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Quantale-valued Cauchy tower spaces and completeness

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Abstract

Generalizing the concept of a probabilistic Cauchy space, we introduce quantale-valued Cauchy tower spaces. These spaces encompass quantale-valued metric spaces, quantale-valued uniform (convergence) tower spaces and quantale-valued convergence tower groups. For special choices of the quantale, classical and probabilistic metric spaces are covered and probabilistic and approach Cauchy spaces arise. We also study completeness and completion in this setting and establish a connection to the Cauchy completeness of a quantale-valued metric space.

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1. Introduction

Cauchy spaces, axiomatized in [17], are a natural setting for studying completeness and completion [32]. In the probabilistic case such spaces are introduced as certain "towers indexed by $[0, 1]$" by Richardson and Kent [33] and by Nusser [27]. The connection to probabilistic metric spaces [34], however, was not clarified there. In this paper, we generalize the approaches of [33, 27]

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by allowing the index set to be a quantale. In this way, a quantale-valued metric space, in particular a probabilistic metric space, possesses a "natural" Cauchy structure, which then can in turn be used to study completeness and completion. Further examples include quantale-valued uniform convergence tower spaces and quantale-valued uniform spaces as well as quantale-valued convergence groups. We study the basic categorical properties of the category of quantale-valued Cauchy tower spaces and characterize those spaces that are quantale-valued metrical. Finally we discuss completeness and completion and we establish a connection with the Cauchy completeness [6] of a quantale-valued metric space.

2. Preliminaries

Let L be a complete lattice with distinct top element $\top$ and bottom element $\bot$. In L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and, for a subset $B \subseteq L$, we have $\alpha \lhd \bigvee_{\beta \in B} \beta$ iff $\alpha \lhd \beta$ for some $\beta \in B$. Sometimes we consider also a weaker relation, the *way-below relation*, $\alpha \ll \beta$ if for all directed subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. The properties of this relation are similar to the properties of the well-below relation, replacing arbitrary subsets by directed subsets. But we also have $\alpha \vee \beta \ll \gamma$ if $\alpha, \beta \ll \gamma$, [9].

A complete lattice is completely distributive, if and only if we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$ [31] and it is *continuous* if and only if we have $\alpha = \bigvee\{\beta : \beta \ll \alpha\}$ for any $\alpha \in L$, [9]. Clearly $\alpha \lhd \beta$ implies $\alpha \ll \beta$ and hence every completely distributive lattice is also continuous. For more results on lattices we refer to [9].

Lemma 2.1. *Let L be a continuous lattice. Then*

(1) $\bigvee_{\delta \ll \alpha}(\delta * \delta) = \alpha * \alpha$.
(2) *If $\epsilon \ll \alpha * \alpha$, then there is $\delta \ll \alpha$ such that $\epsilon \ll \delta * \delta$.*

Proof. (1) We have $\bigvee_{\delta \ll \alpha}(\delta * \delta) \leq \alpha * \alpha = \bigvee_{\delta \ll \alpha} \delta * \bigvee_{\gamma \ll \alpha} \gamma = \bigvee_{\delta, \gamma \ll \alpha} \delta * \gamma \leq \bigvee_{\delta \vee \gamma \ll \alpha}(\delta \vee \gamma) * (\delta \vee \gamma) \leq \bigvee_{\eta \ll \alpha}(\eta * \eta)$.

(2) follows directly from (1) as the set $\{\delta \in L : \delta \ll \alpha\}$ is directed. □

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *commutative and integral quantale* if $(L, *)$ is a commutative semigroup with the top element of L as the unit, and $*$ is distributive over arbitrary joins, i.e. if we have $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J}(\alpha_i * \beta)$ for all $\alpha_i, \beta \in L, i \in J$. In a quantale we can define an *implication operator*, $\alpha \to \beta = \bigvee\{\gamma \in L : \alpha * \gamma \leq \beta\}$, which can be characterized by $\gamma \leq \alpha \to \beta \iff \gamma * \alpha \leq \beta$. A quantale is called *divisible* [11] if for $\beta \leq \alpha$ there exists $\gamma \in L$ such that $\beta = \alpha * \gamma$. In a divisible quantale we have $\bigvee_{\delta \lhd \alpha}(\delta * \delta) = \alpha * \alpha$, see [30].

Prominent examples of quantales are e.g. the unit interval $[0, 1]$ with a left-continuous *t-norm* [34] or *Lawvere's quantale*, the interval $[0, \infty]$ with the opposite order and addition $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + \alpha = \infty$), see e.g. [7]. A further noteworthy example is the quantale of distance distribution functions. A *distance distribution function* $\varphi : [0, \infty] \longrightarrow [0, 1]$, satisfies $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0, \infty]$. The set of all distance distribution functions is denoted by Δ^+. With the pointwise order, the set Δ^+ then becomes a completely distributive lattice [7] with top-element ε_0. A quantale operation on Δ^+, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, is also called a *sup-continuous triangle function* [34].

We consider in the sequel only commutative and integral quantales $\mathsf{L} = (L, \leq, *)$ with underlying complete lattices that are completely distributive.

For a set X, we denote its power set by $\mathsf{P}(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on X by $\mathsf{F}(X)$. The set $\mathsf{F}(X)$ is ordered by set inclusion and maximal elements of $\mathsf{F}(X)$ in this order are called *ultrafilters*. The set of all ultrafilters on X is denoted by $\mathsf{U}(X)$. In particular, for each $x \in X$, the *point filter* $[x] = \{A \subseteq X : x \in A\}$ is an ultrafilter. If $\mathbb{F} \in \mathsf{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathsf{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G$ for some $F \in \mathbb{F}\}$. For filters $\Phi, \Psi \in \mathsf{F}(X \times X)$ we define Φ^{-1} to be the filter generated by the filter base $\{F^{-1} : F \in \Phi\}$ where $F^{-1} = \{(x, y) \in X \times X : (y, x) \in F\}$ and $\Phi \circ \Psi$ to be the filter generated by the filter base $\{F \circ G : F \in \Phi, G \in \Psi\}$, whenever $F \circ G \neq \varnothing$ for all $F \in \Phi, G \in \Psi$, where $F \circ G = \{(x, y) \in X \times X : (x, s) \in F, (s, y) \in G$ for some $s \in X\}$.

For details and notation from category theory we refer to [1] and [29].

Definition 2.2 ([14])**.** Let $\mathsf{L} = (L, \leq, *)$ be a quantale. A pair $(X, \overline{q} = (q_\alpha)_{\alpha \in L})$ is called an L-*convergence tower space* [14], where $(q_\alpha \colon \mathsf{F}(X) \longrightarrow \mathsf{P}(X))_{\alpha \in L}$ is a family of mappings satisfying:

(LCTS1) $x \in q_\alpha([x])$, $\forall x \in X$ and $\forall \alpha \in L$;
(LCTS2) $\forall \mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$, with $\mathbb{F} \leq \mathbb{G}$, and $\alpha \in L$ implies $q_\alpha(\mathbb{F}) \subseteq q_\alpha(\mathbb{G})$;
(LCTS3) $\forall \alpha, \beta \in L$ with $\alpha \leq \beta$ implies $q_\beta(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$, $\forall \mathbb{F} \in \mathsf{F}(X)$;
(LCTS4) $x \in q_\perp(\mathbb{F})$, $\forall x \in X$, $\mathbb{F} \in \mathsf{F}(X)$.

If, moreover, $(X, \overline{q})$ satisfies

(LCTS5) $q_\alpha(\mathbb{F}) \cap q_\alpha(\mathbb{G}) \leq q_\alpha(\mathbb{F} \wedge \mathbb{G})$, $\forall \alpha \in L$, for $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$,

then the pair $(X, \overline{q})$ is called an L-*limit tower space.* If $(X, \overline{q})$ satisfies $x \in q_{\vee A}(\mathbb{F})$ whenever $x \in q_\alpha(\mathbb{F})$ $\forall \alpha \in A$, it is called *left-continuous*. A mapping $f \colon (X, \overline{q}) \longrightarrow \left(X', \overline{q'}\right)$ between L-convergence tower spaces is called *continuous* if, for all $x \in X$, and for all $\mathbb{F} \in \mathsf{F}(X)$, $f(x) \in q'_\alpha(f(\mathbb{F}))$ whenever $x \in q_\alpha(\mathbb{F})$. The category of all L-convergence tower spaces and continuous mappings is denoted by L-CTS.

If $L = \{0, 1\}$, then L-convergence tower spaces can be identified with classical convergence spaces, [5, 29]. If $\mathsf{L} = ([0, \infty], \geq +)$ is Lawvere's quantale, then an L-limit tower space is a limit tower space [4] and a left-continuous L-limit tower space is an approach limit spaces in the sense of Lowen [20]. For $\mathsf{L} = ([0, 1], \leq, *)$, we obtain probabilistic convergence spaces in the sense of Richardson and Kent [33] and if $\mathsf{L} = (\Delta^+ \leq, *)$, then an L-convergence tower space is a probabilistic convergence space in the definition of [12].

3. L-Cauchy tower spaces

Definition 3.1. Let $\mathsf{L} = (L, \leq, *)$ be a quantale. A pair $(X, \overline{C}) = (X, (C_\alpha)_{\alpha \in L})$ is called an L-*Cauchy tower space*, where $C_\alpha \subseteq \mathsf{F}(X)$ for all $\alpha \in L$, if

(LChyTS1) $[x] \in C_\alpha$ for all $x \in X, \alpha \in L$;
(LChyTS2) $\mathbb{G} \geq \mathbb{F} \in C_\alpha$ implies $\mathbb{G} \in C_\alpha$;
(LChyTS3) $\alpha \leq \beta, \mathbb{F} \in C_\beta$ implies $\mathbb{F} \in C_\alpha$;
(LChyTS4) $C_\perp = \mathbb{F}(X)$.
(LChyTS5) $\mathbb{F} \in C_\alpha, \mathbb{G} \in C_\beta, \mathbb{F} \vee \mathbb{G}$ exists, implies $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha * \beta}$.

An L-Cauchy tower space is called *left-continuous* if $\mathbb{F} \in C_\alpha$ for all $\alpha \in A \subseteq L$ implies $\mathbb{F} \in C_{\bigvee A}$. We call a mapping $f : (X, \overline{C}) \longrightarrow (X', \overline{C'})$ *Cauchy-continuous* if $\mathbb{F} \in C_\alpha$ implies $f(\mathbb{F}) \in C'_\alpha$. The category with the L-Cauchy tower spaces as objects and Cauchy-continuous mappings as morphisms is denoted by L-ChyTS.

For $\mathsf{L} = ([0,1], \leq, *)$ with a continous t-norm $*$, an L-Cauchy tower space is a probabilistic Cauchy space in the definition of Nusser [26] and if $* = \min$, then we obtain the probabilistic Cauchy spaces of Kent and Richardson [18]. In case of Lawever's quantale L, an L-Cauchy tower space is a Cauchy tower space in the definition of [25], which can be identified in the left-continuous case with an approach Cauchy space [22].

We note that for $(X, \overline{C}) \in |\mathsf{L}\text{-ChyTS}|$, the "top level" $(X, C_\top)$ is a classical Cauchy space [17].

Proposition 3.2. *The category* L-ChyTS *is topological. If the quantale operation is the minimum, i.e. if $* = \wedge$, then it is also a Cartesian closed category.*

Proof. The topologicalness can be shown in a straight-forward way. We only describe the initial constructions. For $(f_j : X \longrightarrow (X_j, \overline{C^j}))_{j \in J}$ we define for $\mathbb{F} \in \mathsf{F}(X), \mathbb{F} \in C_\alpha \iff f_j(\mathbb{F}) \in C^j_\alpha$ for all $j \in J$. Then $(X, \overline{C})$ is an L-Cauchy tower space and a mapping $g : (Y, \overline{D}) \longrightarrow (X, \overline{C})$ is Cauchy-continuous if and only if $f_j \circ g : (Y, \overline{D}) \longrightarrow (X_j, \overline{C^j})$ is Cauchy-continuous for all $j \in J$.

