In this paper, we continue the study of freezing sets. In particular, we show how to find minimal freezing sets for "thick" convex disks in the digital plane, and we give examples showing the importance of the assumption of convexity in our theorems.

2. Preliminaries

We use $\mathbb{Z}$ to indicate the set of integers and $\mathbb{R}$ for the set of real numbers. For a finite set X, we denote by $\#X$ the number of distinct members of X.

2.1. **Adjacencies.**

Material in this section is quoted or paraphrased from [4].

The c_u-adjacencies are commonly used in digital topology. Let $x, y \in \mathbb{Z}^n$, $x \neq y$, where we consider these points as n-tuples of integers:

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

Let $u \in \mathbb{Z}$, $1 \leq u \leq n$. We say x and y are c_u-*adjacent* if

- there are at most u indices i for which $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Often, a c_u-adjacency is denoted by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

For κ-adjacent x, y, we write $x \leftrightarrow_\kappa y$ or $x \leftrightarrow y$ when κ is understood. We write $x \leftrightarroweq_\kappa y$ or $x \leftrightarroweq y$ to mean that either $x \leftrightarrow_\kappa y$ or $x = y$.

We say subsets A, B of a digital image X are (κ-)adjacent, $A \leftrightarroweq_\kappa B$ or $A \leftrightarroweq B$ when κ is understood, if there exist $a \in A$ and $b \in B$ such that $a \leftrightarroweq_\kappa b$.

We say $\{x_n\}_{n=0}^k \subset (X, \kappa)$ is a κ-*path* (or a *path* if κ is understood) from x_0 to x_k if $x_i \leftrightarroweq_\kappa x_{i+1}$ for $i \in \{0, \ldots, k-1\}$, and k is the *length* of the path.

A subset Y of a digital image (X, κ) is κ-*connected* [15], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a κ-path in Y from a to b.

We define

$$N(X, \kappa, x) = \{y \in X \mid x \leftrightarrow_\kappa y\},$$
$$N^*(X, \kappa, x) = \{y \in X \mid x \leftrightarroweq_\kappa y\} = N(X, \kappa, x) \cup \{x\}.$$

Definition 2.1. Let $X \subset \mathbb{Z}^n$.

- The *boundary of X with respect to the c_i adjacency*, $i \in \{1,2\}$, is

  $$Bd_i(X) = \{x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_i} x\}.$$

  Note $Bd_1(X)$ is what is called the *boundary of X* in [14]. However, for this paper, $Bd_2(X)$ offers certain advantages.
- The *interior of X* with respect to the c_i adjacency is $Int_i(X) = X \setminus Bd_i(X)$.

2.2. **Digitally continuous functions.** Material in this section is quoted or paraphrased from [4].

The following generalizes a definition of [15].

Definition 2.2 ([1])**.** Let (X,κ) and (Y,λ) be digital images. A function $f : X \to Y$ is (κ,λ)*-continuous* if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y. If $(X,\kappa) = (Y,\lambda)$, we say such a function is κ*-continuous*, denoted $f \in C(X,\kappa)$. □

When the adjacency relations are understood, we may simply say that f is *continuous*. Continuity can be expressed in terms of adjacency of points:

Theorem 2.3 ([15, 1])**.** *A function $f : (X,\kappa) \to (Y,\lambda)$ is continuous if and only if $x \leftrightarrow_\kappa x'$ in X implies $f(x) \Leftrightarrow_\lambda f(x')$.*

Similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings* in [10, 11].

Composition preserves continuity, in the sense of the following.

Theorem 2.4 ([1])**.** *Let (X,κ), (Y,λ), and (Z,μ) be digital images. Let $f : X \to Y$ be (κ,λ)-continuous and let $g : Y \to Z$ be (λ,μ)-continuous. Then $g \circ f : X \to Z$ is (κ,μ)-continuous.*

Given $X = \Pi_{i=1}^{v} X_i$, we denote throughout this paper the projection onto the i^{th} factor by p_i; i.e., $p_i : X \to X_i$ is defined by $p_i(x_1,\ldots,x_v) = x_i$, where $x_j \in X_j$.

Given a function $f : X \to X$, we say $x \in X$ is a *fixed point of f* if $f(x) = x$. The set of points $\{x \in X \mid f(x) = x\}$ we denote as $\operatorname{Fix}(f)$.

We use the notation id_X to denote the *identity function*: $\operatorname{id}_X : X \to X$ is the function $\operatorname{id}_X(x) = x$ for all $x \in X$.

Definition 2.5 ([4]). Let (X,κ) be a digital image. We say $A \subset X$ is a *freezing set for X* if given $f \in C(X,\kappa)$, $A \subset \operatorname{Fix}(f)$ implies $f = \operatorname{id}_X$.

2.3. **Digital disks and bounding curves.** Let $\kappa \in \{c_1, c_2\}$, $n > 1$. We say a κ-connected set $S = \{x_i\}_{i=1}^n \subset \mathbb{Z}^2$ is a *(digital) line segment* if the members of S are collinear.

Remark 2.6. A digital line segment must be vertical, horizontal, or have slope of ± 1. We say a segment with slope of ± 1 is *slanted*.

A *(digital) κ-closed curve* is a path $S = \{s_i\}_{i=0}^m$ such that $s_0 = s_m$, and $0 < |i-j| < m$ implies $s_i \neq s_j$. If $i,j < m$ and $s_i \leftrightarrow_\kappa s_j$ implies $|i-j| \bmod m = 1$, S is a *(digital) κ-simple closed curve*. For a simple closed curve $S \subset \mathbb{Z}^2$ we generally assume

- $m \geq 8$ if $\kappa = c_1$, and
- $m \geq 4$ if $\kappa = c_2$.

