

Conception and reflection of a university seminar concerning realistic modelling in mathematics education

Concepción y reflexión de un seminario universitario sobre la modelización realista en la enseñanza de las matemáticas

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Abstract

In this article we primarily want to share our experiences and reflections concerning a special course at the University of Vienna (see chapter 2). The topic of this seminar was “modelling and modelling problems” and the aim was twofold: On the one hand student teachers themselves should gain experience in working on a modelling problem (small groups of 3-5 persons), on the other hand they should get first experiences to create modelling problems for students at school (grade 8-11) and to supervise their modelling processes during a modelling day at school. These skills are crucial for prospective teachers if they should implement the idea of “modelling” later in their professional life as teachers in classroom. In order to make clear our personal view on modelling and modelling problems (What are proper modelling problems? What is the difference between modelling problems and dressed-up word problems? Etc.) –which is necessary to properly understand Section 2 –we first outline our corresponding approach in Section 1.

En este artículo queremos compartir principalmente nuestras experiencias y reflexiones sobre un curso especial en la Universidad de Viena (véase el capítulo 2). El tema de este seminario era “modelización y problemas de modelización” y el objetivo era doble: Por un lado, los propios estudiantes de magisterio debían adquirir experiencia en el trabajo con un problema de modelización (en pequeños grupos de 3 a 5 personas), y por otro lado, debían obtener las primeras experiencias para crear problemas de modelización para los alumnos de la escuela (grado 8 - 11) y supervisar sus procesos de modelización durante una jornada de modelización en la escuela. Estas habilidades son cruciales para los futuros profesores si deben implementar la idea de “modelado” más adelante en su vida profesional como profesores en el aula. Para dejar clara nuestra opinión personal sobre la modelización y los problemas de modelización (¿Qué son los problemas de modelización propiamente dichos? ¿Cuál es la diferencia entre los problemas de modelización y los problemas de palabras disfrazados? etc.) – que es necesario para entender bien la Sección 2 –, primero esbozamos nuestro correspondiente enfoque en la Sección 1.

Keywords: Mathematical modelling, mathematics education, mathematical concepts, tools and processes, school practice, German speaking.

Palabras clave: Modelización matemática, educación matemática, conceptos, herramientas y procesos matemáticos, práctica escolar, habla alemana.

1. “Dressed-up” word problems and modelling problems

The topic connections to real life in mathematics teaching has been important within the German-speaking community of mathematics education for many decades, and slowly, it has been arriving in actual classroom teaching as well. The so-called ISTRON group, for example, has existed since 1990, and it has published a series (from 1993 to 2019 there have been 24 volumes) containing specific lesson plans and other (including theoretical, more scientific) essays on the topics of connections to real life in mathematics teaching or Modelling. After all, connections to reality and modelling problems are important for classroom teaching:

1. **Mathematics can be useful for the world outside mathematics, including one’s own (individual), as well as more general (societal) circumstances: mathematics → reality.** In this sense, we are mostly talking about a rational approach to “non-mathematical situations” (in everyday life, at work, etc.) – in the one case, to one’s own benefit, or, in the case of leaders and decision makers, for society at large (the ability to analyse situations outside mathematics as a basis for choices that have to be made).
2. **The world outside mathematics can enrich and facilitate mathematics and its teaching: reality → mathematics (teaching).** This mainly concerns the ways in which facilitation of mathematical learning can be achieved by using connections to reality (motivation, understanding, retention, meaning, etc.).

In the following, we will lay out our understanding of dressed-up word problems versus modelling problems (see Niss/Blum 2020).

In German speaking countries the traditional way of “applying” mathematics in the classroom has been the so-called “Sachrechnen” (the word is roughly to be translated as “matter/object calculation”), which includes calculations involving measures and weights as well as calculations of percentages, interests and “the rule of three”. In this approach, the “real-life” situations are, in fact, artificially constructed and not very near to reality. The problem itself or its solution are not actually central to the consideration, they are just a verbal/textual “costume” of a formula or a certain calculus¹ (hence the name “dressed-up” word problems), designed to fit the mathematics currently taught in that class (this is the primary focus!). A bit exaggerated: In order to have some variety when practicing a certain calculus, exercise problems are re-written (dressed-up) as texts. This dressing-up of mathematics in texts is an old tradition—documents from sources as early as the Babylonians or Arabs contain such exercises, as mental challenges or for the entertainment of the readers. Most of these “word problems” are not specifically focused on making the presented real-life problem the object of the learning process, on showing, by way of example, that and how mathematics can play an essential role in authentic and realistic situations (which is what real applications or modelling would ideally strive for). Instead, the goal is to

- **translate** into the language of mathematics (= “undressing” the exercise previously dressed-up by the authors) while not focusing on questions of meaning and realism (even though vom Hofe et al. (2006) underline the importance of mental models, so called ‘Grundvorstellungen’, which carry the meaning of mathematical notions and procedures in these translation processes).
- **practice** mathematical concepts and procedures.

¹An example would be “Textgleichungen”—text equations such as “Charlotte is now twice as old as her currently 60-year-old mother was when Charlotte was born.”

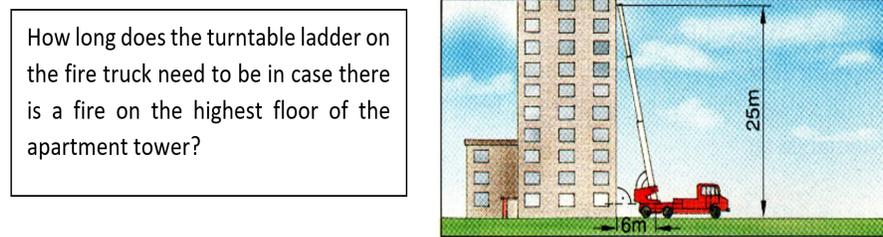


Figure 1 – Dressed-up word problem on the topic of Pythagoras (SINUS programme, Hessen 2002, from a lecture by W. Blum).

These objectives should certainly be warranted in mathematics teaching and thus, at times, one may very well use a not-quite authentic, not-quite true-to-life, not-quite realistic problem which the students otherwise would not be confronted with. But caution is advised, however, this approach must not go too far.