To show the Cartesian closedness, we can follow the proof in [18]. We only state the function space structures. We define for two spaces $(X, \overline{C}), (X', \overline{C'}) \in |\mathsf{L}\text{-ChyTS}|$ the set of all Cauchy-continuous mappings by $Hom(X, X') = \{f : (X, \overline{C}) \longrightarrow (X', \overline{C'}) \ : \ f$ Cauchy-continuous$\}$ and define for $\mathbb{H} \in \mathsf{F}(Hom(X, X'))$,

$$\mathbb{H} \in C^c_\alpha \iff \forall \beta \leq \alpha \forall \mathbb{F} \in C_\beta : ev(\mathbb{H} \times \mathbb{F}) \in C'_\beta.$$

Here, $ev : Hom(X, X') \times X \longrightarrow X'$, $ev(f, x) = f(x)$, is the evaluation mapping. Then $(Hom(X, X'), \overline{C^c})$ is an L-Cauchy tower space and the evaluation mapping is Cauchy-continuous. Furthermore, for a Cauchy continuous mapping $f : (X, \overline{C}) \times (X', \overline{C'}) \longrightarrow (X'', \overline{C''})$ we define $f^* : (X, \overline{C}) \longmapsto (Hom(X, X'), \overline{C^c})$ by $f^*(x) = f_x$ with $f_x(x') = f(x, x')$ for $x' \in X'$. Then f^* is Cauchy-continuous. □

We define, for an L-Cauchy tower space $(X, \overline{C})$, the underlying L-convergence tower space $(X, \overline{q^{\overline{C}}})$ by

$$x \in q_\alpha^{\overline{C}}(\mathbb{F}) \iff \mathbb{F} \wedge [x] \in C_\alpha.$$

If $* = \wedge$, then the axiom (LCTS5) is satisfied: For $\mathbb{F} \wedge [x] \in C_\alpha, \mathbb{G} \wedge [x] \in C_\beta$, then $(\mathbb{F} \wedge [x]) \vee (\mathbb{G} \wedge [x])$ exists and hence $(\mathbb{F} \wedge \mathbb{G}) \wedge [x] = (\mathbb{F} \wedge [x]) \wedge (\mathbb{G} \wedge [x]) \in C_{\alpha \wedge \alpha} = C_\alpha$.

Definition 3.3. An L-Cauchy tower space $(X, \overline{C})$ is called a *T1-space* if $[x] \wedge [y] \in C_\top$ implies $x = y$. It is called a *T2-space* if $\mathbb{F} \wedge [x], \mathbb{F} \wedge [y] \in C_\top$ implies $x = y$.

As the concepts of T1-space and T2-space only involve the Cauchy space $(X, C_\top)$ we immediately have that an L-Cauchy tower space $(X, \overline{C})$ is a T1-space if and only if it is a T2-space.

4. Example: L-metric spaces

An L-*metric space*, see e.g. [7, 19], is a pair (X, d) of a set X and a mapping $d : X \times X \longrightarrow L$ which satisfies the axioms

(LM1) $d(x, x) = \top$ for all $x \in X$;
(LM2) $d(x, y) * d(y, z) \leq d(x, z)$ for all $x, y, z \in X$.

We call an L-metric space *symmetric* if it satisfies

(LMs) $d(x, y) = d(y, x)$ for all $x, y \in X$.

A mapping between two L-metric spaces, $f : (X, d) \longrightarrow (X', d')$ is called an L-*metric morphism* if $d(x_1, x_2) \leq d'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-metric spaces with L-metric morphisms by L-MET, the subcategory with symmetric L-metric spaces is denoted by L-sMET

In case $L = (\{0, 1\}, \leq, \wedge)$, an L-metric space is a preordered set. If $\mathsf{L} = ([0, \infty], \geq, +)$ is Lawvere's quantale, an L-metric space is a quasi-metric space. If $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic quasi-metric space, see [7].

In [14] we defined, for an L-metric space (X, d), the L-convergence tower $\overline{q^d}$ by

$$x \in q_\alpha^d(\mathbb{F}) \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) \geq \alpha.$$

Similarly, we define an L-Cauchy tower, $\overline{C^d}$, by

$$\mathbb{F} \in C_\alpha^d \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{x, y \in F} d(x, y) \geq \alpha.$$

We note that $\mathbb{F} \in C_\top^d$ if and only if for all $\epsilon \lhd \top$ there is $F \in \mathbb{F}$ such that F has a "diameter" $\bigwedge_{x,y \in F} d(x, y) \geq \epsilon$. For Lawvere's quantale this is exactly the definition of a Cauchy filter in a metric space. We note that because $\mathbb{F}$ is a filter, $\bigvee_{F \in \mathbb{F}} \bigwedge_{x,y \in F} d(x, y)$ is a directed join. Hence we can replace $\epsilon \lhd \top$ by $\epsilon \ll \top$ here.

Proposition 4.1. *Let* $(X, d) \in |\mathsf{L\text{-}MET}|$*. Then* $(X, \overline{C^d}) \in |\mathsf{L\text{-}ChyTS}|$.

Proof. (LChyTS1) From $\bigvee_{F\in[x]} \bigwedge_{u,v\in F} d(u, v) \geq d(x, x) \geq \alpha$ it follows $[x] \in C^d_\alpha$.

(LChyTS2), (LChyTS3) and (LChyTS4) are obvious.

(LChyTS5) Let $\mathbb{F} \in C^d_\alpha$, $\mathbb{G} \in C^d_\beta$ and let $\mathbb{F} \vee \mathbb{G}$ exist. Let $\alpha' \lhd \alpha$ and $\beta' \lhd \beta$. Then there are $F \in \mathbb{F}$ such that for all $x, y \in F$ we have $d(x, y) \geq \alpha'$ and $G \in \mathbb{G}$ such that for all $u, v \in G$ we have $d(u, v) \geq \beta'$. As $\mathbb{F}\vee\mathbb{G}$ exists, there is $z \in F\cap G$ and hence we have, for all $x \in F$ and all $v \in G$, $d(x, z) \geq \alpha', d(z, v) \geq \beta'$ and $d(z, x) \geq \alpha'$ and $d(v, z) \geq \beta'$. Hence, for all $x \in F, v \in G$ we conclude $d(x, v) \geq d(x, z) * d(z, v) \geq \alpha' * \beta'$ and for all $x \in G, v \in F$ we conclude likewise $d(x, v) \geq d(x, z) * d(z, v) \geq \beta' * \alpha'$. If both $x, v \in F$, then $d(x, v) \geq \alpha' \geq \alpha' * \beta'$ and if both $x, v \in G$ then $d(x, v) \geq \beta' \geq \alpha' * \beta'$. Hence for all $x, v \in F \cup G$ we have $d(x, v) \geq \alpha' * \beta'$ and as the sets $F \cup G$ with $F \in \mathbb{F}, G \in \mathbb{G}$ form a basis of $\mathbb{F} \wedge \mathbb{G}$ we conclude $\bigvee_{H\in\mathbb{F}\wedge\mathbb{G}} \bigwedge_{x,v\in H} d(x, v) \geq \alpha' * \beta'$. From the complete distributivity finally we obtain $\bigvee_{H\in\mathbb{F}\wedge\mathbb{G}} \bigwedge_{x,v\in H} d(x, v) \geq \alpha * \beta$ and hence $\mathbb{F} \wedge \mathbb{G} \in C^d_{\alpha*\beta}$. □

It is not difficult to show that for a symmetric L-metric space (X, d), the space $(X, \overline{C^d})$ is a T1-space if and only if (X, d) is *separated*, i.e. if $d(x, y) = \top$ implies $x = y$.

Proposition 4.2. *Let* $f : (X, d) \longrightarrow (X', d')$ *be an* $\mathsf{L\text{-}MET}$*-morphism. Then* $f : (X, \overline{C^d}) \longrightarrow (X', \overline{C^{d'}})$ *is an* $\mathsf{L\text{-}ChyTS}$*-morphism.*

Proof. Let $\mathbb{F} \in C^d_\alpha$. Then $\bigvee_{H\in f(\mathbb{F})} \bigwedge_{u,v\in H} d'(u, v) \geq \bigvee_{F\in\mathbb{F}} \bigwedge_{x,y\in F} d'(f(x), f(y)) \geq \bigvee_{F\in\mathbb{F}} \bigwedge_{x,y\in F} d(x, y) \geq \alpha$ and hence $f(\mathbb{F}) \in C^{d'}_\alpha$. □

Hence we have a functor $\mathsf{F} : \mathsf{L\text{-}MET} \longrightarrow \mathsf{L\text{-}ChyTS}$, $\mathsf{F}((X, d)) = (X, \overline{C^d})$, $\mathsf{F}(f) = f$. This functor, restricted on $\mathsf{L\text{-}sMET}$, is injective on objects. Let $(X, d) \neq (X, d')$. Then there are $x, y \in X$ such that $d(x, y) \neq d'(x, y)$, i.e. without loss of generality $d(x, y) \not\leq d'(x, y)$. It is not difficult to show that $[x]\wedge[y] \in C^d_\alpha \iff d(x, y)\wedge d(y, x) = d(x, y) \geq \alpha$. As we have $[x]\wedge[y] \in C^d_{d(x,y)}$ but $[x] \wedge [y] \notin C^{d'}_{d(x,y)}$ we see that $\overline{C^d} \neq \overline{C^{d'}}$.

We define now for $(X, \overline{C}) \in |\mathsf{L\text{-}ChyTS}|$ a mapping $d^{\overline{C}} : X \times X \longrightarrow L$ by

$$d^{\overline{C}}(x, y) = \bigvee_{[x]\wedge[y]\in C_\alpha} \alpha.$$

Proposition 4.3. *Let* $(X, \overline{C}) \in |\mathsf{L\text{-}ChyTS}|$*. Then* $(X, d^{\overline{C}}) \in |\mathsf{L\text{-}sMET}|$*. If* $(X, \overline{C})$ *is a T1-space and is left-continuous, then* $(X, d^{\overline{C}})$ *is separated.*

Proof. (LM1) We have $d^{\overline{C}}(x, x) = \bigvee_{[x]\in C_\alpha} \alpha = \top$ by (LChyTS1).

(LM2) We have by the distributivity of the quantale operation over joins $d^{\overline{C}}(x, y) * d^{\overline{C}}(y, z) = \bigvee_{[x]\wedge[y]\in C_\alpha,[y]\wedge[z]\in C_\beta} \alpha * \beta$. Clearly $([x] \wedge [y]) \vee ([y] \wedge [z])$

exists and by (LChyTS5) and (LChyTS2) we conclude $d^{\overline{C}}(x,y) * d^{\overline{C}}(y,z) \leq \bigvee_{[x]\wedge[z]\in C_{\alpha*\beta}} \alpha*\beta \leq \bigvee_{[x]\wedge[z]\in C_\gamma} \gamma = d^{\overline{C}}(x,z)$.

(LMs) is obvious.

For the separation, let $\epsilon \lhd \top = d^{\overline{C}}(x,y)$. Then $[x]\wedge[y] \in C_\epsilon$. The left-continuity yields $[x]\wedge[y]\in C_\top$ and hence $x=y$ by (T1). □

Proposition 4.4. *Let $f:(X,\overline{C}) \longrightarrow (X',\overline{C'})$ be an* L-ChyTS*-morphism. Then $f:(X,d^{\overline{C}}) \longrightarrow (X',d^{\overline{C'}})$ is an* L-MET*-morphism.*

Proof. We have, using $f([x]\wedge[y]) = f([x])\wedge f([y]) = [f(x)]\wedge[f(y)]$ that $d^{\overline{C}}(x,y) = \bigvee_{[x]\wedge[y]\in C_\alpha} \alpha \leq \bigvee_{[f(x)]\wedge[f(y)]\in C'_\alpha} \alpha = d^{\overline{C'}}(f(x),f(y))$. □

Proposition 4.5. *Let $(X,d) \in |\mathsf{L\text{-}sMET}|$. Then $d^{\overline{C^d}} = d$.*

Proof. We use again $[x]\wedge[y]\in C^d_\alpha \iff d(x,y)\geq\alpha$ and conclude $d^{\overline{C^d}}(x,y) = \bigvee_{[x]\wedge[y]\in C^d_\alpha} \alpha = \bigvee_{d(x,y)\geq\alpha} \alpha = d(x,y)$. □

Hence the functor G which assigns to an L-Cauchy tower space $(X,\overline{C})$ the symmetric L-metric space $(X,d^{\overline{C}})$ and leaves morphisms unchanged, is left-inverse for F, in the sense that we have $\mathsf{G}\circ\mathsf{F}((X,d) = (X,d)$ for all $(X,d)\in|\mathsf{L\text{-}sMET}|$. However, L-sMET is in general not a reflective subcategory of $\mathsf{L}-\mathsf{ChyTS}$. The example in Remark 7.9 [21] can be used as a counterexample in the case of Lawvere's quantale.