These requirements are necessary for the Jordan Curve Theorem of digital topology, below, as a c_1-simple closed curve in $\mathbb{Z}^2$ needs at least 8 points to have a nonempty finite complementary c_2-component, and a c_2-simple closed curve in $\mathbb{Z}^2$ needs at least 4 points to have a nonempty finite complementary c_1-component. Examples in [14] show why it is desirable to consider S and $\mathbb{Z}^2 \setminus S$ with different adjacencies.

Theorem 2.7 ([14], Jordan Curve Theorem for digital topology)**.** *Let $\{\kappa, \kappa'\} = \{c_1, c_2\}$. Let $S \subset \mathbb{Z}^2$ be a simple closed κ-curve such that S has at least 8 points if $\kappa = c_1$ and such that S has at least 4 points if $\kappa = c_2$. Then $\mathbb{Z}^2 \setminus S$ has exactly 2 κ'-connected components.*

One of the κ'-components of $\mathbb{Z}^2 \setminus S$ is finite and the other is infinite. This suggests the following.

Definition 2.8. Let $S \subset \mathbb{Z}^2$ be a c_2-closed curve such that $\mathbb{Z}^2 \setminus S$ has two c_1-components, one finite and the other infinite. The union D of S and the finite c_1-component of $\mathbb{Z}^2 \setminus S$ is a *(digital) disk*. S is a *bounding curve* of D. The finite c_1-component of $\mathbb{Z}^2 \setminus S$ is the *interior of* S, denoted $Int(S)$, and the infinite c_1-component of $\mathbb{Z}^2 \setminus S$ is the *exterior of* S, denoted $Ext(S)$.

Notes:

- If D is a digital disk determined as above by a bounding c_2-closed curve S, then (S, c_1) can be disconnected. See Figure 1.
- There may be more than one closed curve S bounding a given disk D. See Figure 2. Since we are interested in finding *minimal* freezing sets and since it turns out we often compute these from bounding curves, we may prefer those that are *minimal* bounding curves. A bounding curve S for a disk D is *minimal* if there is no bounding curve S' for D such that $\#S' < \#S$.
- In particular, a bounding curve need not be contained in $Bd_1(D)$. E.g., in the disk D shown in Figure 2(i), $(2,2)$ is a point of the bounding curve; however, all of the points c_1-adjacent to $(2,2)$ are members of D, so by Definition 2.1, $(2,2) \notin Bd_1(D)$. However, a bounding curve for D must be contained in $Bd_2(D)$.
- In Definition 2.8, we use c_2 adjacency for S and we do not require S to be simple. Figure 2 shows why these seem appropriate.
  - The use of c_2 adjacency allows slanted segments in bounding curves and makes possible a bounding curve in subfigure (ii) with fewer points than the bounding curve in subfigure (i) in which adjacent pairs of the bounding curve are restricted to c_1 adjacency.

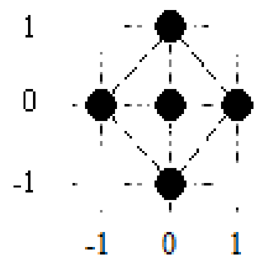

FIGURE 1. The c_1-disk $D = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| < 2\}$. The bounding curve $S = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| = 1\} = D \setminus \{(0, 0)\}$ is not c_1-connected.

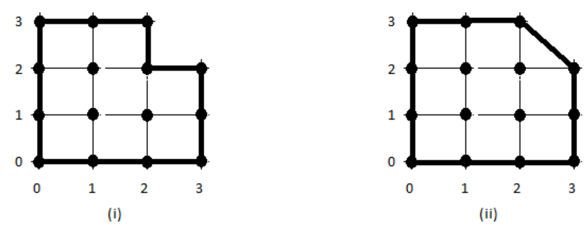

FIGURE 2. Two views of $[0, 3]_{\mathbb{Z}}^2 \setminus \{(3, 3)\}$, which can be regarded as a c_1-disk with either of the closed curves shown in dark as a bounding curve.
(i) The dark line segments show a c_1-simple closed curve S that is a bounding curve for D.
(ii) The dark line segments show a c_2-closed curve S that is a minimal bounding curve for D.
Note the point $(2, 2)$ in the bounding curve shown in (i). By Definition 2.1, $(2, 2) \notin Bd_1(D)$; however, $(2, 2) \in Bd_2(D)$.

  - Neither of the bounding curves shown in Figure 2 is a c_2-simple closed curve. E.g., non-consecutive points of each of the bounding curves, $(0, 1)$ and $(1, 0)$, are c_2-adjacent. The bounding curve shown in Figure 2(ii) is clearly also not a c_1-simple closed curve.
- A closed curve that is not simple may be the boundary Bd_2 of a digital image that is not a disk. This is illustrated in Figure 3.

More generally, we have the following.

Definition 2.9. let $X \subset \mathbb{Z}^2$ be a finite, c_i-connected set, $i \in \{1, 2\}$. Suppose there are pairwise disjoint c_2-closed curves $S_j \subset X$, $1 \leq j \leq n$, such that

- $X \subset S_1 \cup Int(S_1)$;
- for $j > 1$, $D_j = S_j \cup Int(S_j)$ is a digital disk;

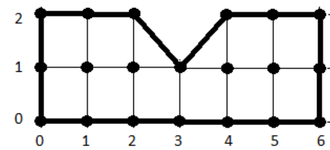

FIGURE 3. $D = [0,6]_{\mathbb{Z}} \times [0,2]_{\mathbb{Z}} \setminus \{(3,2)\}$ shown with a bounding curve S in dark segments. D is not a disk with either the c_1 or the c_2 adjacency, since with either of these adjacencies, $\mathbb{Z}^2 \setminus S$ has two bounded components, $\{(1,1),(2,1)\}$ and $\{(4,1),(5,1)\}$.