An extreme example: The ranch of an American cattle farmer is shaped like a regular hexagon with specified side length a . What is the side length of the neighbour's ranch, which is equal in area and shaped like a regular pentagon? This is definitely overdone! – a harmful **fake application**. At best, such tasks should be used as brainteasers. However, it must not happen to misuse them as feigned realistic modelling tasks! In the following, we state conditions under which, according to our judgment, “dressed-up word problems” are regarded as valuable:

- The exercises are set in a sensible context (no harmful pseudo applications, the context should be at least plausible and not too far-fetched – of course, drawing this line is always subjective and therefore variable).
- Honest communication regarding the exercises: Admitting that they are, in fact, “just dressed-up” exercises for translation and practice purposes – as opposed to authentic questions or problems that we might encounter and be interested in for their solutions.
- The connections to reality cannot be restricted to the work on dressed-up word problems, i.e. there should also be more realistic and authentic problems (see e.g. Niss/Blum 2020 or Vos 2011) to be dealt with, so that students can experience: “Mathematics helps with problems”.

We consider dressed-up word problems which meet the above conditions important to classroom teaching as they constitute a crucial aspect of mathematics and its teaching. But the orientation towards applications in mathematics teaching in classroom cannot be limited to solving some dressed-up word problems (see e.g. Blum/Niss 1991) every now and then. By being confronted with authentic (not too complex) situations, students should also frequently experience that mathematics is useful when analysing, describing and solving problems in authentic situations that might actually happen to a person. If this is the case, we speak of modelling: The situation is at the centre of interest, not the practice of some procedure for which we assign some quickly translatable word problems that students have to deal with. This means that this has virtually nothing to do with practicing some concepts or procedures, since in many cases, it is unclear at the beginning, which kind of mathematics will play a role. The following example should illustrate the difference between a dressed-up word problem and a modelling problem in a concise manner: Figure 1 shows a short dressed-up word problem on the topic of Pythagoras. Here, everything is clear and straightforward, so that one has to (is expected to) use the Pythagorean theorem once and thus “practice” it.

Fire department

The Munich fire department acquired a new turntable ladder in 2004. The vehicle can be used to rescue people from great heights, using a cage at the end of the turntable ladder. According to regulations, the vehicle has to keep a minimum distance of 12 m from the burning building.

The technical characteristics of the vehicle are:

Type: Daimler Chrysler AG Econic 18/28 LL - Diesel
 Year of manufacture: 2004
 Engine power: 205 kW (279 hp)
 Engine displacement: 6374 cm³
 Vehicle measurements: length 10 m; width 2.5 m; height 3.19 m
 Ladder measurements: 30 m length
 Unladen weight: 15540 kg
 Gross weight: 18000 kg



What is the maximum height from which the Munich fire department can rescue people using this turntable ladder?

Figure 2 – Modelling problem (Holzäpfel/Leiß2014).

Of course, the task in Figure 2 also concerns the Pythagorean theorem, but at first, it is not that obvious: The significant sketch of the right triangle is not given. Also, some additional assumptions have to be made (→ Idealising: turning capacity of the ladder; suitability of the terrain; which side of the vehicle will be facing the building? etc.), there is superfluous information (→ Structuring), making it necessary to consider what is actually needed for a solution. Then one can mathematise (translate into a mathematical/geometric model: Pythagoras) and, as a next step, perform calculations (i.e. arrive at the mathematical solution). Finally, the solution has to be interpreted and possibly validated. All of these are actions that are described in many cycles of modelling (e.g. Schupp 1988, Blum/Leiß2005 and some others). They may differ in some details, but all have in common that it is necessary to distinguish between reality and mathematics, and that processes of translation between these two worlds have to take place (mathematising, interpreting). As a modelling problem, this problem should not necessarily be used at the time when Pythagoras is discussed in class, since it is precisely not the point of this task to practice the theorem. To summarize: In the case of dressed-up word problems the mathematics defines the context (one uses contexts in order to practice specific mathematical contents), while with modelling problems, the authentic context defines the mathematics (which does not necessarily have to go hand in hand with the domain currently discussed). After describing our notion of modelling and our understanding of modelling problems, we are now prepared to present the conception of a university seminar concerning the work on modelling problems at school and university level. Its aim is to interweave pre-service teachers' own modelling activities with their role as coaches for students' modelling activities at school.

2. A seminar on Modelling at the University of Vienna

For many years we (usually, together) have conducted a seminar on the topic of Modelling with student teachers of mathematics. This seminar is not a mandatory subject for students² but

²At times, this paper talks about both adult students at university and younger students in school. Usually, the context makes clear which one is meant. When ambiguity could arise, we refer to them as student teachers and school students, respectively.

an elective one within a module. It has two pillars (goals): On the one hand, the students spend the whole semester working in groups (of 3-5, ideally 4) on rather complex modelling problems, thus gaining experience of their own in modelling. The results have to be demonstrated in a final presentation and handed in as a comprehensive seminar paper. On the other hand, the participants should also take on the role of teachers who accompany school students as they work on modelling problems. For this purpose, the student teachers develop problems which can be used at a modelling day at a school. After final selection and editing by ourselves, these become the basis for an actual modelling day at a local school. The year levels taking part vary considerably, in most cases, we worked with students of grades 9-11, but we have also had younger students (from grades 6, 7 or 8). Naturally, the information about the cooperating year levels was communicated to the student teachers beforehand, so that their modelling problems could be adapted accordingly.

Prior to the modelling day, the student teachers received instructions about how they should act when mentoring the school students while they work on the problems and what they should be considering (see Section 3.2 and Section 3.3; each student teacher is mentoring one group of school students at the modelling day). At the end-of-day reflection, the student teachers always report that it was an experience they valued, meaning that they learnt a lot which could be useful for their future (teaching practice).

Two of the modelling problems for school students essentially designed by student teachers are described in Section 3.1, now we present two problems we have used (multiple times) as problems to be solved by the student teachers. Doing so, we want to demonstrate with concrete examples, which kinds of problems our student teachers work on during the seminar.

2.1. The optimal throw when playing darts

Statement of task:

Darts has many variations of the game. The usual one starts with all players having 301 or 501 points. Each round, every player throws three darts and the achieved number of points is subtracted from the player's score. The goal is to reach exactly 0 points. In the beginning, one should therefore try to score as many points as possible. This beginning of the game is the situation we discuss now.

Since, at least for the untrained hobby player, it is not possible to hit exactly the point that was aimed at, the following question arises: Which region should a player aim at in order to score as many points as possible?

The Bull's Eye is quite small! Maybe one should rather aim for the 20- but there looms the 1! Or should we aim somewhere completely different?