We consider now the following axiom for $(X,\overline{C}) \in |\mathsf{L\text{-}ChyTS}|$.

$$(LChyM) \qquad \mathbb{F}\in C_\alpha \iff \forall\epsilon\lhd\alpha\exists F_\epsilon\in\mathbb{F}\forall x,y\in F_\epsilon : [x]\wedge[y]\in C_\epsilon.$$

Proposition 4.6. *Let $(X,d)\in|\mathsf{L\text{-}sMET}|$. Then $(X,\overline{C^d})$ satisfies (LChyM).*

Proof. Let first $\mathbb{F}\in C^d_\alpha$ and let $\epsilon\lhd\alpha$. Then there is $F_\epsilon\in\mathbb{F}$ such that for all $x,y\in F_\epsilon$ we have $d(x,y)\geq\epsilon$. The latter is equivalent to $[x]\wedge[y]\in C^d_\epsilon$.

Let now for all $\epsilon\lhd\alpha$ exist $F_\epsilon\in\mathbb{F}$ such that for all $x,y\in F_\epsilon$ we have $[x]\wedge[y]\in C^d_\epsilon$. Then

$$\bigvee_{F\in\mathbb{F}} \bigwedge_{u,v\in F} d(u,v) \geq \bigwedge_{u,v\in F_\epsilon} d(u,v) \geq \bigwedge_{[u]\wedge[v]\in C^d_\epsilon} d(u,v) = \bigwedge_{d(u,v)\geq\epsilon} d(u,v) \geq \epsilon.$$

The complete distributivity yields $\bigvee_{F\in\mathbb{F}}\bigwedge_{u,v\in F} d(u,v)\geq\alpha$, i.e. $\mathbb{F}\in C^d_\alpha$. □

Proposition 4.7. *Let $(X,\overline{C})\in|\mathsf{L\text{-}ChyTS}|$. If $(X,\overline{C})$ satisfies (LChyM), then $C^{d^{\overline{C}}}_\alpha = C_\alpha$ for all $\alpha\in L$.*

Proof. Let first $\mathbb{F}\in C^{d^{\overline{C}}}_\alpha$. Then $\bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F}\bigvee_{[x]\wedge[y]\in C_\beta}\beta\geq\alpha$. Let $\epsilon\lhd\alpha$. Then there is $F\in\mathbb{F}$ such that for all $x,y\in F$ there is $\beta\geq\epsilon$ with $[x]\wedge[y]\in C_\beta\subseteq C_\epsilon$. The axiom $(LChyM)$ then implies $\mathbb{F}\in C_\alpha$.

Conversely, let $\mathbb{F}\in C_\alpha$. The axiom $(LChyM)$ implies that for all $\epsilon\lhd\alpha$ there is $F_\epsilon\in\mathbb{F}$ such that for all $x,y\in F_\epsilon$ we have $[x]\wedge[y]\in C_\epsilon$. Therefore $\bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F}\bigvee_{[x]\wedge[y]\in C_\beta}\beta \geq \bigwedge_{x,y\in F_\epsilon}\bigvee_{[x]\wedge[y]\in C_\beta}\beta\geq\epsilon$ and again the complete distributivity yields $\mathbb{F}\in C^{d^{\overline{C}}}_\alpha$. □

From this we conclude the following result.

Theorem 4.8. *The subcategory of* L-ChyTS *with objects the spaces that satisfy the axiom* $(LChyM)$ *is isomorphic to the category* L-sMET.

Now we shall look into alternative characterizations of the L-Cauchy tower of an L-metric space in terms of the L-convergence tower $(X, \overline{q^d})$.

Proposition 4.9. *Let* $(X,d) \in |\mathsf{L}\text{-}\mathsf{MET}|$. *Then the following are equivalent.*
(i) For all $\epsilon \lhd \alpha$ *there is* $x \in X$ *such that* $B^d(x,\epsilon) \in \mathbb{F}$;
(ii) $\bigvee_{x\in X} \bigvee_{x \in q^d_\beta(\mathbb{F})} \beta \geq \alpha$.

Proof. (i)⇒(ii): Let $\epsilon \lhd \alpha$. Then there is $x \in X$ and $F \in \mathbb{F}$ such that $F \subseteq B^d(x,\epsilon)$. Hence for all $y \in F$ we have $d(x,y) \gg \epsilon$ and we conclude $\bigvee_{F\in\mathbb{F}} \bigwedge_{y \in F} d(x,y) \geq \epsilon$. This is equivalent to $x \in q^d_\epsilon(\mathbb{F})$ and hence we conclude $\bigvee_{x\in q^d_\beta(\mathbb{F})} \beta \geq \epsilon$. Therefore $\bigvee_{x\in X}\bigvee_{x\in q^d_\beta(\mathbb{F})} \beta \geq \epsilon$ and the complete distributivity yields $\bigvee_{x\in X}\bigvee_{x\in q^d_\beta(\mathbb{F})} \beta \geq \alpha$.

(ii)⇒(i): Let $\epsilon \lhd \alpha$. Then there is $x \in X$ and $\beta \rhd \epsilon$ such that $x \in q^d_\beta(\mathbb{F})$. Hence we have $\bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F} d(x,y) \geq \beta \rhd \epsilon$ and there is $F \in \mathbb{F}$ such that for all $y \in F$ we have $d(x,y) \rhd \epsilon$. As $\alpha \lhd \beta$ implies $\alpha \ll \beta$, we conclude $F \subseteq B^d(x,\epsilon)$, i.e. $B^d(x,\epsilon) \in \mathbb{F}$. □

Proposition 4.10. *Let* $(X,d) \in |\mathsf{L}\text{-}\mathsf{MET}|$ *and consider the following statements:*
(i) $\mathbb{F} \in C^d_\alpha$;
(ii) $\bigvee_{x\in X}\bigvee_{x\in q^d_\beta(\mathbb{F})} \beta \geq \alpha$;
(iii) $\mathbb{F} \in C^d_{\alpha*\alpha}$.
Then (i) implies (ii). If L *is divisible and* (X,d) *is symmetric, (ii) implies (iii).*

Proof. Let first $\mathbb{F} \in C^d_\alpha$. For $F \in \mathbb{F}$ we fix $x_F \in F$. Then $\alpha \leq \bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F} d(x,y) \leq \bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F} d(x_F,y)$. This shows $x_F \in q^d_\alpha(\mathbb{F})$ for all $x_F \in F$ and we conclude $\bigvee_{x\in X}\bigvee_{x\in q^d_\beta(\mathbb{F})}\beta \geq \alpha$.

Let now L be divisible and (X,d) be symmetric and let $\bigvee_{x\in X}\bigvee_{x\in q^d_\beta(\mathbb{F})}\beta \geq \alpha \rhd \epsilon$. Then there is $x_\epsilon \in X$ and $\beta \rhd \epsilon$ such that $x \in q^d_\beta(\mathbb{F})$. This means that $\bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F} d(x_\epsilon,y) \geq \beta \rhd \epsilon$ and hence there is $F_\epsilon \in \mathbb{F}$ such that for all $y \in F_\epsilon$ we have $d(x_\epsilon, y) \geq \epsilon$. For $u,v \in F_\epsilon$ we have by symmetry and the triangle inequality $d(u,v) \geq d(u,x_\epsilon) * d(x_\epsilon, v) \geq \epsilon * \epsilon$. Hence $\bigvee_{F\in\mathbb{F}}\bigwedge_{u,v\in F} d(u,v) \geq \epsilon*\epsilon$ and the complete distributivity yields $\mathbb{F} \in C^d_{\alpha*\alpha}$. □

Remark 4.11. (1) For Lawvere's quantale $\mathsf{L} = ([0,\infty], \geq, +)$ and the idempotent $\top$-element $\top = 0$, the above characterization yields, using $x \in q^d_\beta(\mathbb{F}) \iff \inf_{F\in\mathbb{F}}\sup_{y\in F} d(x,y) \leq \beta$,

$$\mathbb{F} \in C^d_0 \iff \inf_{x\in X}\inf_{F\in\mathbb{F}}\sup_{y\in F} d(x,y) = 0.$$

This is the definition of a Cauchy filter in the metric space viewed as an approach space [20].

(2) A sequence (x_n) in a probabilistic metric space [34] is called a *strong Cauchy sequence* if for all $t > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d(x_n, x_m)(t) > 1 - t$. This definition employs the strong uniformity, i.e. the system of entourages generated by the sets $U(t) = \{(x, y) \in X \times X \,:\, d(x, y)(t) > 1 - t\}$, $t > 0$. Equivalently, as is pointed out in [34], we can also use the system of entourages generated by the sets $U(t, \epsilon) = \{(x, y) \in X \times X \,:\, d(x, y)(t) > 1 - \epsilon\}$, $t, \epsilon > 0$. As in [16], using the quantale of distance distribution functions $(\Delta^+, \leq, *)$, we can see that the sets $B^d(\varphi) = \{(x, y) \in X \times X \,:\, d(x, y) \rhd \varphi\}$, $\varphi \lhd \epsilon_0$ generate the same uniformity. From this we can deduce that a sequence (x_n) is a strong Cauchy sequence in the probabilistic metric space (X, d) if and only if the generated filter is in $C^d_{\epsilon_0}$.

Note that for a divisible quantale L, a symmetric L-metric space (X, d) and an idempotent element $\alpha \in L$, in Proposition 4.10, (i) and (ii) are equivalent. In particular, for a frame L, we can characterize the L-Cauchy tower of a symmetric L-metric space by (ii) or by the equivalent statement (i) in Proposition 4.9. The following example shows that we cannot omit the idempotency here.

Example 4.12. Let $X = \mathbb{R}$ and $\mathsf{L} = ([0, \infty], \geq, +)$ be Lawvere's quantale and d the usual metric on $\mathbb{R}$. We consider, for $\alpha > 0$, the sequence $x_n = \frac{\alpha}{2} + \frac{1}{n}$ if n is even and $x_n = -\frac{\alpha}{2} - \frac{1}{n}$ if n is odd and denote $\mathbb{F}$ the filter generated by this sequence. Then

$$\bigvee_{F \in \mathbb{F}} \bigwedge_{x,y \in F} d(x, y) = \inf_{k \in \mathbb{N}} \sup_{m,n \geq k} \left| \frac{\alpha}{2} + \frac{1}{n} - (-\frac{\alpha}{2} - \frac{1}{m}) \right| = \inf_{k \in \mathbb{N}} \left| \alpha + \frac{2}{k} \right| = \alpha$$

and hence we have $\mathbb{F} \in C^d_\alpha$.

However, as $\bigvee_{x \in X} \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) = \inf_{x \in \mathbb{R}} \inf_{k \in \mathbb{N}} \sup_{m \geq k} (d(x, \frac{\alpha}{2} + \frac{1}{m}) \vee d(x, -\frac{\alpha}{2} - \frac{1}{m})) = \inf_{x \in \mathbb{R}} (\left| x - \frac{\alpha}{2} \right| \vee \left| x + \frac{\alpha}{2} \right|) = \frac{\alpha}{2}$, we have $\bigvee_{x \in X} \bigvee_{x \in q^d_\beta(\mathbb{F})} \beta = \frac{\alpha}{2}$ but $\mathbb{F} \notin C^d_{\alpha/2}$.