- no two of
$$S_1 \cup Ext(S_1), D_2, \ldots, D_n$$
are c_1-adjacent or c_2-adjacent; and
- we have
$$\mathbb{Z}^2 \setminus X = Ext(S_1) \cup \bigcup_{j=2}^{n} Int(S_j).$$

Then $\{S_j\}_{j=1}^{n}$ is a *set of bounding curves of* X.

Note: As above, a digital image $X \subset \mathbb{Z}^2$ may have more than one set of bounding curves.

A set X in a Euclidean space $\mathbb{R}^n$ is *convex* if for every pair of distinct points $x, y \in X$, the line segment $\overline{xy}$ from x to y is contained in X. The *convex hull of* $Y \subset \mathbb{R}^n$, denoted $hull(Y)$, is the smallest convex subset of $\mathbb{R}^n$ that contains Y. If $Y \subset \mathbb{R}^2$ is a finite set, then $hull(Y)$ is a single point if Y is a singleton; a line segment if Y has at least 2 members and all are collinear; otherwise, $hull(Y)$ is a polygonal disk, and the endpoints of the edges of $hull(Y)$ are its *vertices*.

A digital version of convexity can be stated for subsets of the digital plane $\mathbb{Z}^2$ as follows. A finite set $Y \subset \mathbb{Z}^2$ is *(digitally) convex* if either

- Y is a single point, or
- Y is a digital line segment, or
- Y is a digital disk with a bounding curve S such that the endpoints of the maximal line segments of S are the vertices of $hull(Y) \subset \mathbb{R}^2$.

Let s_1 and s_2 be sides of a digital disk $X \subset \mathbb{Z}^2$, i.e., maximal digital line segments in a bounding curve S of X, such that $s_1 \cap s_2 = \{p\} \subset X$. The *interior angle of* X *at* p is the angle formed by s_1, s_2, and $Int(X)$.

Remark 2.10. Let (X, κ) be a digital disk in $\mathbb{Z}^2$, $\kappa \in \{c_1, c_2\}$. Let s_1 and s_2 be sides of X such that $s_1 \cap s_2 = \{p\} \subset X$. Then the interior angle of X at p is well defined.

Proof. If there exists $q \in X \setminus (s_1 \cup s_2)$ such that $q \leftrightarrow_{c_2} p$, then the interior angle of X at p is the angle obtained by rotating s_1 about p through q to reach s_2.

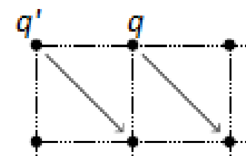

FIGURE 4. Illustration of Lemma 2.13. Arrows show the images of q, q' under $f \in C(X, c_2)$. Since $f(q)$ is to the right of q and $q' \leftrightarrow_{c_1,c_2} q$ with q' to the left of q, f pulls q' to the right so that $f(q')$ is to the right of q'.

Otherwise, the angles formed by s_1 and s_2 measure 45° ($\pi/4$ radians) and 315° ($7\pi/4$ radians). Since X is convex, the 45° angle determined by s_1 and s_2 is the interior angle of X at p. □

Remark 2.11. It follows from Remark 2.6 that every interior angle measures as a multiple of 45° ($\pi/4$ radians). For a convex disk, an interior angle must be 45° ($\pi/4$ radians), 90° ($\pi/2$ radians), or 135° ($3\pi/4$ radians).

2.4. **Tools for determining fixed point sets.** The following assertions will be useful in determining fixed point and freezing sets.

Proposition 2.12 (Corollary 8.4 of [9]). *Let (X, κ) be a digital image and $f \in C(X, \kappa)$. Suppose $x, x' \in \mathrm{Fix}(f)$ are such that there is a unique shortest κ-path P in X from x to x'. Then $P \subset \mathrm{Fix}(f)$.*

Lemma 2.13 below is in the spirit of "pulling" as introduced in [13]. We quote [4]:

> The following assertion can be interpreted to say that in a c_u-adjacency, a continuous function that moves a point p also [pulls along] a point that is "behind" p. E.g., in $\mathbb{Z}^2$, if q and q' are c_1- or c_2-adjacent with q left, right, above, or below q', and a continuous function f moves q to the left, right, higher, or lower, respectively, then f also moves q' to the left, right, higher, or lower, respectively.

Lemma 2.13 ([4]). *Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \le u \le n$. Let $q, q' \in X$ be such that $q \leftrightarrow_{c_u} q'$. Let $f \in C(X, c_u)$.*

(1) *If $p_i(f(q)) > p_i(q) > p_i(q')$ then $p_i(f(q')) > p_i(q')$.*
(2) *If $p_i(f(q)) < p_i(q) < p_i(q')$ then $p_i(f(q')) < p_i(q')$.*

Figure 4 illustrates Lemma 2.13.

Theorem 2.14 ([4]). *Let $X \subset \mathbb{Z}^n$ be finite. Then for $1 \le u \le n$, $Bd_1(X)$ is a freezing set for (X, c_u).*

Remark 2.15. A similar proof can be used to show that if $X \subset \mathbb{Z}^2$ is finite, then a set of bounding curves for X is a freezing set for (X, c_i), $i \in \{1, 2\}$.