Solution approaches:

1. Add weighted values from directly adjacent sectors. A very easy option would be to calculate using a throw precision that is good enough to hit either the target (hit probability p) or at least one of the directly adjacent sectors (hit probability $\frac{(1-p)}{2}$ each). For example, if one is aiming at the 20 sector, it's possible to land in the neighbouring sectors 1 or 5. Thus, the expected value in this case is $\mu = 20p + \frac{(1-p)}{2} + 5\frac{(1-p)}{2} = 20p + 3(1-p) = 3 + 17p$. This calculation would have to be performed for each of the twenty sectors in order to compare and determine the sector for which the expected value (depending on p) is highest. This solution strategy takes neither double and triple sectors, the bulls eye, or the question about which point within a sector one should aim at, into consideration, nor does it deal

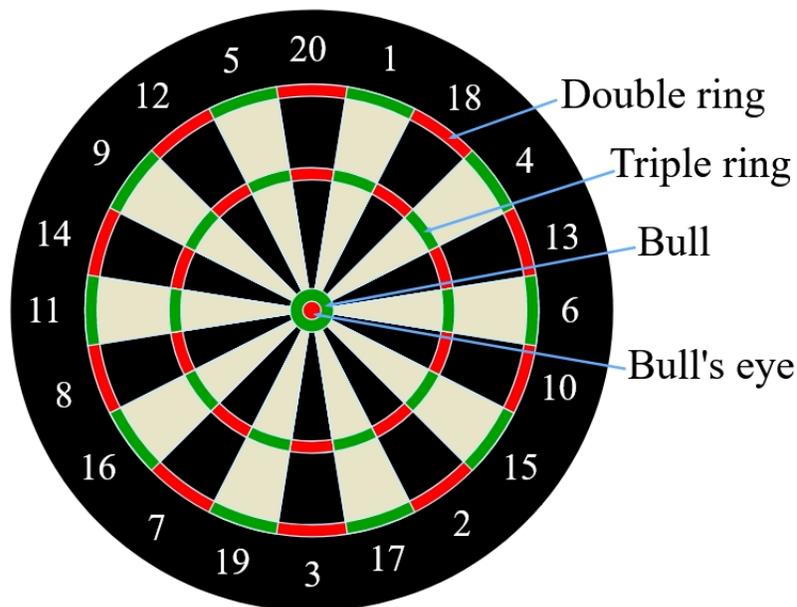


Figure 3 – Dartboard with point values and special areas (Source: <https://commons.wikimedia.org/wiki/File:Dartboard.svg>).

with the fact that the sectors become smaller as they approach the centre, which increases the probability of hitting not directly adjacent sectors.

2. Add weighted values from parts of adjacent sectors (proportional to their size). The model becomes somewhat more precise when one applies the idea described above not to entire sectors, but to parts of sectors. Aiming at the Triple-20-sector, for example, one should assign positive hit probabilities (e.g. proportional to their area) to directly or diagonally adjacent sector parts (Single-20, Single 5, Triple-5, Single-1, Triple-1).
3. Lay circles over a sort of discretised dartboard. First, one places a chequered grid on the dartboard (see Fig. 4 on the left, arbitrarily small grid distance) and assigns each small square its respective point value. Then a circle is placed on this grid, such that the centre of the circle lies exactly on the point that is aimed at and all point values within the circle are added up (see Fig. 4 in the middle). The smaller the radius of the circle, the higher the assumed hit precision of the throwing player. This is done for various points to be aimed at, and the resulting sums should be compared. Of course, this optimisation problem can be solved using a computer, which moves the circle across the grid in pre-defined increments and presents the sum of the point values inside the circle for each centre point. This can be implemented in Excel, for example: The grid with the point values of the entire dartboard is entered into a data sheet, the circle with all entries “1” or certain weights (see Fig. 4 middle and right) into a second data sheet. Finally, the sum of the cell-by-cell products of the two data sheets is calculated. This gives the expected value of the points achieved when aiming at the centre of the circle.
4. Two-dimensional normal distribution. This solution approach expands on the idea of 3. by “moving” the density function f of the two-dimensional normal distribution across the dartboard instead of a circle (see Fig. 5). The functional equation of f can be found in the literature, for the expected value $\mu = (\mu_x | \mu_y)$, the standard deviations ς_x and ς_y and the correlation coefficient $\rho = 0$, it is:

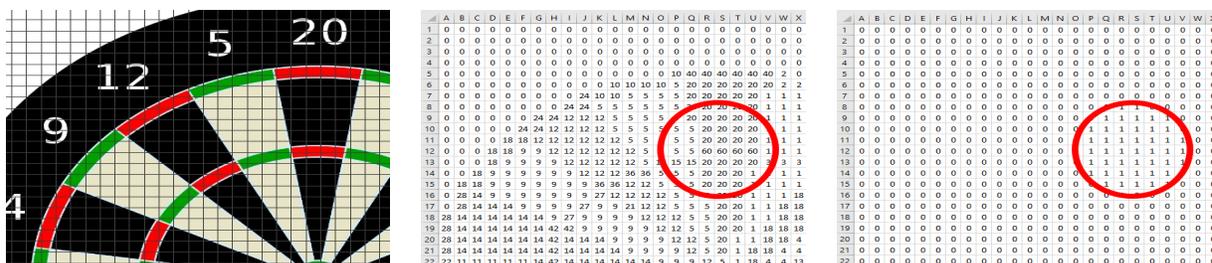


Figure 4 – Discretised dartboard (left), implementation in Excel (middle), “1-circle” (right).

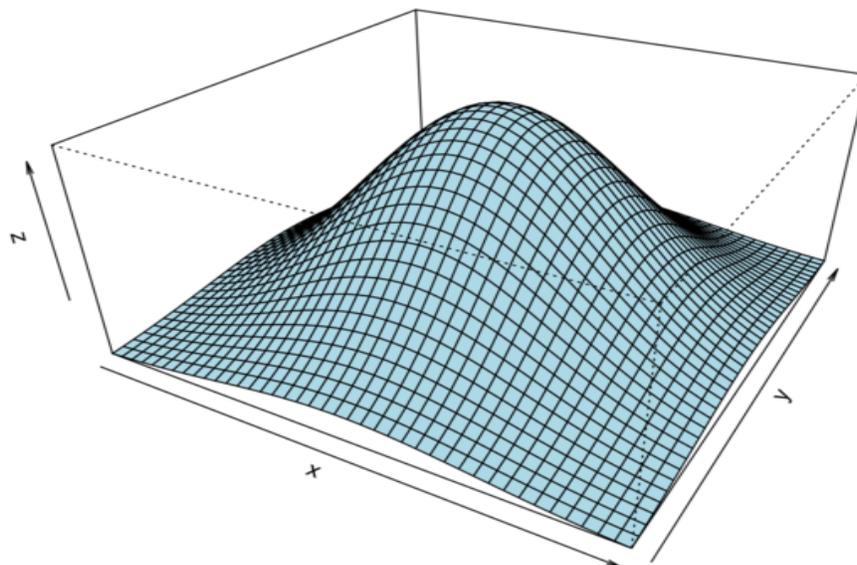


Figure 5 – Graph of the density function f of the two-dimensional normal distribution (Source: https://de.wikipedia.org/wiki/Mehrdimensionale_Normalverteilung).

$$f(x, y) = \frac{1}{(2\pi\zeta_x\zeta_y)} \exp\left(-\frac{1}{2}\left(\frac{(x - \mu_x)^2}{\zeta_x^2} + \frac{(y - \mu_y)^2}{\zeta_y^2}\right)\right).$$

Why is this a proper problem?