Proposition 4.13. *Let $(X, d) \in |\mathsf{L}\text{-}\mathsf{MET}|$. Then $q^{\overline{C^d}}_\alpha(\mathbb{F}) \subseteq q^d_\alpha(\mathbb{F})$ and if (X, d) is symmetric, we have $q^d_\alpha(\mathbb{F}) \subseteq q^{\overline{C^d}}_{\alpha * \alpha}(\mathbb{F})$.*

Proof. Let first $x \in q^{\overline{C^d}}_\alpha(\mathbb{F})$. Then $\mathbb{F} \wedge [x] \in C^d_\alpha$. Let $\epsilon \lhd \alpha$. Then there is $F \in \mathbb{F}$ such that for all $u, v \in F \cup \{x\}$ we have $d(u, v) \geq \epsilon$. We conclude for $u = x$ that $d(x, v) \geq \epsilon$ for all $v \in F$ and hence $\bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) \geq \epsilon$. The complete distributivity implies $x \in q^d_\alpha(\mathbb{F})$.

Let now (X, d) be symmetric and $x \in q^d_\alpha(\mathbb{F})$. Let $\epsilon \ll \alpha$. Noting that the set $D = \{\bigwedge_{y \in F} d(x, y) \,:\, F \in \mathbb{F}\}$ is directed, there is $F \in \mathbb{F}$ such that for all $y \in F$ we have $d(x, y) \geq \epsilon$. Let $u, v \in F \cup \{x\}$. We distinguish four cases.

Case 1: $u \neq x, v \neq x$. Then $d(x, u), d(x, v) \geq \epsilon$ and hence $d(u, v) \geq d(u, x) * d(x, v) \geq \epsilon * \epsilon$.

Case 2: $u = x, v \neq x$. Then $d(u,v) = d(x,v) \geq \epsilon \geq \epsilon * \epsilon$.
Case 3: $u \neq x, v = x$. Similar to case 2.
Case 4: $u = v = x$. Then $d(u,v) = d(x,x) = \top \geq \epsilon * \epsilon$.
Hence we have $\bigvee_{F\in\mathbb{F}} \bigwedge_{u,v\in F\cup\{x\}} d(u,v) \geq \epsilon*\epsilon$ and the continuity of L yields $\mathbb{F} \wedge [x] \in C^d_{\alpha*\alpha}$, i.e. $x \in q^{\overline{C^d}}_{\alpha*\alpha}(\mathbb{F})$. □

Again, for idempotent $\alpha \in L$, i.e. if $\alpha*\alpha = \alpha$, we have the equality $q^{\overline{C^d}}_{\alpha}(\mathbb{F}) = q^d_\alpha(\mathbb{F})$. In particular in a frame L, this equality is guaranteed for all $\alpha \in L$.

Example 4.14. We use Example 4.12 and show that $\frac{\alpha}{4} \in q^{\overline{C^d}}_{\alpha}(\mathbb{F})$. To this end, we consider the sequence $(y_n) = (x_1, \frac{\alpha}{4}, x_2, \frac{\alpha}{4}, x_3, \frac{\alpha}{4}, ...)$, the generated filter of which is $\mathbb{G} = \mathbb{F} \wedge [\frac{\alpha}{4}]$. We then have $\bigvee_{G\in\mathbb{G}} \bigwedge_{x,y\in G} d(x,y) = \inf_{k\in\mathbb{N}} \sup_{m,n\geq k} |y_n - y_m| = \alpha$ and hence we have $\mathbb{F} \wedge [\frac{\alpha}{4}] \in C^d_\alpha$, i.e. $\frac{\alpha}{4} \in q^{\overline{C^d}}_{\alpha}(\mathbb{F}) = q^{\overline{C^d}}_{\alpha/2+\alpha/2}(\mathbb{F})$. However, we see in a similar way that $\inf_{k\in\mathbb{N}} \sup_{n\geq k} \left|\frac{\alpha}{4} - x_n\right| = \frac{3}{4}\alpha \not\leq \frac{\alpha}{2}$, so that $\frac{\alpha}{4} \notin q^d_{\alpha/2}(\mathbb{F})$. So we have $q^{\overline{C^d}}_{\alpha/2+\alpha/2}(\mathbb{F}) \not\subseteq q^d_\alpha(\mathbb{F})$.

5. Example: L-uniform convergence tower spaces and L-uniform tower spaces

Definition 5.1 ([15])**.** A pair $\big(X, \overline{\Lambda} = (\Lambda_\alpha)_{\alpha\in L}\big)$, is called an L-*uniform convergence tower space*, if $\Lambda_\alpha \subseteq \mathsf{F}(X\times X)$, $\alpha \in L$, satisfy the following:

(LUCTS1) $[(x,x)] \in \Lambda_\alpha$ for all $x \in X$, $\alpha \in L$;
(LUCTS2) $\Phi \in \Lambda_\alpha$ whenever $\Phi \leq \Psi$ and $\Psi \in \Lambda_\alpha$;
(LUCTS3) $\Phi, \Psi \in \Lambda_\alpha$ implies $\Phi \wedge \Psi \in \Lambda_\alpha$;
(LUCTS4) $\Lambda_\beta \subseteq \Lambda_\alpha$ whenever $\alpha \leq \beta$;
(LUCTS5) $\Phi^{-1} \in \Lambda_\alpha$ whenever $\Phi \in \Lambda_\alpha$;
(LUCTS6) $\Phi\circ\Psi \in \Lambda_{\alpha*\beta}$ whenever $\Phi \in \Lambda_\alpha, \Psi \in \Lambda_\beta$ and $\Phi\circ\Psi$ exists;
(LUCTS7) $\Lambda_\perp = \mathsf{F}(X\times X)$.

A mapping $f\colon \big(X, \overline{\Lambda}\big) \longrightarrow \big(X', \overline{\Lambda'}\big)$ is called *uniformly continuous* if, for all $\Phi \in \mathsf{F}(X\times X)$, $(f\times f)(\Phi) \in \Lambda'_\alpha$, whenever $\Phi \in \Lambda_\alpha$. The category of L-uniform convergence tower spaces and uniformly continuous mappings is denoted by $\mathsf{L\text{-}UCTS}$.

If $\mathsf{L} = ([0,1], \leq, *)$, then we obtain Nusser's probabilistic uniform convergence spaces [26, 27]. For $\mathsf{L} = (\Delta^+, \leq, *)$ we obtain the probabilistic uniform convergence spaces in [2].

For $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$, $\mathbb{F} \in \mathsf{F}(X)$, $x \in X$ and $\alpha \in L$ we define

$$x \in q^{\overline{\Lambda}}_\alpha(\mathbb{F}) \iff \mathbb{F}\times[x] \in \Lambda_\alpha.$$

It is not difficult to show that $(X, \overline{q^{\overline{\Lambda}}} = (q^{\overline{\Lambda}}_\alpha)_{\alpha\in L}) \in |\mathsf{L\text{-}CTS}|$ is an L-limit tower space.

For $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$ and $\alpha \in L$, a filter $\mathbb{F} \in \mathsf{F}(X)$ is called an α-*Cauchy filter*, written as $\mathbb{F} \in C^{\overline{\Lambda}}_\alpha$ if and only if $\mathbb{F}\times\mathbb{F} \in \Lambda_\alpha$.

Proposition 5.2. *Let* $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$*. Then* $(X, \overline{C^{\overline{\Lambda}}}) \in |\mathsf{L\text{-}ChyTS}|$.

Proof. (LChyTS1) Since for all $x \in X$ and $\alpha \in L$, $[x] \times [x] \in \Lambda_\alpha$, we have $[x] \in C^{\overline{\Lambda}}_\alpha$.
(LChyTS2) Let $\mathbb{G} \geq \mathbb{F}$ with $\mathbb{F} \in C^{\overline{\Lambda}}_\alpha$. Then $\mathbb{F} \times \mathbb{F} \in \Lambda_\alpha$ and hence, by (LUCTS2), $\mathbb{G} \times \mathbb{G} \in \Lambda_\alpha$, i.e. $\mathbb{G} \in C^{\overline{\Lambda}}_\alpha$.
(LChyTS3) Let $\alpha \leq \beta$ and $\mathbb{F} \in C^{\overline{\Lambda}}_\beta$. Then $\mathbb{F} \times \mathbb{F} \in \Lambda_\beta$, then by (LUCTS4), $\mathbb{F} \times \mathbb{F} \in \Lambda_\alpha$ which in turn yields $\mathbb{F} \in C^{\overline{\Lambda}}_\alpha$.
(LChyTS4) Follows at once from the definition.
(LChyTS5) Let $\alpha, \beta \in L$, and let $\mathbb{F} \in C^{\overline{\Lambda}}_\alpha$ and $\mathbb{G} \in C^{\overline{\Lambda}}_\beta$ such that $\mathbb{F} \vee \mathbb{G}$ exists. Then $\mathbb{F} \times \mathbb{F} \in \Lambda_\alpha$ and $\mathbb{G} \times \mathbb{G} \in \Lambda_\beta$. Then $\mathbb{F} \times \mathbb{G} = (\mathbb{F} \times \mathbb{G}) \circ (\mathbb{F} \times \mathbb{G}) \in \Lambda_{\alpha * \beta}$. Also, $\mathbb{G} \times \mathbb{F} \in \Lambda_{\alpha * \beta}$. Since

$$(\mathbb{F} \wedge \mathbb{G}) \times (\mathbb{F} \wedge \mathbb{G}) = (\mathbb{F} \times \mathbb{F}) \wedge (\mathbb{F} \times \mathbb{G}) \wedge (\mathbb{G} \times \mathbb{F}) \wedge (\mathbb{G} \times \mathbb{G}) \in \Lambda_{\alpha * \beta},$$

this implies $\mathbb{F} \wedge \mathbb{G} \in C^{\overline{\Lambda}}_{\alpha * \beta}$. □

Proposition 5.3. *Let* $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$*. Then* $q^{\overline{C^{\overline{\Lambda}}}}_\alpha(\mathbb{F}) \subseteq q^{\overline{\Lambda}}_\alpha(\mathbb{F}) \subseteq q^{\overline{C^{\overline{\Lambda}}}}_{\alpha * \alpha}(\mathbb{F})$.

Proof. Let $x \in q^{\overline{C^{\overline{\Lambda}}}}_\alpha(\mathbb{F})$, then $\mathbb{F} \wedge [x] \in C^{\overline{\Lambda}}_\alpha$. This implies that $(\mathbb{F} \wedge [x]) \times (\mathbb{F} \wedge [x]) \in \Lambda_\alpha$. As $(\mathbb{F} \wedge [x]) \times (\mathbb{F} \wedge [x]) \leq \mathbb{F} \times [x]$, (LUCTS2) implies $x \in q^{\overline{\Lambda}}_\alpha(\mathbb{F})$.

If $x \in q^{\overline{\Lambda}}_\alpha(\mathbb{F})$, then $\mathbb{F} \times [x] \in \Lambda_\alpha$ and with (LUCTS5) then also $[x] \times \mathbb{F} = (\mathbb{F} \times [x])^{-1} \in \Lambda_\alpha$. From (LUCTS6) we obtain $\mathbb{F} \times \mathbb{F} = (\mathbb{F} \times [x]) \circ ([x] \times \mathbb{F}) \in \Lambda_{\alpha * \alpha}$. This yields with (LUCTS3) $(\mathbb{F} \wedge [x]) \times (\mathbb{F} \wedge [x]) = (\mathbb{F} \times \mathbb{F}) \wedge (\mathbb{F} \times [x]) \wedge ([x] \times \mathbb{F}) \wedge ([x] \times [x]) \in \Lambda_{\alpha * \alpha}$. Thus, $\mathbb{F} \wedge [x] \in C^{\overline{\Lambda}}_{\alpha * \alpha}$ which means $x \in q^{\overline{C^{\overline{\Lambda}}}}_{\alpha * \alpha}(\mathbb{F})$. □

Again, for idempotent $\alpha \in L$, we have equality, $q^{\overline{C^{\overline{\Lambda}}}}_\alpha(\mathbb{F}) = q^{\overline{\Lambda}}_\alpha(\mathbb{F})$. In particular this is the case if L is a frame.