Theorem 2.16. *Let D be a digital disk in $\mathbb{Z}^2$. Let S be a bounding curve for D. Then S is a freezing set for (D, c_1) and for (D, c_2).*

Proof. This is like the proof of Theorem 2.14 in [4]. Let $\kappa \in \{c_1, c_2\}$. Let $f \in C(D, \kappa)$ such that $S \in \text{Fix}(f)$. Suppose there exists $x \in D$ such that $f(x) \neq x$. Then x lies on a horizontal segment $\overline{ab}$ and on a vertical segment $\overline{cd}$ such that $\{a, b, c, d\} \subset S$, $p_1(a) < p_1(b)$, and $p_2(c) < p_2(d)$.

- If $p_1(f(x)) > p_1(x)$ then by Lemma 2.13, $p_1(f(a)) > p_1(a)$, contrary to $a \in S \subset \text{Fix}(f)$.
- If $p_1(f(x)) < p_1(x)$ then by Lemma 2.13, $p_1(f(b)) < p_1(b)$, contrary to $b \in S \subset \text{Fix}(f)$.
- If $p_2(f(x)) > p_2(x)$ then by Lemma 2.13, $p_1(f(c)) > p_1(c)$, contrary to $c \in S \subset \text{Fix}(f)$.
- If $p_2(f(x)) < p_2(x)$ then by Lemma 2.13, $p_1(f(d)) < p_1(d)$, contrary to $d \in S \subset \text{Fix}(f)$.

In all cases, we have a contradiction brought on by assuming $x \notin \text{Fix}(f)$. Therefore, $f = \text{id}_D$, so S is a freezing set for (D, κ). □

We will use the following.

Definition 2.17. Let (X, κ) be a digital image. Let $p, q \in X$ such that

$$N(X, p, \kappa) \subset N^*(X, q, \kappa).$$

Then q is a *close κ-neighbor* of p.

Lemma 2.18. *Let (X, κ) be a digital image. Let $p, q \in X$ such that q is a close κ-neighbor of p. Then there is a κ-retraction $r : X \to X$ of X to $X \setminus \{p\}$.*

Proof. This was shown in the proof of Lemma 4.8 of [9]. □

Lemma 2.19. *Let (X, κ) be a digital image. Let $p, q \in X$ such that q is a close κ-neighbor of p. Then p belongs to every freezing set of (X, κ).*

Proof. Let A be a freezing set of (X, κ). It follows from Lemma 2.18 that $A \setminus \{p\}$ is not a freezing set of (X, κ). The assertion follows. □

3. c_1-Freezing sets for disks in $\mathbb{Z}^2$

The following can be interpreted as stating that the set of "corner points" form a freezing set for a digital cube with the c_1 adjacency.

Theorem 3.1 ([4])**.** *Let $X = \Pi_{i=1}^n [0, m_i]_{\mathbb{Z}}$. Let $A = \Pi_{i=1}^n \{0, m_i\}$. Then A is a freezing set for (X, c_1); minimal for $n \in \{1, 2\}$.*

Remark 3.2. Example 5.16 of [4] shows that the freezing set of Theorem 3.1 is not minimal for $n = 3$.

The argument used to prove Theorem 3.1 may lead one to ask if this theorem can be generalized for $n = 2$ as follows:

> Given a digital disk $D \subset \mathbb{Z}^2$ such that all of the maximal segments of a bounding curve S of D are horizontal or vertical, is the set of the endpoints of the maximal segments of S a minimal freezing set for (D, c_1)?

The following provides a negative answer to this question.

Example 3.3. Let $D = [0,3]_{\mathbb{Z}} \times [0,6]_{\mathbb{Z}} \setminus \{(3,3)\})$. Then

$$A = \{(0,0),(3,0),(3,2),(3,4),(3,6),(0,6)\}$$

(see Figure 5(i)) is a minimal freezing set for (D, c_1). Note $(2,2)$ and $(2,4)$ are endpoints of maximal horizontal and vertical bounding segments of D and are not members of A. While $(2,2)$ and $(2,4)$ are members of a bounding curve for D, they are not members of a minimal bounding curve, which includes edges from $(3,4)$ to $(2,3)$ and from $(2,3)$ to $(3,2)$ (see Figure 5(ii)).

Proof. Let $f \in C(D, c_1)$ such that $A \subset \mathrm{Fix}(f)$. It follows from Proposition 2.12 that the vertical segments $\{0\} \times [0,6]_{\mathbb{Z}}$, $\{3\} \times [0,2]_{\mathbb{Z}}$, and $\{3\} \times [4,6]_{\mathbb{Z}}$, the horizontal segments $[0,3]_{\mathbb{Z}} \times \{0\}$ and $[0,3]_{\mathbb{Z}} \times \{6\}$, and the path

$$\{(3,2),(2,2),(2,3),(2,4),(3,4)\}$$

are all subsets of $\mathrm{Fix}(f)$. Since the union of these paths is a bounding curve S for D, we have $S \subset \mathrm{Fix}(f)$. That A is a freezing set follows from Theorem 2.16.