The problem meets all the requirements for a good modelling problem. It allows solution approaches of differing complexity and thus connects concepts from various mathematical domains (geometry, stochastics, analysis, discrete mathematics). Student teachers who work on the problem usually procure a real dartboard immediately, in order to gain an intuitive sense for the situation. This increases motivation and the relevance of the problem for the students. Of course, the opportunity for gathering empirical data is offered here, which enables one to obtain realistic values for the radius in approach 3 or the standard deviation in approach 4. Especially in the case of these latter two solution approaches, the use of a computer (spreadsheets or CAS) makes sense and allows applying and expanding technological competence. Additionally, the scope of the problem can be modified: Students could, for example, also investigate the question of which initial throwing angle and which initial throwing velocity minimises the standard deviations. Further ideas and solution strategies to this problem can be found in Bracke et al. (2013, German language).



Figure 6 – Sedimentation tank in a purification plant (https://www.bmlrt.gv.at/fotoservice/wasser/abwasserwirtschaft/IKB_6205_bah.html, with kind permission of the BMLRT, Austria).

2.2. Division of a sedimentation tank

Statement of task:

In the sedimentation tank of a purification plant, wastewater is clarified before being channelled into a river. Currently, about 6 m^3 of wastewater are accumulated every day. The average concentration of contaminants is $c_0 = 3,06 \text{ kg/m}^3$. Legal regulations determine that a maximum of 4 kg of contaminants may be released into the river per day. The clarification takes place via controlled chemical and biological reactions during the water's dwell time in the tank with the dimensions $L \times W \times H = 10 \text{ m} \times 4 \text{ m} \times 2 \text{ m}$.

In order to describe the process of decomposition (without in-or outflow), 10 different series of measurements M_1, \dots, M_{10} were established, i.e. the relative concentration of contaminants was measured after 20, 40, \dots , 400 hours (see Table 1, which contains not real, but “fictional” data which could look similar in reality – “exponential decay”, cf. microbiology).

Problem: At the moment, the purification plant stays just within legal regulations during long-term operations – how can this be reconstructed using a mathematical model? If the volume of wastewater was increased or the regulations were tightened, however, enormous difficulties would arise, since an expansion of the tank would only be feasible under great costs and efforts and thus practically cannot take place.

Statement of task: Experiments suggest that dividing the tank into multiple sub-basins would lead to a decrease in contaminant concentration in the outward flow (see Fig. 7). The operating company of the purification plant is thus highly interested in finding out whether this effect can also be shown in a mathematical model it has to be attributed to measurement errors. In the former case, the company would further like to determine how many divisions would result in optimal values and where the separating walls should be positioned.

Solution approaches: First, students have to arrive at a model description of the process of decay, e.g. by averaging of the 10 measurement series and recognising that approximately an

Time (h)	M1	M2	M3	M4	M5	M6	M7	M8	M9	M10
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
20	0.785	0.775	0.776	0.778	0.773	0.780	0.776	0.768	0.769	0.774
40	0.597	0.605	0.596	0.608	0.602	0.605	0.604	0.600	0.600	0.602
60	0.470	0.471	0.473	0.475	0.470	0.469	0.474	0.474	0.458	0.474
80	0.365	0.357	0.366	0.361	0.360	0.358	0.365	0.362	0.361	0.367
100	0.290	0.274	0.279	0.285	0.282	0.286	0.284	0.293	0.282	0.283
120	0.218	0.217	0.221	0.215	0.222	0.225	0.221	0.212	0.220	0.216
140	0.176	0.170	0.179	0.175	0.171	0.164	0.170	0.168	0.168	0.168
160	0.130	0.142	0.133	0.132	0.142	0.126	0.137	0.134	0.120	0.135
180	0.103	0.101	0.103	0.103	0.097	0.102	0.106	0.108	0.109	0.104
200	0.083	0.080	0.085	0.086	0.084	0.080	0.086	0.078	0.078	0.079
220	0.058	0.066	0.062	0.053	0.065	0.058	0.060	0.060	0.067	0.065
240	0.050	0.046	0.043	0.048	0.045	0.046	0.050	0.053	0.043	0.041
260	0.044	0.033	0.035	0.033	0.035	0.037	0.040	0.037	0.041	0.042
280	0.033	0.027	0.032	0.023	0.029	0.025	0.030	0.032	0.031	0.029
300	0.029	0.023	0.023	0.025	0.022	0.024	0.028	0.027	0.027	0.021
320	0.025	0.020	0.019	0.016	0.017	0.018	0.020	0.019	0.022	0.020
340	0.015	0.019	0.015	0.012	0.017	0.017	0.009	0.018	0.021	0.016
360	0.010	0.015	0.011	0.011	0.007	0.014	0.009	0.014	0.014	0.015
380	0.009	0.012	0.010	0.006	0.007	0.013	0.008	0.004	0.008	0.012
400	0.003	0.009	0.006	0.004	0.006	0.005	0.005	0.003	0.008	0.011

Table 1 – Data of measurement series

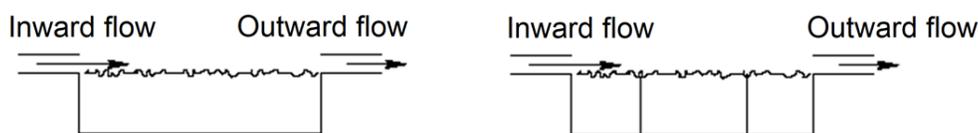


Figure 7 – Installation of separating walls.

exponential decay takes place. Using Excel and “Add trend line”, one obtains the respective parameters of $C(t) = C(0) \exp(-\lambda t)$, where $C(t)$ is the concentration at time t . After this, students have multiple options for continuing. On the one hand, one can think about this in a discrete fashion (in time steps, difference equation), on the other hand continuously (differential equation). Another possible distinction would be whether one tends to think in terms of concentrations (i.e. relative amounts of contaminants) or rather in terms of absolute amounts of contaminants. Experience shows that student teachers find it easier to think in terms of absolute quantities of contaminants, however, confusion about this frequently arises and student teachers whose second subject is not physics often struggle with the distinction.