Definition 5.4 ([15])**.** Let $\mathsf{L} = (L, \leq, *)$ be a quantale. A pair $\big(X, \overline{\mathcal{U}}\big)$ with $\overline{\mathcal{U}} = (\mathcal{U}_\alpha)_{\alpha \in L}$ a family of filters on $X \times X$ is called an L*-uniform tower space* if for all $\alpha \in L$ the following holds:

(LUTS1) $\mathcal{U}_\alpha \leq [\Delta]$ with $[\Delta] = \bigwedge_{x \in X} [(x, x)]$;
(LUTS2) $\mathcal{U}_\alpha \leq (\mathcal{U}_\alpha)^{-1}$;
(LUTS3) $\mathcal{U}_{\alpha * \beta} \leq \mathcal{U}_\alpha \circ \mathcal{U}_\beta$;
(LUTS4) $\mathcal{U}_\alpha \leq \mathcal{U}_\beta$ whenever $\alpha \leq \beta$;
(LUTS5) $\mathcal{U}_\perp = \bigwedge \mathsf{F}(X \times X)$;
(LUTS6) $\mathcal{U}_{\bigvee A} \leq \bigvee_{\alpha \in A} \mathcal{U}_\alpha$ whenever $\varnothing \neq A \subseteq L$.

A mapping $f \colon \big(X, \overline{\mathcal{U}}\big) \longrightarrow \big(X', \overline{\mathcal{U}'}\big)$ is called *uniformly continuous* if $\mathcal{U}'_\alpha \leq (f \times f)(\mathcal{U}_\alpha)$ for all $\alpha \in L$. The category with objects all L-uniform tower spaces and uniformly continuous mappings as morphisms is denoted by $\mathsf{L\text{-}UTS}$.

If $\mathsf{L} = ([0,1], \leq, *)$ with a t-norm $*$, then we obtain Florescu's probabilistic uniform spaces [8], for $\mathsf{L} = (\Delta^+, \leq, *)$ we obtain the probabilistic uniform spaces in [2]. For Lawvere's quantale an L-uniform tower space is an approach uniform space [23].

For $\big(X,\overline{\mathcal{U}}=(\mathcal{U}_\alpha)_{\alpha\in L}\big)\in|\mathsf{L}-\mathsf{UTS}|$ and $\alpha\in L$, a filter $\mathbb{F}\in\mathsf{F}(X)$ is called an *α-Cauchy filter* written as $\mathbb{F}\in C_\alpha^{\overline{\mathcal{U}}}$ if and only if $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_\alpha$.

Lemma 5.5. *Let $(X,\overline{\mathcal{U}})\in|\mathsf{L\text{-}UTS}|$. Then $(X,C^{\overline{\mathcal{U}}})\in|\mathsf{L\text{-}ChyTS}|$.*

Proof. (LChyTS1) Clearly by (LUTS1), $\mathcal{U}_\alpha\leq[x]\times[x]$, implies $[x]\in C_\alpha^{\overline{\mathcal{U}}}$, for all $x\in X$ and $\alpha\in L$. (LChyTS2) Let $\mathbb{F},\mathbb{G}\in\mathsf{F}(X)$ with $\mathbb{F}\leq\mathbb{G}$, and $\mathbb{F}\in C_\alpha^{\overline{\mathcal{U}}}$. Then $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_\alpha$ but then $\mathbb{G}\times\mathbb{G}\geq\mathcal{U}_\alpha$, which in turn implies $\mathbb{G}\in C_\alpha^{\overline{\mathcal{U}}}$. (LChyTS3) Let $\alpha\leq\beta$ and $\mathbb{F}\in C_\beta^{\overline{\mathcal{U}}}$, implying $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_\beta$. But then $\mathcal{U}_\beta\geq\mathcal{U}_\alpha$ by (LUTS4) for $\alpha\leq\beta$. So, $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_\alpha$ which gives $\mathbb{F}\in C_\alpha^{\overline{\mathcal{U}}}$. The condition (LChyTS4) is trivially true. Finally, we check the condition (LChyTS5): let $\mathbb{F}\in C_\alpha^{\overline{\mathcal{U}}}$ and $\mathbb{G}\in C_\beta^{\overline{\mathcal{U}}}$ such that $\mathbb{F}\vee\mathbb{G}$ exists. Then $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_\alpha$ and $\mathbb{G}\times\mathbb{G}\geq\mathcal{U}_\beta$. Then $(\mathbb{F}\times\mathbb{F})\circ(\mathbb{G}\times\mathbb{G})\geq\mathcal{U}_\alpha\circ\mathcal{U}_\beta$ which in view of the condition (LUTS3) yields that $\mathbb{F}\times\mathbb{G}=(\mathbb{F}\times\mathbb{F})\circ(\mathbb{G}\times\mathbb{G})\geq\mathcal{U}_{\alpha*\beta}$. This means that $\mathbb{F}\times\mathbb{G}\geq\mathcal{U}_{\alpha*\beta}$. Furthermore, note that we also have: $\mathbb{G}\times\mathbb{F}\geq\mathcal{U}_{\alpha*\beta}$, $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_{\alpha*\beta}$, and $\mathbb{G}\times\mathbb{G}\geq\mathcal{U}_{\alpha*\beta}$; this is due to the condition (LUTS5) and the fact that $\alpha*\beta\leq\alpha,\beta$. Hence we arrive at the following:

$$(\mathbb{F}\wedge\mathbb{G})\times(\mathbb{F}\wedge\mathbb{G})=(\mathbb{F}\times\mathbb{F})\wedge\mathbb{G}\times\mathbb{G})\wedge(\mathbb{F}\times\mathbb{G})\wedge(\mathbb{G}\times\mathbb{F})\geq\mathcal{U}_{\alpha*\beta},$$

that is, $(\mathbb{F}\wedge\mathbb{G})\times(\mathbb{F}\wedge\mathbb{G})\geq\mathcal{U}_{\alpha*\beta}$ which in turn implies that $\mathbb{F}\wedge\mathbb{G}\in C_{\alpha*\beta}^{\overline{\mathcal{U}}}$. □

Given an L-metric space (X,d) we define $\Lambda_\alpha^d\subseteq\mathsf{F}(X\times X)$ for $\alpha\in L$, by [15]

$$\Phi\in\Lambda_\alpha^d\Longleftrightarrow\bigvee_{F\in\Phi}\bigwedge_{(x,y)\in F}\ d(x,y)\geq\alpha.$$

Theorem 5.6. *The L-Cauchy tower space $(X,\overline{C^d})$ of an L-metric space (X,d) is the same as the L-Cauchy tower space $(X,C^{\overline{\Lambda^d}})$ induced by the L-uniform convergence tower space $(X,\overline{\Lambda^d})$ of an L-metric space. That is, $\overline{C^d}=\overline{C^{\overline{\Lambda^d}}}$.*

Proof. This follows from $\bigvee_{H\in\mathbb{F}\times\mathbb{F}}\bigwedge_{(x,y)\in H}d(x,y)=\bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F}d(x,y)$ for $\mathbb{F}\in\mathsf{F}(X)$, as the sets $F\times F$ with $F\in\mathbb{F}$ are a basis of $\mathbb{F}\times\mathbb{F}$. □

For an L-metric space (X,d) we define $U_\epsilon=\{(x,y)\in X\times X\ :\ d(x,y)\gg\epsilon\}$. For $\epsilon\ll\alpha$, these sets form the basis of a filter $\mathcal{U}_\alpha^d$ and we have $\mathcal{U}_\alpha^d=\bigwedge_{\Phi\in\Lambda_\alpha^d}\Phi$, see [15]. We call $(X,\overline{\mathcal{U}^d})$ the L-uniform tower space induced by the L-metric space (X,d).

Proposition 5.7. *Let $(X,d)\in|\mathsf{L\text{-}MET}|$. Then $C_\alpha^{\overline{\mathcal{U}^d}}=C_\alpha^d$ for all $\alpha\in L$.*

Proof. Let $\mathbb{F}\in C_\alpha^{\overline{\mathcal{U}^d}}$. Then $\mathbb{F}\times\mathbb{F}\in\mathcal{U}_\alpha^d$ and for all $\epsilon\ll\alpha$ there is $F\in\mathbb{F}$ such that $F\times F\subseteq U_\epsilon$. Hence for all $\epsilon\ll\alpha$ there is $F\in\mathbb{F}$ such that $\bigwedge_{x,y\in F}d(x,y)\geq\epsilon$. This implies $\bigwedge_{x,y\in F}d(x,y)\geq\alpha$ from which we conclude $\mathbb{F}\in C_\alpha^d$.

Conversely, we note that the set $\{\bigwedge_{x,y\in F}d(x,y)\ :\ F\in\mathbb{F}\}$ is directed. If $\mathbb{F}\in C_\alpha^d$, then $\bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F}d(x,y)\geq\alpha$. Hence, for $\epsilon\ll\alpha$ there is $F\in\mathbb{F}$ such that for all $x,y\in F$ we have $d(x,y)\gg\epsilon$. This means $F\times F\subseteq U_\epsilon$ and we have $\mathbb{F}\times\mathbb{F}\geq\mathcal{U}_\alpha^d$. □

6. Example: L-limit tower groups

Let $(X, \cdot)$ be a group with identity element $e \in X$ and let $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathsf{F}(X)$. We define $\mathbb{F} \odot \mathbb{G}$ as the filter with filter basis $\{F \cdot G \,:\, F \in \mathbb{F}, G \in \mathbb{G}\}$ and $\mathbb{F}^{-1}$ as the filter with filter basis $\{F^{-1} \,:\, F \in \mathbb{F}\}$. Then, for $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathsf{F}(X)$ and $x, y \in X$, the following properties are easily verified: $\mathbb{F} \odot \mathbb{F}^{-1} \leq [e]$; $[x] \odot [x]^{-1} = [e]$; $[x^{-1}] = [x]^{-1}$; $[x \cdot y] = [x] \cdot [y]$; $(\mathbb{F} \odot \mathbb{G}) \odot \mathbb{H} = \mathbb{F} \odot (\mathbb{G} \odot \mathbb{H})$; $(\mathbb{F}^{-1})^{-1} = \mathbb{F}$; $(\mathbb{F} \odot \mathbb{G})^{-1} = \mathbb{G}^{-1} \odot \mathbb{F}^{-1}$; $[e] \odot \mathbb{F} = \mathbb{F} \odot [e] = \mathbb{F}$; $(\mathbb{F} \wedge \mathbb{G})^{-1} = \mathbb{F}^{-1} \wedge \mathbb{G}^{-1}$ and $(\mathbb{F} \wedge \mathbb{G}) \odot \mathbb{H} = (\mathbb{F} \odot \mathbb{H}) \wedge (\mathbb{G} \odot \mathbb{H})$.

Definition 6.1. Let $\mathsf{L} = (L, \leq, *)$ be a quantale, and $(X, \cdot)$ be a group with identity e. Then a triple $(X, \cdot, \overline{q} = (q_\alpha)_{\alpha \in L})$ is called an L-*convergence tower group* (respectively, an L-*limit tower group*) if the following conditions are fulfilled:

(LCTG) If $(X, \overline{q})$ is an L-convergence tower space (resp. an L-limit tower space)
(LCTGM) $x \in q_\alpha(\mathbb{F})$, $y \in q_\beta(\mathbb{G})$ implies $xy \in q_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$, for all $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$, for all $\alpha, \beta \in L$, and $x, y \in X$;
(LCTGI) $x \in q_\alpha(\mathbb{F})$ implies $x^{-1} \in q_\alpha(\mathbb{F}^{-1})$, for all $\mathbb{F} \in \mathsf{F}(X)$, $x \in X$ and $\alpha \in L$.

The category of all L-convergence tower groups and continuous group homomorphisms is denoted by L-CTGrp (respectively, the category of all L-limit tower groups and continuous group homomorphisms is denoted by L-LIMGrp.)