To show A is a minimal freezing set, we observe that the following are pairs (p,q) such that $p \in A$ and q is a close c_1-neighbor of p (see Figure 5):

$$((0,0),(1,1)), \quad ((3,0),(2,1)), \quad ((3,2),(2,1)), \quad ((3,4),(2,5)),$$
$$((3,6),(2,5)), \quad \text{and} \quad ((0,6),(1,5)).$$

By Lemma 2.19, every member of A must belong to every freezing set of (X, c_1). It follows that A is a minimal freezing set. □

Definition 3.4. Let $X \subset \mathbb{Z}^2$ be a digital disk. We say X is *thick* if the following are satisfied. For some bounding curve S of X,

- for every slanted segment S of $Bd_2(X)$, if $p \in S$ is not an endpoint of S, then there exists $c \in X$ such that (see Figure 6)

$$c \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} c, \tag{3.1}$$

  and
- if p is the vertex of a $90°$ ($\pi/2$ radians) interior angle θ of S, then there exists $q \in Int(X)$ such that
  - if θ has horizontal and vertical sides then $q \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} q$ (see Figure 7);
  - if θ has slanted sides then $q \leftrightarrow_{c_1} p$ (see Figure 8);

  and
- if p is the vertex of a $135°$ ($3\pi/4$ radians) interior angle θ of S, there exist $b, b' \in X$ such that b and b' are in the interior of θ and (see Figure 9)

$$b \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} b \quad \text{and} \quad b' \leftrightarrow_{c_1} p.$$

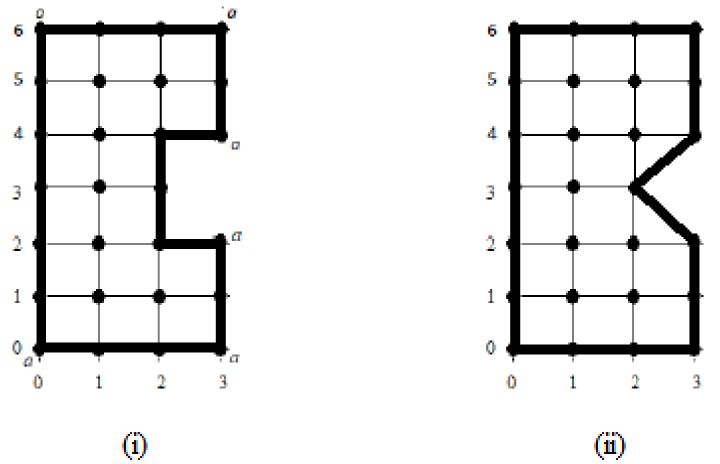

Figure 5. There are distinct boundary curves for the disk D that contain the horizontal segments from $(0,0)$ to $(3,0)$ and from $(0,6)$ to $(3,6)$; and vertical segments from $(0,0)$ to $(0,6)$, from $(3,0)$ to $(3,2)$, and from $(3,4)$ to $(3,6)$.
(i) We can complete a boundary curve by using the horizontal segments from $(2,2)$ to $(3,2)$ and from $(2,4)$ to $(3,4)$ and the vertical segment from $(2,2)$ to $(2,4)$, as shown in dark. This lets us view D as a disk with horizontal and vertical sides. Members of the minimal freezing set A for (D,c_1), determined in Example 3.3, are marked "a". Note $\{(2,2),(2,4)\} \cap A = \varnothing$. $(2,2)$ and $(2,4)$ are endpoints of a maximal horizontal segment of a bounding curve, but not of the minimal bounding curve S; the latter is shown in (ii). Indeed, by Definition 2.8, $\{(2,2),(2,4)\} \subset Int(D)$.
(ii) Alternately, we can complete a boundary curve by using the slanted line segments from $(2,3)$ to $(3,4)$ and from $(2,3)$ to $(3,2)$. This is a minimal boundary curve S that lets us view D as in Example 4.1. A minimal freezing set for (D,c_2) is $S \setminus \{(2,3)\}$.

Examples of digital images that fail to be thick are shown in Figure 10.

The following expands on the dimension 2 case of Theorem 3.1 to give a subset of $Bd(X)$ that is a freezing set.

Theorem 3.5. *Let X be a finite digital image in $\mathbb{Z}^2$ with a set of bounding curves S_i, $1 \le i \le n$, as in Definition 2.9. Let A_1 be the set of points $x \in \bigcup_{i=1}^n S_i$ such that x is an endpoint of a maximal horizontal or a maximal vertical edge of $\bigcup_{i=1}^n S_i$. Let A_2 be the union of slanted line segments in $\bigcup_{i=1}^n S_i$. Then $A = A_1 \cup A_2$ is a freezing set for (X, c_1).*

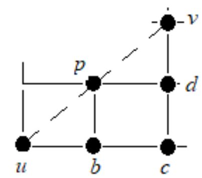

FIGURE 6. $p \in \overline{uv}$ in a bounding curve, with $\overline{uv}$ slanted. Note $u \not\leftrightarrow_{c_1} p \not\leftrightarrow_{c_1} v$, $p \leftrightarrow_{c_2} c \not\leftrightarrow_{c_1} p$, $\{p, c\} \subset N(\mathbb{Z}^2, c_1, b) \cap N(\mathbb{Z}^2, c_1, d)$. If X is thick then $c \in X$. (Not meant to be understood as showing all of X.)

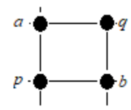

FIGURE 7. $\angle apb$ is a 90° ($\pi/2$ radians) angle of a bounding curve of X at $p \in A_1$, with horizontal and vertical sides. If X is thick then $q \in Int(X)$. (Not meant to be understood as showing all of X.)

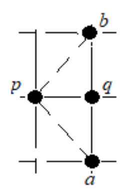

FIGURE 8. $\angle apb$ is a 90° ($\pi/2$ radians) angle between slanted segments of a bounding curve. If X is thick then $q \in Int(X)$. (Not meant to be understood as showing all of X).

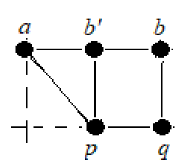

FIGURE 9. $\angle apq$ is an angle of 135° degrees ($3\pi/4$ radians) of a bounding curve of X at p, with $\overline{ap} \cup \overline{pq}$ a subset of the bounding curve. If X is thick then $b, b' \in X$. (Not meant to be understood as showing all of X.)