Since we are talking about a long-term operation of the tank, neglecting the process of filling the tank initially seems to be appropriate: Which concentration does the tank tend to in the long term (“equilibrium”, “limit”)? With this in mind, the inflow into the basin, the mixing process and the outflow from the basin have to be modelled (discretely or continuously), and this is not easy. If one works discretely, using spreadsheets is of great advantage, which is appropriate for modelling problems. Most students started off working discretely, since their experience with differential equations (setting up, solving) is limited. This is not so important, however, since working discretely yields also proper solutions: How does the concentration (amount of contamination) one time step ahead follow from the current concentration (amount of contamination)? What are possible limits? In each time step Δt (20 h according to Table 1,

24 h for an entire day, or even 1 h are specific and nearby values), the decay described above can be brought into effect; the addition of new wastewater and the required outflow from the basin (long-term operation!) can be thought of as happening at the beginning of such a time step, or at the end. This gives a lower vs. an upper bound of the concentration.

Once one has determined such a limiting concentration (long-term operation) inside the tank, one will be interested in how the process of dividing the tank with separating walls will affect the concentration of the outflow. One will find that separating the basin precisely in the middle has the best effect for maintaining a minimal long-term concentration in the outward flow from the second sub-basin. This can be achieved by formal limit considerations, but even without formal limits spreadsheet calculations can lead to this insight. Considering two separating walls (3 sub-basins), one will once more expect optimal results for a uniform separation of the tank. This is easily reasoned if one knows that for two sub-basins, separation in the middle is the best choice. The result is even better than with only one separating wall. Does this continue to be the case? Should one install as many separating walls as possible? At this point, it quickly becomes apparent that the separating walls themselves have a certain volume (students can reasonably assume this) and are therefore using up a finite volume. A smaller gross volume, however, obviously leads to less decomposition. Thus – as is the case in many modelling problems – two phenomena striving in opposite directions are at work: One the one hand, multiple separating walls are beneficial, on the other hand, each separating wall decreases the gross volume available and thus the dwell time in the tank, leading to decreased decomposition of contaminants. Therefore, there exists a limited optimal number of separation walls and sub-basins.

As has been shown, this modelling problem can be solved in many different ways and on various levels (discretely, continuously; using formal limit considerations or merely spreadsheet calculations on an experimental basis; focusing on concentrations or on absolute quantities of contaminants, etc.) as well. All student teachers who have worked on this problem so far believed they had “learned a lot”, and that is certainly not a minor result! For a specific elaboration of this problem and its many opportunities, see Bracke & Humenberger 2012 (in German).

3. Modelling days at schools

3.1. Modelling problems for school students

The problems in this section are mainly the work of the student teachers who attended our seminar, we were only involved in the final wording and editing process.

Airplane and sunset, for students in grades 9/10

Statement of task: When travelling by airplane at dawn or at dusk, one can often see spectacular sunrises and sunsets (Fig. 8). Among others, these related questions may arise:

1. Is it possible to “outrun” a sunrise or sunset via airplane?
2. Is the answer the same at every point of the globe?
3. What’s the situation when considering a jet fighter?

Here, the school students first must realise what it means to “outrun” a sunrise or sunset. One would have to fly at such a speed that the earth’s rotation was “compensated for”, which would stop the sun rising or setting further when viewed from the airplane. Additionally, one



Figure 8 – ©Stefan Jud, with kind permission.

has to consider that an airplane would have the best chance when flying precisely in an east-west direction, since otherwise a part of the velocity (namely the component in the north-south direction) would be lost unnecessarily. The students have to notice by themselves, that and why this phenomenon differs around the globe, depending on the latitude. How fast (in km/h) is the earth's rotation at different latitudes? Next, one has to investigate the speed of a regular passenger airplane or a military jet fighter. At this point, students are busy "researching" but also need to be able to visualize the actual processes of flying and the concepts of sunrise and sunset (both from a spatial and a physics perspective). At which latitude is the earth's rotation slow enough for an airplane to keep up? Where does this threshold lie for fighter jets? How fast would one have to be able to fly in order to achieve this at the equator?

While this problem does not allow that many mathematical possibilities, when groups of students tackle the problem on their own, their approaches vary considerably. In our opinion, it is an appealing problem, both because of its direct connection to reality and its authenticity (sitting inside an airplane, one might reasonably wonder about this), and because it is not difficult and requires students to realize independently what actually happens at sunrises and sunsets – a concept that is important even for people with no pronounced interest in astronomy.

Minigolf, grade 10

Statement of task: A typical minigolf course involves a station at which the ball is shot across a ramp while aiming at a net. What is the necessary velocity at which the ball has to be driven off in order to hit precisely the middle of the ring?

Solution approaches:

1. Separate consideration of the horizontal and vertical trajectories of the ball. It makes sense to start by drawing a simplified sketch of this situation (see Fig. 9, on the right). The origin was placed at the upper end of the ramp. This facilitates later calculations. First, we only consider the throw trajectory, i.e. the movement of the ball from the point $(0,0)$ to the

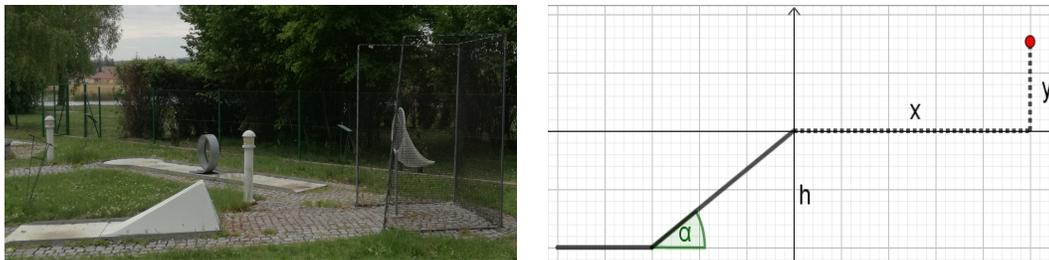


Figure 9 – Minigolf station (left), mathematical model (right).