If $L = \{0, 1\}$, then we obtain classical convergence groups [28]. If $\mathsf{L} = ([0,1], \leq, *)$ with a continuous t-norm $*$, then we get probabilistic convergence group under a t-norm in the sense of [13]. If $\mathsf{L} = ([0, \infty], \geq, +)$, then a left-continuous L-convergence tower group is an approach group [24]. If $\mathsf{L} = (\Delta^+, \leq, *)$, we obtain a probabilistic convergence group in the definition of [3].

For $(X, \cdot, \overline{q}) \in |\mathsf{L} - \mathsf{LIMTGrp}|$, a filter $\mathbb{F} \in \mathsf{F}(X)$ is called an α-*Cauchy filter*, written $\mathbb{F} \in C_\alpha^{\overline{q}}$, if and only if $e \in q_\alpha\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$.

Proposition 6.2. *Let* $(X, \cdot, \overline{q}) \in |\mathsf{L}\text{-}\mathsf{LIMGrp}|$. *Then* $\left(X, \overline{C^q}\right) \in |\mathsf{L}\text{-}\mathsf{ChyTS}|$.

Proof. (LChyTS1) Since $e \in q_\alpha([x]^{-1} \odot [x])$ implies $[x] \in C_\alpha^q$.
(LChyTS2) Let $\mathbb{F} \leq \mathbb{G}$ with $\mathbb{F} \in C_\alpha^q$. This implies $e \in q_\alpha\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$. But then $e \in q_\alpha\left(\mathbb{G}^{-1} \odot \mathbb{G}\right)$ which gives $\mathbb{G} \in C_\alpha^q$.
(LChyTS3) Let $\alpha, \beta \in L$ with $\alpha \leq \beta$ and $\mathbb{F} \in C_\beta^q$. Then $e \in q_\beta\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$ implies $e \in q_\alpha\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$. Hence $\mathbb{F} \in C_\alpha^q$.
(LChyTS4) Obvious.
(LChyTS5) Let $\mathbb{F} \in C_\alpha$ and $\mathbb{G} \in C_\beta^q$ such that $\mathbb{F} \vee \mathbb{G}$ exists. It is not difficult to prove that $\mathbb{F} \odot \mathbb{G}^{-1} \leq [e]$. As $e \in q_\alpha\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$ and $e \in q_\beta\left(\mathbb{G}^{-1} \odot \mathbb{G}\right)$, we obtain with condition (LCTGM) $e = ee \in q_{\alpha*\beta}\left(\mathbb{F}^{-1} \odot \mathbb{F} \odot \mathbb{G}^{-1} \odot \mathbb{G}\right)$ which implies that $e \in q_{\alpha*\beta}\left(\mathbb{F}^{-1} \odot \mathbb{G}\right)$. Similarly, we can get $e \in q_{\alpha*\beta}\left(\mathbb{G}^{-1} \odot \mathbb{F}\right)$. Since

$$(\mathbb{F} \wedge \mathbb{G})^{-1} \odot (\mathbb{F} \wedge \mathbb{G}) = (\mathbb{F}^{-1} \wedge \mathbb{G}^{-1}) \odot (\mathbb{F} \wedge \mathbb{G})$$
$$= (\mathbb{F}^{-1} \odot \mathbb{F}) \wedge (\mathbb{F}^{-1} \odot \mathbb{G}) \wedge (\mathbb{G}^{-1} \odot \mathbb{F}) \wedge (\mathbb{G}^{-1} \odot \mathbb{G}),$$

we conclude from condition (LCTS5) $e \in q_{\alpha*\beta}\left((\mathbb{F}\wedge\mathbb{G})^{-1}\odot(\mathbb{F}\wedge\mathbb{G})\right)$. Hence $\mathbb{F}\wedge\mathbb{G}\in C^q_{\alpha*\beta}$. □

Definition 6.3 ([14])**.** A triple $(X,\cdot,d)$ is called an L-*metric group* if d is invariant, i.e., $d(x,y) = d(xz,yz) = d(zx,zy)$ for all $x,y,z \in X$. A group homomorphism $f\colon (X,\cdot) \longrightarrow (X',\cdot')$ between the L-metric groups $(X,\cdot,d)$ and $(X',\cdot',d')$ is called an L-METGrp-*morphism* if it is an L-MET-morphism between (X,d) and $(X'd')$. The category of L-metric groups is denoted by L-METGrp.

Lemma 6.4. *Let $(X,\cdot,d)$ be a symmetric L-metric group. Then $(X,\cdot,\overline{q^d})$ is an L-convergence tower group.*

Proof. We need to show the conditions (LCTGM) and (LCTGI). For (LCTGM), let $x \in q^d_\alpha(\mathbb{F})$ and $y \in q^d_\beta(\mathbb{G})$. For $\epsilon \lhd \alpha$ and $\delta \lhd \beta$, there is $F \in \mathbb{F}$ and $G \in \mathbb{G}$ such that for all $u \in F$ and for all $v \in G$ we have $d(x,u) \geq \epsilon$ and $d(y,v) \geq \delta$. Then $d(e,x^{-1}u) \geq \epsilon$ and $d(yv^{-1},e) \geq \delta$ and hence $d(yv^{-1},x^{-1}u) \geq \epsilon*\delta$. This implies $d(xy,uv) \geq \epsilon*\delta$ for all $u \in F, v \in G$ and hence $d(xy,h) \geq \epsilon*\delta$ for all $h \in F\odot G$. Therefore $\bigvee_{H\in\mathbb{F}\odot\mathbb{G}}\bigwedge_{h\in H} d(xy,h) \geq \epsilon*\delta$ and from the complete distributivity we obtain $xy \in q^d_{\alpha*\beta}(\mathbb{F}\odot\mathbb{G})$.

(LCTGI) follows with the symmetry from $d(x,y) = d(xy^{-1},e) = d(y^{-1},x^{-1})$. □

Proposition 6.5. *Let (X,d) ba a symmetric L-metric group. Then $C^d_\alpha = C^{\overline{q^d}}_\alpha$ for all $\alpha \in L$.*

Proof. $\mathbb{F} \in C^{\overline{q^d}}_\alpha$ is equivalent to $\bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F} d(e,x^{-1}y) \geq \alpha$, which is, by the invariance of the L-metric, equivalent to $\bigvee_{F\in\mathbb{F}}\bigwedge_{x,y\in F} d(x,y) \geq \alpha$, i.e. equivalent to $\mathbb{F} \in C^d_\alpha$. □

7. Completeness and completion

Following [33, 27] we call $(X,\overline{C}) \in |\mathsf{L\text{-}ChyTS}|$ *complete* if for all $\alpha \in L$, $\mathbb{F} \in C_\alpha$ implies the existence of $x \in X$ such that $\mathbb{F}\wedge[x] \in C_\alpha$.

With this definition, the "point of convergence" $x = x(\mathbb{F},\alpha)$ not only depends on the filter $\mathbb{F}$ but may also depend on the "level" $\alpha \in L$. In the left-continuous case, we can omit this extra dependency.

Proposition 7.1. *Let $(X,\overline{C}) \in |\mathsf{L\text{-}ChyTS}|$ be left-continuous. Then $(X,\overline{C})$ is complete if for all $\mathbb{F} \in \mathsf{F}(X)$ there is $x = x(\mathbb{F})$ such that for all $\alpha \in L$, $\mathbb{F} \in C_\alpha$ implies $\mathbb{F}\wedge[x] \in C_\alpha$.*

Proof. Let $\mathbb{F} \in \mathbb{F}(X)$ and define $A = \{\alpha \in L : \mathbb{F} \in C_\alpha\}$ and $\delta = \bigvee A$. By left-continuity $\mathbb{F} \in C_\delta$ and hence there is $x = x(\mathbb{F},\delta)$ such that $\mathbb{F}\wedge[x] \in C_\delta$. If $\mathbb{F} \in C_\alpha$, then $\alpha \leq \delta$ and hence $\mathbb{F}\wedge[x] \in C_\delta \subseteq C_\alpha$ and we can choose x for each α. □

The following examples show that the left-continuity is essential.

Example 7.2. Let $\mathsf{L} = \{\bot, \alpha, \beta, \top\}$ with α, β incomparable, $* = \wedge$ and let X be an infinite set. There is an ultrafilter $\mathbb{U} \in \mathsf{U}(X)$ with $\mathbb{U} \not\geq [x]$ for all $x \in X$. We fix $x_0, y_0 \in X$ with $x_0 \neq y_0$ and define an L-Cauchy tower as follows. We let $C_\bot = \mathsf{F}(X)$, $\mathbb{F} \in C_\alpha$ iff $\mathbb{F} \geq \mathbb{U} \wedge [x_0]$ or if $\mathbb{F} = [x]$ for some $x \in X$. Similarly, $\mathbb{F} \in C_\beta$ iff $\mathbb{F} \geq \mathbb{U} \wedge [y_0]$ or $\mathbb{F} = [x]$ for some $x \in X$, and finally $\mathbb{F} \in C_\top$ if $\mathbb{F} = [x]$ for some $x \in X$. We only show (LChyTS5) as the other axioms are easy. Let $\mathbb{F} \in C_\gamma$, $\mathbb{G} \in C_\delta$ and $\mathbb{F} \vee \mathbb{G}$ exist. We distinguish three cases.

Case 1: $\gamma = \alpha, \delta = \beta$. Then $\alpha \wedge \beta = \bot$ and hence $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha \wedge \beta}$.

Case 2: $\mathbb{F}, \mathbb{G} \in C_\alpha$. If $\mathbb{F} = [x]$ and $\mathbb{G} = [y]$ then $x = y$ because $\mathbb{F} \vee \mathbb{G}$ exists and hence $\mathbb{F} \wedge \mathbb{G} = [x] \in C_\alpha$. If $\mathbb{F} = [x]$ and $\mathbb{G} = \mathbb{U} \wedge [x_0]$, then $[x] \vee (\mathbb{U} \wedge [x_0])$ exists. If $x \neq x_0$ we define $F = X \setminus \{x\}$. Then either $F \in \mathbb{U}$ or its complement $F^c \in \mathbb{U}$ and also $F \in [x_0]$. If $F \in \mathbb{U}$ then $F \in \mathbb{U} \wedge [x_0]$ and hence also $F \in [x] \vee (\mathbb{U} \wedge [x_0])$, in contradiction to $F^c \in [x]$ and $F \cap F^c = \varnothing$. If $F \notin \mathbb{U}$ then $F^c \in \mathbb{U}$ and hence $\mathbb{U} = [x]$, again a contradiction. We conclude $x = x_0$, i.e. $\mathbb{F} = [x_0]$ and $\mathbb{G} = \mathbb{U} \wedge [x_0]$ and therefore $\mathbb{F} \wedge \mathbb{G} = \mathbb{G} \in C_\alpha = C_{\alpha \wedge \alpha}$.

Case 3: $\mathbb{F}, \mathbb{G} \in C_\beta$. This is similar.

We note that $(X, \overline{C})$ is complete but since we have $\mathbb{U} \in C_\alpha$, $\mathbb{U} \in C_\beta$ and $\mathbb{U} \notin C_{\alpha \vee \beta} = C_\top$, it is not left-continuous.

Clearly, $\mathbb{U} \in C_\alpha$ and $\mathbb{U} \wedge [x_0] \in C_\alpha$. Likewise, $\mathbb{U} \in C_\beta$ and $\mathbb{U} \wedge [y_0] \in C_\beta$. As we have seen above $\mathbb{U} \wedge [x] \in C_\alpha$ implies $x = x_0$ and hence in particular $\mathbb{U} \wedge [y_0] \notin C_\alpha$. This shows that the point of convergence for $\mathbb{U}$ is different for α and β.