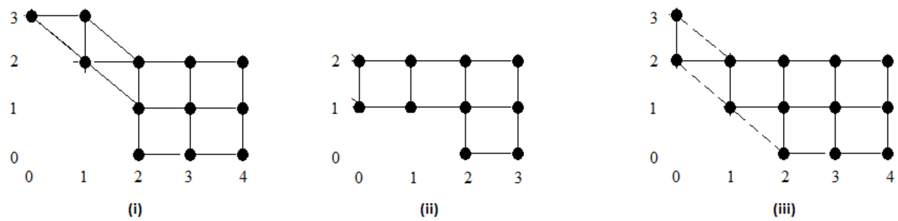

Figure 10. Digital disks that are not thick.
(i) $(1, 2)$ is a non-endpoint of a slanted boundary segment for which there is no point corresponding to c of Figure 6.
(ii) $(0, 1)$ is the vertex of a 90°-degree ($\pi/2$ radians) interior angle but is not c_2-adjacent to any member of the interior of the disk. Note the segment from (1,1) to (2,0) belongs to a minimal bounding curve, so (2,1) is an interior point. Therefore, this image really is a disk.
(iii) $(0, 2)$ is the vertex of a 135° interior angle of a bounding curve for which there is no point corresponding to b of Figure 9.

Proof. Let $f \in C(X, c_1)$ such that $A \subset \text{Fix}(f)$. Let x, x' be distinct members of A_1 that are endpoints of the same maximal horizontal or vertical edge E in some S_i. Then E is the unique shortest c_1-path in X from x to x'. By Proposition 2.12, $E \subset \text{Fix}(f)$. It follows that every horizontal and every vertical side of S_i belongs to $\text{Fix}(f)$. By hypothesis we also have that $A_2 \subset \text{Fix}(f)$, so $S_i \subset \text{Fix}(f)$. Therefore, $Bd_2(X) \subset \text{Fix}(f)$. By Remark 2.15, $f = \text{id}_X$. Thus A is a freezing set for (X, c_1). □

Remark 3.6. The set A of Theorem 3.5 need not be minimal. This is shown in Example 3.3, where $(2, 3)$, as a member of a slanted edge of a minimal bounding curve (see Figure 5), is a member of the set A of Theorem 3.5, but is not a member of the minimal freezing set.

Theorem 3.7. *Let X be a thick convex disk with a bounding curve S. Let A_1 be the set of points $x \in S$ such that x is an endpoint of a maximal horizontal or a maximal vertical edge of S. Let A_2 be the union of slanted line segments in S. Then $A = A_1 \cup A_2$ is a minimal freezing set for (X, c_1) (see Figure 11(ii)).*

Proof. That A is a freezing set follows as in the proof of Theorem 3.5. To show A is minimal, we must show that if we remove a point p from A, the remaining set $A \setminus \{p\}$ is not a freezing set.

We start by considering $p \in A_1$. Since X is convex, the interior angle of S at p must be 45° ($\pi/4$ radians), 90° ($\pi/2$ radians), or 135° ($3\pi/4$ radians).

- Suppose the interior angle of S at p is 45° ($\pi/4$ radians). Let b be a point of S that is c_1-adjacent to p on the horizontal or vertical edge of this angle (see Figure 12). Then b is a close c_1-neighbor of p in X. By Lemma 2.19, $X \setminus \{p\}$ is not a freezing set for (X, c_2).

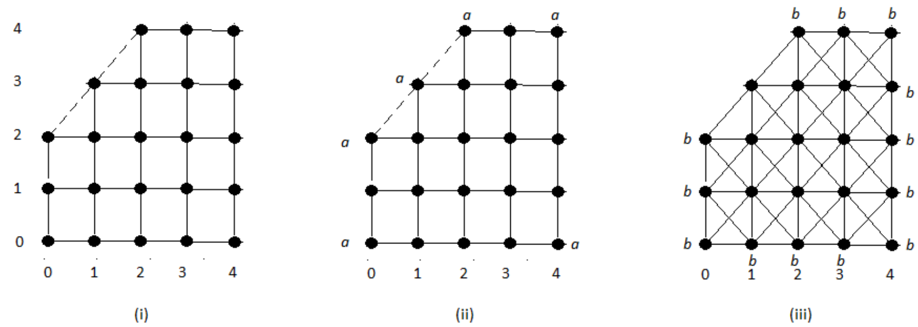

Figure 11. The convex disk $D = [0,4]_{\mathbb{Z}}^2 \setminus \{(0,3),(0,4),(1,4)\}$. The dashed segment from $(0,2)$ to $(2,4)$ shown in (i) and (ii) indicates part of the bounding curve and not c_1-adjacencies.
(i) D with a c_2 bounding curve.
(ii) (D, c_1) with members of a minimal freezing set A marked "a" - these are the endpoints of the maximal horizontal and vertical segments of the bounding curve, and all points of the slanted segment of the bounding curve, per Theorem 3.5.
(iii) (D, c_2) with members of a minimal freezing set B marked "b" - these are the endpoints of the maximal slanted edge and all the points of the horizontal and vertical edges of the bounding curve, per Theorem 4.2.

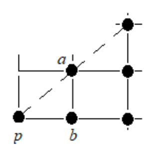

Figure 12. $\angle apb$ is a $45°$ ($\pi/4$ radians) interior angle of a bounding curve at $p \in A_1$. (Not meant to be understood as showing all of X.)