midpoint of the ring (in Fig. 9), on the right, this is the point (x, y) . The air resistance is not a factor in this model, the ball is assumed to be point-shaped and the loss of velocity from rolling the ball up the ramp is disregarded initially. The horizontal movement of the ball is a uniform movement, therefore, the following holds for the flight time t^* of the ball from $(0, 0)$ to (x, y) : $t^* = \frac{x}{v_x}$ (where v_x is the x -component of the velocity at the point $(0, 0)$, v_x being constant for the entire flight). Considering the vertical movement of the ball, $v_y(t) = v_y(0) - gt$ holds for the y -component of the velocity (where g is earth's gravitational acceleration). Thus, the position of the ball at time t is given by: $y(t) = v_y(0)t - \frac{1}{2}gt^2$. Since the velocity vector at $(0, 0)$ points toward the (elongated) ramp, one has $v_y(0) = v_x \tan(\alpha)$ and thus $y(t) = v_x \tan(\alpha)t - \frac{1}{2}gt^2$. Letting $y(t^*)$ equal y and replacing t^* in this equation by $\frac{x}{v_x}$, one can solve this equation for v_x : $v_x = \sqrt{\frac{g}{(2x \tan(\alpha)) - 2y}}x$. Using $v_y(0) = v_x \tan(\alpha)$, the length of the velocity vector at the point $(0, 0)$ is $v_0 = \sqrt{v_x^2 + (v_y(0))^2}$. At this point, we are only missing the loss of velocity caused by the ball's rolling up the ramp. Doing so, the ball gains the potential energy mgh , while the kinetic energy decreases by the same amount. Therefore, after rearranging the equation $m\frac{(v_0+v)^2}{2} = m\frac{v_0^2}{2} + mgh$, we obtain $v = \sqrt{v_0^2 + 2gh} - v_0$. This is the loss of velocity in question. Hence, the golf player must drive off the ball with a velocity of $\sqrt{v_0^2 + 2gh}$ in order to hit precisely the middle of the ring.

Students would probably start off by estimating the quantities from the given photograph (e.g. h , α , x and y) and then using these numbers for their calculations. In principle, there's nothing wrong with this approach, although it makes the situation somewhat less clear than the general case.

- Equation of the trajectory parabola (projectile motion). While there's no solution to be found on the internet for the minigolf problem, there is of course one for the (time-independent) equation of the trajectory parabola: $y(x) = -\frac{(x^2g)}{2v_0^2(\cos(\alpha)^2)} + x \tan(\alpha)$. Some school students may know this equation from their physics classes. This is not a problem, however, since enough remains to be done. In order to solve the problem, students have to attribute the values in the formula to the values in their task and understand how the equation can assist them in finding a solution. Letting $y(x) = y$, the equation $y = -\frac{(x^2g)}{2v_0^2(\cos(\alpha)^2)} + x \tan(\alpha)$ can be solved for v_0 . The loss of velocity from the ramp can then be calculated as demonstrated above.
- Approximation using GeoGebra. Students might also try to draw the trajectory parabola from 2. using GeoGebra. The value of the desired initial velocity v_0 should be allowed to vary using a slider. Changing this value, one can adjust the parabola until it passes through the point (x, y) (see Fig. 10).

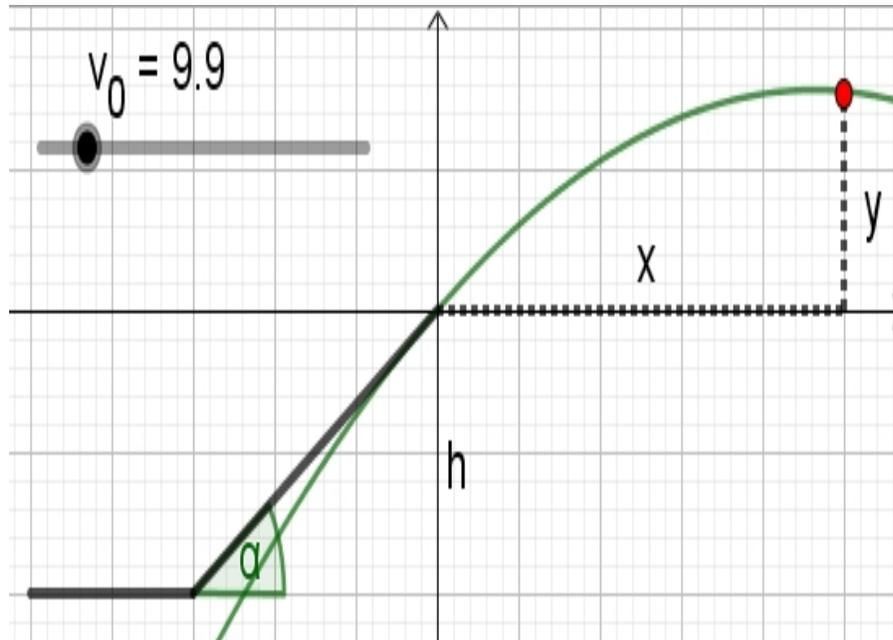


Figure 10 – Trajectory parabola, which – for $v_0 = 9,9m/s$ – passes through the points $(0,0)$ and (x,y) .

Why is this a proper problem?

As with the others, the problem offers various ways of approaching it (e.g. including technology) on different levels. The model described above could also be developed further (e.g. considering air resistance, friction when rolling up the ramp). Modifying and allowing the ball to hit any point inside the ring (i.e. not necessarily having to hit the middle point), one obtains an interval within which the velocity v_0 has to lie.

The problem is appealing because it allows testing the result at a minigolf course for its usability. The velocity of the initial hit can be measured using photoelectric sensors from the physics department.

3.2. Organising a modelling day at school

One should allocate at least five hours for the modelling day at a school to take place. After introducing the problems, sorting students into groups and assigning the problems to each group (about 30 min total), the actual work on the problems begins. Usually, the school students require around three hours to find a solution (depending on the difficulty level of the problem and the depth of their work) and 30 min to prepare their presentations. The short presentations should be allocated altogether approximately one hour. In the context of our university seminar, each student teacher supervises only one group of school students, and this is enough taking into account that it is their first experience in mentoring school students working on modelling problems. In case of in-service teachers wanting to organise a modelling day for their own class, one might consider having all of the groups work on the same problem, which facilitates the simultaneous mentoring and supervision. We recommend group sizes of 3 to 4 students; the composition of the groups can be determined by the students' interest to work on specific problems or on their individual performance levels (we recommend groups that are homogenous in performance in order to allow and force all members to participate equally). Reserving a computer room for the modelling day is advisable in order to make sure that sufficient PCs are available for investigations and the preparation of the presentations.

Alternatively, laptops can be used. When selecting problems, one must make sure that solutions to the respective problems are not available on the internet. Prohibiting the use of computers for this reason, however, would not be a solution, since GeoGebra and spreadsheet calculations are ideal tools for modelling.