Example 7.3. We consider $X = [0, \infty)$ and Lawvere's quantale $\mathsf{L} = ([0, \infty], \geq, +)$. Again we choose an ultrafilter $\mathbb{U} \in \mathsf{U}(X)$ with $\mathbb{U} \neq [x]$ for all $x \in [0, \infty)$. We define $C_\infty = \mathsf{F}(X)$ and for $\alpha < \infty$ we define $\mathbb{F} \in C_\alpha$ if $\mathbb{F} = [x]$ for some $x \in X$ or if $\mathbb{F} \geq \mathbb{U} \wedge [x]$ for some $0 < x < \alpha$. Then $([0, \infty), \overline{C})$ is an L-Cauchy tower space. We again only show (LChyTS5). Let $\mathbb{F} \in C_\alpha, \mathbb{G} \in C_\beta$ and let $\mathbb{F} \vee \mathbb{G}$ exist. The case $\mathbb{F} = [x]$ and $\mathbb{G} = [y]$ implies $x = y$ and then $\mathbb{F} \wedge \mathbb{G} = [x] \in C_{\alpha \wedge \beta}$. If $\mathbb{F} = [x]$ and $\mathbb{G} \geq \mathbb{U} \wedge [y]$ with $0 < y < \beta$ implies again $x = y$ and we obtain $\mathbb{F} \wedge \mathbb{G} = \mathbb{G} \in C_\beta \subseteq C_{\alpha + \beta}$. It remains the case $\mathbb{G} \geq \mathbb{U} \wedge [x], \mathbb{G} \geq \mathbb{U} \wedge [y]$. If $x = y$, trivially $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha + \beta}$. If $x \neq y$, then we choose two sets $F_1, F_2 \subseteq [0, 1]$ with non-empty and finite complement and $F_1 \cap F_2 = \varnothing$ such that $x \in F_1, y \in F_2$. Then $F_2 \in \mathbb{U} \wedge [x]$, as $F_2 \in [x]$ and if $F_2 \notin \mathbb{U}$, then the complement $F_2^c \in \mathbb{U}$ and as F_2^c is a finite set, then $\mathbb{U} = [z]$ for $z \in F_2^c$. Similarly, $F_1 \in \mathbb{U} \wedge [y]$. As $\mathbb{H} = (\mathbb{U} \wedge [x]) \vee (\mathbb{U} \wedge [y])$ exists, we get the contradiction $\varnothing = F_1 \cap F_2 \in \mathbb{H}$.

As $\mathbb{U} \in C_\alpha$ for all $0 < \alpha$ but $\mathbb{U} \notin C_0$ we see that $([0, \infty), \overline{C})$ is not left-continuous. Clearly, the space is complete. However, the point of convergence varies with the level α: We have $\mathbb{U} \in C_\alpha$ for all $0 < \alpha < \infty$. We fix $\alpha \in (0, \infty)$. Then there is x with $0 < x < \alpha$ such that $\mathbb{U} \wedge [x] \in C_\alpha$. For β with $0 < \beta < x$, however, $\mathbb{U} \wedge [x] \notin C_\beta$, because $\mathbb{U} \wedge [x] \not\geq \mathbb{U} \wedge [y]$ for $x \neq y$. Hence we cannot choose for each level the same point of convergence for $\mathbb{U}$.

We note that for $(X, d) \in |\mathsf{L}\text{-}\mathsf{MET}|$, the space $(X, \overline{C^d})$ is left-continuous.

Proposition 7.4. *Let $(X_j, \overline{C^j})$ be complete* L*-Cauchy tower spaces for all $j \in J$. Then the product space $(\prod_{j\in J} X_j, \overline{\pi\text{-}C})$ is also complete.*

Proof. Let $\mathbb{F} \in (\pi\text{-}C)_\alpha$. Then, for all $j \in J$, $pr_j(\mathbb{F}) \in C^j_\alpha$. By completeness there is, for each $j \in J$, a point $x_j \in X_j$ such that $pr_j(\mathbb{F}) \wedge [x_j] \in C^j_\alpha$. Hence with $x = (x_j)_{j\in J}$, then for all $j \in J$, $pr_j(\mathbb{F} \wedge [x]) = pr_j(\mathbb{F}) \wedge [pr_j(x)] = pr_j(\mathbb{F}) \wedge [x_j] \in C^j_\alpha$ which implies $\mathbb{F} \wedge [x] \in (\pi\text{-}C)_\alpha$. □

Remark 7.5 (Completeness in L-UTS). We define for an L-uniform convergence tower space $(X, \overline{\mathcal{U}})$ the underlying L-convergence tower $\overline{q^{\overline{\mathcal{U}}}}$ by $x \in q^{\overline{\mathcal{U}}}_\alpha(\mathbb{F}) \iff \mathbb{F} \times [x] \geq \mathcal{U}_\alpha$. Noting that $(\mathbb{F}\wedge[x]) \times (\mathbb{F}\wedge[x]) = (\mathbb{F}\times\mathbb{F}) \wedge (\mathbb{F}\times[x]) \wedge (\mathbb{F}\times[x])^{-1} \wedge [(x,x)]$ and $\mathbb{F}\times\mathbb{F} = (\mathbb{F}\times[x]) \circ ([x]\times\mathbb{F})$ we immediately obtain $x \in q^{\overline{\mathcal{C}^{\overline{\mathcal{U}}}}}_\alpha(\mathbb{F}) \iff x \in q^{\overline{\mathcal{U}}}_{\alpha*\alpha}(\mathbb{F})$. For the definition of completeness however, we can use either L-convergence tower, i.e. if we define $(X, \overline{\mathcal{U}})$ *complete* if $(X, \overline{\mathcal{C}^{\overline{\mathcal{U}}}})$ is complete, then this is equivalent to demanding that $\mathbb{F}\times\mathbb{F} \geq \mathcal{U}_\alpha$ implies $\mathbb{F}\times[x] \geq \mathcal{U}_\alpha$ for some $x \in X$.

Remark 7.6 (Completeness in L-UCTS.). Similarly, if we define in L-UCTS that a space $(X, \overline{\Lambda})$ is complete if $(X, \overline{C^{\overline{\Lambda}}})$ is complete, then this is equivalent to $\mathbb{F}\times\mathbb{F} \in \Lambda_\alpha$ implies $\mathbb{F}\times[x] \in \Lambda_\alpha$ for some $x \in X$.

Let $(X, \overline{C})$ be a non-complete L-Cauchy tower space. We call a pair $((X', \overline{C'}), \kappa)$ with a complete L-Cauchy tower space $(X', \overline{C'})$ and an initial and injective mapping $\kappa : (X, \overline{C}) \longrightarrow (X', \overline{C'})$ such that $\kappa(X)$ is dense in $(X', \overline{C'})$, a completion of $(X, \overline{C})$. Here, a set $A \subseteq X$ is called *dense in* $(X, \overline{C})$ if for all $x \in X$ there is $\mathbb{F} \in \mathsf{F}(X)$ such that $A \in \mathbb{F}$ and $\mathbb{F}\wedge[x] \in C_\top$ and a mapping $\kappa : (X, \overline{C}) \longrightarrow (X', \overline{C'})$ is *initial* if $\mathbb{F} \in C_\alpha$ if and only if $\kappa(\mathbb{F}) \in C'_\alpha$.

In the sequel, we describe a completion construction which goes back to [18] and [27]. The proofs in [27] can simply be adapted, replacing the quantale $\mathsf{L} = ([0,1], \leq, *)$ with a continuous t-norm $*$ on $[0,1]$ by an arbitrary quantale. For this reason, they are not presented.

We consider a non-complete L-Cauchy tower space $(X, \overline{C})$ and define $\mathcal{N}_{\overline{C}} = \{\mathbb{F} \in \mathsf{F}(X) \ : \ \mathbb{F} \in C_\top, \mathbb{F}\wedge[x] \notin C_\top \forall x \in X\}$. Furthermore, we consider the following equivalence relation on $C_\top$: $\mathbb{F} \sim \mathbb{G} \iff \mathbb{F}\wedge\mathbb{G} \in C_\top$ and we denote the equivalence class of $\mathbb{F} \in C_\top$ by $\langle\mathbb{F}\rangle = \{\mathbb{G} \in C_\top \ : \ \mathbb{F} \sim \mathbb{G}\}$. We define $X^* = \{\langle[x]\rangle \ : \ x \in X\} \cup \{\langle\mathbb{F}\rangle \ : \ \mathbb{F} \in \mathcal{N}_{\overline{C}}\}$ and denote the inclusion mapping $\iota_X = \iota : X \longrightarrow X^*$, $x \longmapsto \iota(x) = \langle[x]\rangle$. We note that if $(X, \overline{C})$ is a T1-space, then ι is an injection. In fact, if $\iota(x) = \iota(y)$, then $[x]\wedge[y] \in C_\top$ and by (T1) then $x = y$.

Let $\Phi \in \mathsf{F}(X^*)$ and let $\alpha \neq \bot$. We define $\Phi \in C^*_\alpha$ if there is $\mathbb{F} \in C_\alpha$ such that $\Phi \geq \iota(\mathbb{F})$ or if there is $\mathbb{F} \in C_\alpha$ and there are $\mathbb{F}_1, ..., \mathbb{F}_n \in \mathcal{N}_{\overline{C}}$ such that $\mathbb{F}\vee\mathbb{F}_i$ exists for all $i = 1, ..., n$ and $\Phi \geq \iota(\mathbb{F}) \wedge \bigwedge_{i=1}^n [\langle\mathbb{F}_i\rangle]$. Furthermore, we put $C^*_\bot = \mathsf{F}(X)$.

We will consider the *completion axiom* (LCA): for all $\mathbb{F} \in C_\alpha$ with $\mathbb{F}\wedge[x] \notin C_\alpha$ for all $x \in X$, there is $\mathbb{V} \in \mathcal{N}_{\overline{C}}$ such that $\mathbb{F}\wedge\mathbb{V} \in C_\alpha$.

Then $((X^*, \overline{C^*}), \iota)$ is a completion of $(X, \overline{C})$ if (LCA) is true. We can say more.

Proposition 7.7 ([27, 18]). *Let $(X, \overline{C}) \in |\mathsf{L\text{-}ChyTS}|$. Then $(X, \overline{C})$ has a completion if and only if it satisfies (LCA).*

For the completion $((X^*, \overline{C^*}), \iota)$, the following universal property is true.

Theorem 7.8 ([27, 18]). *Let $(X, \overline{C}), (Y, \overline{D}) \in |\mathsf{L\text{-}ChyTS}|$ and let $f : X \longrightarrow Y$ be Cauchy-continuous. Then there exists a Cauchy-continuous mapping $f^* : (X^*, \overline{C^*}) \longrightarrow (Y^*, \overline{D^*})$ such that the following diagram commutes:*

$$\begin{array}{ccc} (X, \overline{C}) & \xrightarrow{f} & (Y, \overline{D}) \\ \iota_X \downarrow & & \downarrow \iota_Y \\ (X^*, \overline{C^*}) & \xrightarrow{f^*} & (Y^*, \overline{D^*}) \end{array}$$

Corollary 7.9 ([27]). *Let $(X, \overline{C}), (Y, \overline{D}) \in |\mathsf{L\text{-}ChyTS}|$ and let $(X, \overline{C})$ satisfy (LCA) and let $(Y, \overline{D})$ be a complete T1-space. If $f : X \longrightarrow Y$ is Cauchy-continuous then there is a unique Cauchy-continuous extension $f^* : (X^*, \overline{C^*}) \longrightarrow (Y, \overline{D})$ such that $f^* \circ \iota = f$.*

It is at present not clear if for an L-metric space (X, d) the completion $(X, \overline{(C^d)^*})$ is again L-metrical, i.e. if there is an L-metric e in X^* such that $(C^d)^*_\alpha = C^e_\alpha$ for all $\alpha \in L$. We can, however, show the one half of the axiom (LChyM).