- Suppose the interior angle of S at p is $90°$ ($\pi/2$ radians). Let a, b be the points of S that are c_1-adjacent to p on the horizontal and vertical edges of this angle and let q be the point of $Int(X)$ that is c_1-adjacent to each of a and b (see Figure 7).

  Then q is a close c_1-neighbor of p in X. Thus, $A \setminus \{p\}$ is not a freezing set for (X, c_1).
- Suppose the interior angle of S at p is $135°$ ($3\pi/4$ radians). Let $a, q \in S$ be such that a and q are the members of this angle that are c_2-adjacent to p, where $\overline{ap}$ is slanted and $\overline{pq}$ is horizontal or vertical. Since X is thick, Definition 3.4 yields that there exists $b \in X$ such that $b \leftrightarrow_{c_2} p$ (as

in Figure 9). Then b is a close c_1-neighbor of p in X. By Lemma 2.19, $A \setminus \{p\}$ is not a freezing set for (X, c_1).

Thus we have shown that if $p \in A_1$ then $A \setminus \{p\}$ is not a freezing set for (X, c_1).

Now we wish to show if $p \in A_2$ then $A \setminus \{p\}$ is not a freezing set for (X, c_1). Let s be a slanted segment of $Bd_2(X)$ containing p.

If p is not an endpoint of s, then from the assumption (3.1) there exist $b, c, d \in X$ such that $p \leftrightarrow_{c_2} c$, $p \not\leftrightarrow_{c_1} c$, and $b \leftrightarrow_{c_1} c \leftrightarrow_{c_1} d$ (see Figure 6). Then c is a close c_1-neighbor of p in X. By Lemma 2.19, $A \setminus \{p\}$ is not a freezing set for (X, c_1).

If p is an endpoint of s, let s' be the other maximal segment of $Bd_2(X)$ for which p is an endpoint. If s' is horizontal or vertical, then $p \in A_1$, hence, as discussed above, $A \setminus \{p\}$ is not a freezing set for (X, c_1). Therefore, we assume s' is slanted. Since X is convex and both s and s' are slanted, the interior angle of S at p must be $90°$ ($\pi/2$ radians). There exists $q \in Int(X)$ such that $q \leftrightarrow_{c_1} p$ (see Figure 8). Then q is a close c_1-neighbor of p in X. By Lemma 2.19, $A \setminus \{p\}$ is not a freezing set for (X, c_1). □

4. c_2-Freezing sets for disks in $\mathbb{Z}^2$

For disks in $\mathbb{Z}^2$, we obtain results for the c_2 adjacency that are dual to those obtained for the c_1 adjacency in the previous section.

As was true of the c_1 adjacency and Theorem 3.5, we see, by comparing Example 4.1 and Theorem 4.3 below, that with c_2 adjacency, convexity can affect determination of a minimal freezing set for a digital image in $\mathbb{Z}^2$.

Example 4.1. Let $D = [0,3]_{\mathbb{Z}} \times [0,6]_{\mathbb{Z}} \setminus \{(3,3)\})$. (This is the set used in Example 3.3. See Figure 5.) Let S be the bounding curve shown in Figure 5(ii), i.e., S is the union of the vertical segments $\{0\} \times [0,6]_{\mathbb{Z}}$, $\{3\} \times [0,2]_{\mathbb{Z}}$, and $\{3\} \times [4,6]_{\mathbb{Z}}$; the horizontal segments $[0,3]_{\mathbb{Z}} \times \{0\}$ and $[0,3]_{\mathbb{Z}} \times \{6\}$; and the

slanted segments $\overline{(2,3)(3,4)}$ and $\overline{(2,3)(3,2)}$.

Let $B = S \setminus \{(2,3)\}$. Then B is a minimal freezing set for (D, c_2).

Proof. Let $f \in C(D, c_2)$ be such that

$$f|_B = \text{id}_B\,. \tag{4.1}$$

Let $p = (2,3)$, $q = (3,2) \in B$, $s = (3,4) \in B$. Note the following:

- If $p_1(f(p)) > p_1(p)$ then by Lemma 2.13, $p_1(f(1,3)) > 1$ and therefore $p_1(f(0,3)) > 0$, contrary to (4.1).
- If $p_1(f(p)) < p_1(p)$ then by Lemma 2.13, $p_1(f(q)) < 3$, contrary to (4.1).
- If $p_2(f(p)) > p_2(p)$ then by Lemma 2.13, $p_2(f(q)) > 2$, contrary to (4.1).
- If $p_2(f(p)) < p_2(p)$ then by Lemma 2.13, $p_1(f(s)) < 4$, contrary to (4.1).

It follows that $p \in \text{Fix}(f)$. Thus $B \cup \{p\} = Bd_1(D) \subset \text{Fix}(f)$. By Theorem 2.14, $f = \text{id}_D$. This establishes that B is a freezing set.

To show B is minimal, for $b \in B$ notice that there is a c_2-close neighbor $q \in X$:

$(1,1)$ is a c_2-close neighbor of $(0,0) \in B$;
$(i,1)$ is a c_2-close neighbor of $(i,0) \in B$, $i \in \{1,2\}$;
$(1,j)$ is a c_2-close neighbor of $(0,j) \in B$, $1 < j < 5$;
$(1,5)$ is a c_2-close neighbor of $(0,6) \in B$;
$(i,5)$ is a c_2-close neighbor of $(i,6) \in B$, $i \in \{1,2\}$;
$(2,5)$ is a c_2-close neighbor of $(3,6) \in B$;
$(2,j)$ is a c_2-close neighbor of $(3,j) \in B$, $j \in \{1,2,4,5\}$.