3.3. Considerations for the implementation at a school from the perspective of didactics and learning psychology

When selecting problems for a modelling day, one needs to consider whether the problems really do allow solutions on different levels of competence and with varying mathematical tools. The ability to quickly comprehend students' solution processes is one of the central competencies required when supervising modelling problems. It is therefore advisable to spend some time and consider beforehand which parts might cause difficulties (for school students). Conversely, this must not lead to a mentor eliminating all of the challenges in advance or urging students to take on a specific approach. In general, modelling requires a principle of controlled assistance, since the focus should be the independent work of students and not the micromanagement of their progress or a solution according to the instructions of a teacher. It is also helpful in this regard to take on the role of a customer ("client") who is commissioning the students for the solution of a real problem ("product development"). For the minigolf problem mentioned above, this would mean that the supervising student teachers themselves want to know how to improve their own driveoff in order to hit the net. When supervising, this leads to mainly asking questions like "Have you considered ...?", "What happens if you take...into account?" and asking the students to explain the solution approaches they have tried so far. The students should be convincing the client of their solution approach and not the other way around! During these conversations, the students automatically notice the current strengths and weaknesses of their model and the steps which have to be taken next. In our experience, students' motivation is raised by using contexts as realistic as possible which are accessible from the students' realities (cf. Maaß and Mischo 2012). An attractive image, little text and appealing questions increase the likelihood of students engaging with the idea of modelling. During the process of modelling, there will always be phases that require students to tolerate a certain degree of frustration. This is handled more easily when a person is generally interested in the problem and considers finding a solution to be useful. In this context, it is important to make sure that single students do not disengage from the actual work on the mathematical problem early on and, for example, start working on their final presentations after merely one hour, before the problem was worked on comprehensively. In case a student group is finished after a short amount of time (or seems to be done, in any case), it makes sense to have some follow-up questions relating to the problem at one's disposal.

The presentations at the end of the modelling day should definitely begin with a short introduction of the problem itself, especially when students have worked on different problems in their groups. The focus should then be the central ideas of tackling the problem (ideally visualized in some way) and less on the techniques and the formalisms involved. Instead of getting lost in detail, the main steps toward the solution and the insights gained should be presented as a kind of "promotional presentation for the developed product". If students are inexperienced with this format, it is advisable for the supervising student teachers to have a look at the posters and slides beforehand.

4. Developing modelling problems for school students

Experience shows that many student teachers, but also many teachers, struggle to develop their own modelling problems for students. The problems should provide opportunities for a

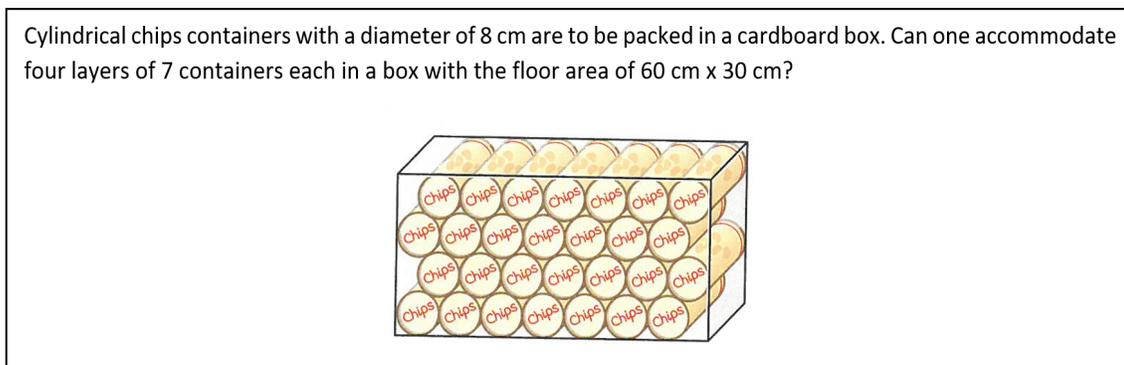


Figure 11 – Task Chips containers (from Böer et al. 2008).

variety of solution approaches and mathematical instruments, allow being considered at different levels of complexity, touch upon the students’ realities of and yield solution processes within a reasonable timeframe. For this reason, we next provide some practical tips for the development of modelling problems, which have proven to be useful during our modelling seminars with student teachers.

4.1. Opening textbook problems by reducing information

The textbook series “mathe live” includes the following problem (where floor area presumably means the front boundary surface of the cardboard box in Fig. 11):

Ultimately, this problem is “just” about recognising that the second and the fourth row from below protrude by exactly half a diameter from the first and third row and that the inequality $78 + 4 \leq 60$ holds, and then checking whether it works in the other dimension as well: Four bags stacked on top of each other would normally require 32 cm, and the question is whether 30 cm are enough using this “offset” positioning. The problem’s context, however, makes it appealing to generate an optimisation problem out of it. One could, for example, remove the image and the measurements of the containers and the cardboard box. Then, the task could look like this:

Statement of task: Customary, cylindrical chips containers are to be packed in cardboard boxes of 28 containers each. What is an optimal way of constructing the cardboard boxes if the goal is to use as little packaging material as possible? Justify your answer using mathematical methods! Students would now have to do investigations by themselves to find out about usual measurements for cylindrical chips containers. Furthermore, different methods of stacking would need to be considered by comparing the respective surface areas of the resulting cardboard packaging. The problem leaves open questions about considering adhesive folds, whether certain cardboard dimensions are better for transport purposes than others, or whether two cylinders are allowed to be stacked top-to-bottom. This allows for a lot of leeway in the students’ modelling endeavours. Further suggestions for the “opening” of problems can be found in mathematics education literature (cf. Bruder 2000, Herget 2013, Köhler 1998).

4.2. Opening textbook problems by removing references to mathematical tools

The following problem can be found in the textbook series “Mathematik verstehen” within the topic of “exponential functions” (Fig. 12).

In this problem, the functional equation describing the process of cooling down is explicitly stated. The task aims at practicing calculations using this function. It can be opened and reworded into a modelling problem if, instead of the functional equation, only single values of

A solid is placed in a cooling chamber and cools down exponentially, where $T(t) = 65 \cdot 0.86^t$ ($^{\circ}\text{C}$) holds for its temperature $T(t)$ after t minutes:

- Which temperature does the solid have in the beginning?
- To which fraction does the temperature of the solid sink per minute, to which fraction after 5 minutes?
- By how many percent does the temperature decrease per minute, by how many percent does it decrease in 10 min?
- Draw the graph of the function T which assigns each point in time t (with $0 \leq t \leq 10$) the temperature $T(t)$ of the solid at that time!