Proposition 7.10. *Let $(X, d) \in |\mathsf{L\text{-}MET}|$. If $\Phi \in (C^d)^*_\alpha$ then for all $\epsilon \lhd \alpha$ there is $\phi_\epsilon \in \Phi$ such that for all $x^*, y^* \in \phi_\epsilon$ we have $[x^*] \wedge [y^*] \in (C^d)^*_\epsilon$.*

Proof. Let $\Phi \in (C^d)^*_\alpha$. Then there are $\mathbb{F} \in C^d_\alpha$ and $\mathbb{F}_1, ..., \mathbb{F}_n \in \mathcal{N}_{\overline{C}}$ such that $\mathbb{F} \vee \mathbb{F}_k$ exists for $k = 1, ..., n$, $n \geq 0$, and $\Phi \geq \iota(\mathbb{F}) \bigwedge_{k=1}^n [\langle \mathbb{F}_k \rangle]$. Let $\epsilon \lhd \alpha$. By the axiom (LChyM) there is $F_\epsilon \in \mathbb{F}$ such that for all $x, y \in F_\epsilon$ we have $[x] \wedge [x] \in C^d_\epsilon$. We define $\phi_\epsilon = \iota(F_\epsilon) \cup \{\langle \mathbb{F}_1 \rangle, ..., \langle \mathbb{F}_n \rangle\} \in \Phi$. Let $x^*, y^* \in \phi_\epsilon$. We distinguish three cases.

Case 1: $x^*, y^* \in \iota(F_\epsilon)$. Then $x^* = \iota(x), y^* = \iota(y)$ for $x, y \in F_\epsilon$ and as $[x] \wedge [y] \in C^d_\epsilon$ we conclude $[x^*] \wedge [y^*] = \iota([x] \wedge [y]) \in (C^d)^*_\epsilon$.

Case 2: $x^* = \langle \mathbb{F}_k \rangle, y^* = \langle \mathbb{F}_l \rangle$. Then trivially $[\langle \mathbb{F}_k \rangle] \wedge [\langle \mathbb{F}_l \rangle] \geq \iota(\mathbb{F}) \wedge \bigwedge_{k=1}^n [\langle \mathbb{F}_k \rangle]$ and hence $[x^*] \wedge [y^*] \in (C^d)^*_\epsilon$.

Case 3: $x^* = \langle [x] \rangle \in \iota(F_\epsilon), y^* = \langle \mathbb{F}_k \rangle$. We have $[F_\epsilon] = \{G \subseteq X : F_\epsilon \subseteq G\} \leq \mathbb{F}$ and $[F_\epsilon] \leq [x]$ as $x \in F_\epsilon$. As $\mathbb{F} \vee \mathbb{F}_k$ exists therefore also $\mathbb{F} \vee [F_\epsilon]$ exists. Moreover, we have $\bigvee_{G \in [F_\epsilon]} \bigwedge_{u,v \in G} d(u, v) \geq \bigwedge_{u,v \in F_\epsilon} d(u, v) \geq \epsilon$ as $d(u, v) \geq \epsilon$ is equivalent to $[u] \wedge [v] \in C^d_\epsilon$. This shows $[F_\epsilon] \in C^d_\epsilon$. We conclude $[x^*] \wedge [y^*] = \iota([x]) \wedge [\langle \mathbb{F}_k \rangle] \geq \iota([F_\epsilon]) \wedge [\langle \mathbb{F}_k \rangle]$, i.e. $[x^*] \wedge [y^*] \in (C^d)^*_\epsilon$. □

8. The L-metric case: Cauchy completeness

We follow concepts and notations introduced in [6], see also [10]. Let $(X, d) \in |\mathsf{L\text{-}MET}|$. A mapping $\Phi : X \longrightarrow L$ is an *order ideal* if $d(y, x) * \Phi(x) \leq \Phi(y)$ for all $x, y \in X$. It is called an *order filter* if $\Phi(x) * d(x, y) \leq \Phi(y)$ for all $x, y \in X$.

Clearly, $d(y,x) \leq \Phi(x) \to \Phi(y)$ and using the L-metric $d_L : L \times L \longrightarrow L$ defined by $d_L(\alpha,\beta) = \alpha \to \beta$ for $\alpha,\beta \in L$, we see that an order ideal is an L-metric morphism from (X, d^{op}) to (L, d_L) and similarly, an L-order filter is an L-metric morphism from (X,d) to (L,d_L). The following lemmas give important examples.

Lemma 8.1 ([6])**.** *Let (X,d) be an L-metric space and $a \in X$. Then $\downarrow(a)$ defined by $\downarrow(a)(x) = d(x,a)$ is an order ideal and $\uparrow(a)$ defined by $\uparrow(a)(x) = d(a,x)$ is an order filter.*

Lemma 8.2. *Let (X,d) be an L-metric space and let $\mathbb{F}$ be a filter on X. Then $\Phi : X \longrightarrow L$, defined by $\Phi(x) = \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y)$ is an order ideal and $\Psi : X \longrightarrow L$ defined by $\Psi(x) = \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(y,x)$ is an order filter.*

Proof. We only show the first case. We have for $x,y \in X$

$$\begin{aligned} d(y,x) * \Phi(x) &= d(y,x) * \bigvee_{F\in\mathbb{F}} \bigwedge_{z\in F} d(x,z) = \bigvee_{F\in\mathbb{F}} d(y,x) * \bigwedge_{z\in F} d(x,z) \\ &\leq \bigvee_{F\in\mathbb{F}} \bigwedge_{z\in F} d(y,x) * d(x,z) \leq \bigvee_{F\in\mathbb{F}} \bigwedge_{z\in F} d(y,z) = \Phi(y). \end{aligned}$$

□

Definition 8.3 ([6])**.** Let (X,d) be an L-metric space and let $\Phi : X \longrightarrow L$ be an order ideal and $\Psi : X \longrightarrow L$ be an order filter. The pair (Φ,Ψ) is called a *cut* on X

(1) $\top = \bigvee_{x\in X} \Phi(x) * \Psi(x)$;
(2) $\Phi(x) * \Psi(y) \leq d(x,y)$ for all $x,y \in X$.

We note that for a given $a \in X$, the pair $(\downarrow(a), \uparrow(a))$ is a cut on X.

Proposition 8.4. *Let (X,d) be an L-metric space and let $\mathbb{F} \in C^d_\top$. Then (Φ,Ψ) as defined in Lemma 8.2 is a cut on X.*

Proof. Let $\epsilon \ll \top$ and choose $\delta \ll \top$ such that $\epsilon \ll \delta * \delta$. Then $\delta \ll \top = \bigvee_{F\in\mathbb{F}} \bigwedge_{x,y\in F} d(x,y)$ and hence there is $F \in \mathbb{F}$ such that for all $x,y \in F$ we have $d(x,y) \gg \delta$ and $d(y,x) \gg \delta$. We conclude for all $x \in F$

$$\begin{aligned} \epsilon &\ll \delta * \delta \leq \bigwedge_{y\in F} d(x,y) * \bigwedge_{y\in F} d(y,x) \\ &\leq \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) * \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(y,x) = \Phi(x) * \Psi(x). \end{aligned}$$

Hence $\epsilon \ll \bigvee_{x\in X} \Phi(x) * \Psi(x)$ and taking the join for all $\epsilon \ll \top$ we obtain $\top = \bigvee_{x\in X} \Phi(x) * \Psi(x)$.

Moreover, we have

$$\begin{aligned} \Phi(x) * \Psi(y) &= \bigvee_{F\in\mathbb{F}} \bigwedge_{z\in F} d(x,z) * \bigvee_{G\in\mathbb{F}} \bigwedge_{z\in G} d(z,y) \\ &\leq \bigvee_{F,G\in\mathbb{F}} \bigwedge_{z\in F\cap G} d(x,z) * d(z,y) \leq d(x,y). \end{aligned}$$

Hence (Φ, Ψ) is a cut on X. $\square$

We call a symmetric L-metric space (X, d) *complete* if for $\mathbb{F} \in C^d_\top$ there is $a \in X$ such that $a \in q^d_\top(\mathbb{F})$. From Proposition 4.3 we know that $q^d_\top(\mathbb{F}) = q^{\overline{C^d}}_\top(\mathbb{F})$ and hence the completeness of (X, d) is the completeness of the Cauchy space $(X, C^d_\top)$. Clearly, the requirement that $(X, \overline{C^d})$ is complete is stronger.

We shall now establish a relation to the concept of Cauchy completeness of an L-metric space as defined by Lawvere [19], see also [6].

Definition 8.5. Let (X, d) be an L-metric space. Then (X, d) is called *Cauchy complete* if for all cuts (Φ, Ψ) there is $a \in X$ that *represents the cut* (Φ, Ψ) in the sense that $\Phi = \downarrow(a)$ and $\Psi = \uparrow(a)$.

Theorem 8.6. *Let (X, d) be a symmetric L-metric space. Then (X, d) is complete if and only if (X, d) is Cauchy complete.*

Proof. Let (X, d) be Cauchy complete and let $\mathbb{F} \in C^d_\top$. We consider the order ideal Φ and the order filter Ψ defined in Lemma 8.2. From Proposition 8.4 we see that (Φ, Ψ) is a cut on X. By assumption, there is $a \in X$ such that $\Phi(x) = d(x, a)$ for all $x \in X$ and $\Psi(x) = d(a, x)$ for all $x \in X$. Then $\top = \Psi(a) \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{z \in F} d(z, a)$. which means that $a \in q^d_\top(\mathbb{F})$ and (X, d) is complete.

Conversely, let (X, d) be complete and let (Φ, Ψ) be a cut. For $\epsilon \ll \top$ we define $F_\epsilon = \{x \in : \Phi(x) * \Psi(x) \gg \epsilon\}$. From $\top = \bigvee_{x \in X} \Phi(x) * \Psi(x)$ we conclude that $F_\epsilon \neq \varnothing$ and hence $\{F_\epsilon : \epsilon \ll \top\}$ is a basis for a filter $\mathbb{F}$. We show that $\mathbb{F} \in C^d_\top$. Let $\epsilon \ll \top$ and choose $\delta \ll \top$ such that $\epsilon \ll \delta * \delta$. For all $x, y \in F_\delta$ we then have

$$\delta * \delta \leq \Phi(x) * \Psi(x) * \Phi(y) * \Psi(y) \leq \Phi(x) * \Psi(y) \leq d(x, y),$$

and hence

$$\epsilon \ll \delta * \delta \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{x, y \in F} d(x, y).$$

Taking the join over all $\epsilon \ll \top$ yields $\mathbb{F} \in C^d_\top$. Hence there is $a \in X$ such that $a \in q^d_\top(\mathbb{F})$.

We note that this means

$$\top = \bigvee_{\epsilon \ll \top} \bigwedge_{z \in F_\epsilon} d(z, a)$$

and, $\Psi : (X, d) \longrightarrow (L, d_L)$ being an $\mathsf{L}-\mathsf{MET}$-morphism, this yields

$$\top = \bigvee_{\epsilon \ll \top} \bigwedge_{z \in F_\epsilon} (\Psi(z) \to \Psi(a)) = \bigvee_{\epsilon \ll \alpha} ((\bigvee_{z \in F_\epsilon} \Psi(z)) \to \Psi(a)).$$

For $\delta \ll \top$ there is $\epsilon_\delta \ll \top$ such that

$$\delta * \bigvee_{z \in F_{\epsilon_\delta}} \Psi(z) \leq \Psi(a).$$

As $\epsilon_\delta \leq \epsilon_\delta \vee \delta \ll \top$ we have $F_{\epsilon_\delta \vee \delta} \subseteq F_{\epsilon_\delta}$ and hence

$$\delta * \bigvee_{z \in F_{\epsilon_\delta \vee \delta}} \Psi(z) \leq \Psi(a).$$

For $z \in F_{\epsilon_\delta \vee \delta}$ we know $\Psi(z) \geq \Phi(z) * \Psi(z) \geq \epsilon_\delta \vee \delta \geq \delta$ and we conclude $\delta * \delta \leq \Psi(a)$. Taking the join over all $\delta \ll \top$ we obtain $\top = \Psi(a)$. Similarly we can show $\top = \Phi(a)$. Φ being an order ideal implies $d(x,a) = d(x,a) * \Phi(a) \leq \Phi(x)$ for all $x \in X$ and from $\Phi(x) = \Phi(x) * \Psi(a) \leq d(x,a)$ we obtain $\Phi(x) = d(x,a)$ for all $x \in X$. Hence $\Phi = \downarrow (a)$. Similarly we can show $\Psi = \uparrow (a)$ and hence $a \in X$ represents the cut (Φ, Ψ) and (X,d) is Cauchy complete. □

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