By Lemma 2.19, $B \setminus \{b\}$ is not a freezing set for (D, c_2). The assertion follows. □

Theorem 4.2. *Let X be a finite digital image in $\mathbb{Z}^2$ with a set of bounding curves S_i, $1 \le i \le n$, as in Definition 2.9. Let B_1 be the set of points $x \in \bigcup_{i=1}^n S_i$ such that x is an endpoint of a maximal slanted edge in $\bigcup_{i=1}^n S_i$. Let B_2 be the union of maximal horizontal and maximal vertical line segments in $\bigcup_{i=1}^n S_i$. Let $B = B_1 \cup B_2$. Then B is a freezing set for (X, c_2).*

Proof. Let $f \in C(X, c_2)$ such that $f|_B = \text{id}_B$.

Let p be a point of a slanted edge E of $\bigcup_{i=1}^n S_i$ such that $p \notin B_1$. Let s and s' be the endpoints of E. If $f(p) \neq p$, it follows from Lemma 2.13 that either $f(s) \neq s$ or $f(s') \neq s'$, a contradiction since by hypothesis we have $\{s, s'\} \subset \text{Fix}(f)$. Therefore, $p \in \text{Fix}(f)$; hence, every slanted edge of $\bigcup_{i=1}^n S_i$ is a subset of $\text{Fix}(f)$. Since by hypothesis all horizontal and vertical edges of $\bigcup_{i=1}^n S_i$ belong to $\text{Fix}(f)$, we conclude that $\bigcup_{i=1}^n S_i \subset \text{Fix}(f)$. It follows from Remark 2.15 that $f = \text{id}_X$. Thus, B is a freezing set for (X, c_2). □

Theorem 4.3. *Let X be a thick convex disk with a bounding curve S. Let B_1 be the set of points $x \in S$ such that x is an endpoint of a maximal slanted edge in S. Let B_2 be the union of maximal horizontal and maximal vertical line segments in S. Let $B = B_1 \cup B_2$. Then B is a minimal freezing set for (X, c_2) (see Figure 11(iii)).*

Proof. That B is a freezing set follows as in the proof of Theorem 4.2. To show B is a minimal freezing set, we must show that $B \setminus \{p\}$ is not a freezing set for every $p \in B$.

We start with $p \in B_1$. Since X is a convex disk, we only have the following possibilities to consider.

- X has an interior angle θ at p of $45°$ ($\pi/4$ radians). Let $a \in X$ be such that $a \leftrightarrow_{c_2} p$ and a is adjacent to p on, say, the slanted edge of θ (see Figure 12). Then a is a c_2-close neighbor of p in X. By Lemma 2.19, $B \setminus \{p\}$ is not a freezing set for (X, c_2).
- X has an interior angle at p of $90°$ ($\pi/2$ radians). Then there is a point $q \in Int(X)$ such that $p \leftrightarrow_{c_1} q$ as in Figure 8. Then q is a c_2-close

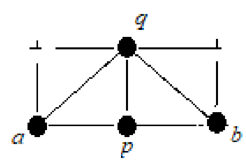

FIGURE 13. $p \in \overline{ab}$, a segment of the bounding curve S. $q \in Int(X)$. $p \leftrightarrow_{c_1} q$. (Not meant to be understood as showing all of X.)

neighbor of p in X. By Lemma 2.19, $B \setminus \{p\}$ is not a freezing set for (X, c_2).

- X has an interior angle at p of 135° ($3\pi/4$ radians). Since X is thick, there is a point b' as in Figure 9 that is a close c_2-neighbor of p. By Lemma 2.19, $B \setminus \{p\}$ is not a freezing set for (X, c_2).

Now consider p as a member of B_2. Since X is convex, this leaves only the following possibilities.

- X has an interior angle at p of 45° ($\pi/4$ radians). Then $p \in B_1 \cap B_2 \subset B_1$. As discussed above, $B \setminus \{p\}$ is not a freezing set for (X, c_2).
- X has an interior angle at p of 90° ($\pi/2$ radians). Let a and b be the points of the horizontal and vertical segments of $Bd(X)$ such that $a \leftrightarrow_{c_1} p \leftrightarrow_{c_1} b$ and let $q \in Int(X)$ be the point such that $a \leftrightarrow_{c_1} q \leftrightarrow_{c_1} b$ (see Figure 7). Then q is a close c_2-neighbor of p in X. By Lemma 2.19, $B \setminus \{p\}$ is not a freezing set for (X, c_2).
- X has an interior angle at p of 135° ($3\pi/4$ radians). Then $p \in B_1 \cap B_2 \subset B_1$. As shown above, $B \setminus \{p\}$ is not a freezing set for (X, c_2).
- p is not an endpoint of its segment of $Bd(X)$. Then p has a c_1-close neighbor $q \in X$ (see Figure 13). By Lemma 2.19, $B \setminus \{p\}$ is not a freezing set for (X, c_2).

We have shown that for all $p \in B$, $B \setminus \{p\}$ is not a freezing set for (X, c_2). Therefore, B is a minimal freezing set for (X, c_2). □

5. Further remarks

Let X be a thick convex digital disk in $\mathbb{Z}^2$. We have shown how to find minimal freezing sets for (X, c_1) and for (X, c_2). We have given examples showing that our assertions do not extend to non-convex disks in $\mathbb{Z}^2$. However, for non-convex disks in $\mathbb{Z}^2$ we have shown how to obtain smaller freezing sets than were previously known. We have also obtained results for freezing sets of c_1- and c_2-connected finite subsets of $\mathbb{Z}^2$ bounded by multiple bounding curves.

We have left unanswered the following.

Question 5.1. *Is every convex disk in $\mathbb{Z}^2$ thick?*

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