Figure 12 – Problem cool down process (from Malle et al. 2010).

t (in minutes after serving)	4	7	10	15
T (in $^{\circ}\text{C}$)	72.3	66.6	60.8	53.1

Table 2 – Coffee temperatures

measurement are given which do not obviously aim at a modelling process using an exponential function. At the same time, the context can be swapped to give the problem a situational background:

Statement of task: Freshly served coffee is usually too hot to be drunk immediately. First it has to cool down to an appropriate drinking temperature. Table 2 shows the measured temperature of coffee in a cup at different points of time:

Of course, the room temperature plays a role in this problem as well. Consider a reasonable room temperature and answer the following questions: Which temperature did the coffee have when it was served? When does the coffee reach drinking temperature?

With this wording, students have to consider for themselves that the temperature of the coffee can be modelled by an exponential function that is shifted upwards by the room temperature ("exponential assimilation to room temperature"). Alternatively, instead of the data above, data gathered independently by the students can be used (the physics department usually has an infrared thermometer). The model developed by the students can conversely also be validated by retrospective measurements on a real cup of coffee.

4.3. Adequately reducing the complexity of real problems

Real optimisation problems are often too complex to be used in the classroom. By reducing the complexity in an adequate manner, however, they can definitely be made accessible to students. When doing so, it is of course necessary to make sure that the problems maintain their original character and the questions and tasks remain meaningful (cf. Kaiser et al. 2013). As an example, one might wonder how two existing railway networks could be connected in an ideal fashion (in Austria, for instance, the Semmeringbasis tunnel is currently being constructed which will connect existing tracks in Gloggnitz and Mürzzuschlag).

Here, the terrain, restrictions regarding development plans, the characteristics of the soil, financial aspects, etc. play an important role. In the following problem, this application problem was reduced to a manageable situation, a certain structure was imposed by providing a graph, and hints were given regarding the relevant aspects of the problem for finding a solution: **Statement of task:** Two already existing and slightly offset, parallel railway tracks are to be connected in a way that is as smooth as possible.

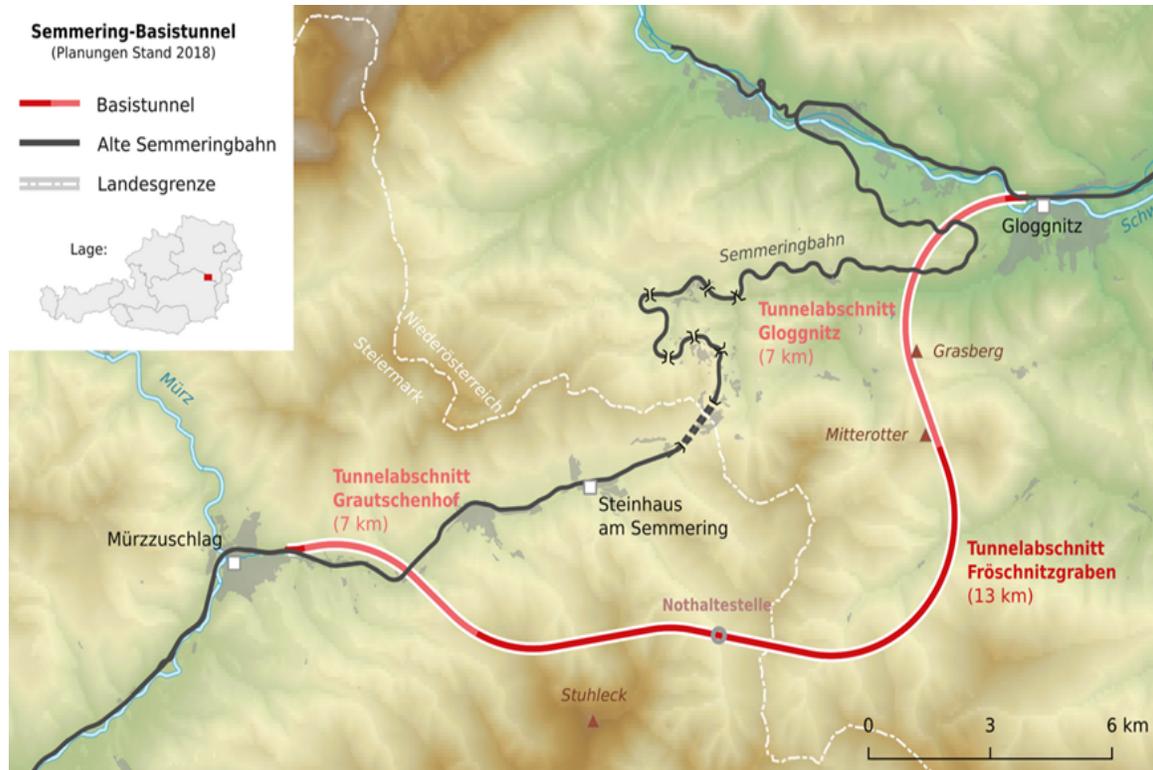


Figure 13 – Routing of the Semmeringbasis tunnel.

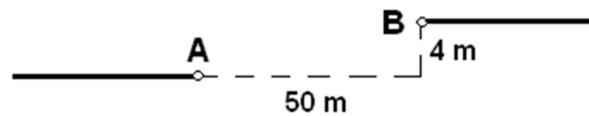


Figure 14 – Connection of tracks.

What would be a possible routing of the connection tracks? Develop a mathematical model! Which curves or functional graphs can be considered for this? What needs to be considered at the connection points A and B (Fig. 14)?

5. Conclusion

As already said above, we consider dressed-up word problems which meet the conditions mentioned in chapter 1 as important to classroom teaching, because for teaching it is crucial to offer many opportunities for translating written text into the language of mathematics and practicing mathematical concepts and procedures. But the orientation towards applications in mathematics teaching in classroom cannot be limited to solving some dressed-up word problems (see e.g. Blum and Niss 1991) every now and then, also authentic modelling problems should be dealt with, in order that students can make the experience: mathematics can help us to analyse and solve realistic and authentic problems. Whether or not and to which extent modelling problems are suitable for assessment purposes (especially in standardized tests) is a difficult and controversial question. In our opinion rather process oriented authentic modelling problems (cf. e.g. Kaiser and Schwarz 2010) should not be part of examinations and assessments, because

they need a lot of time, the possibility to think deeply and in various directions, the possibility to change the approach, to explore situations, to communicate with other students etc. For examination and assessment purposes one should rather use problems in which it is clear what sort of mathematical concept should be used here (a special mathematical concept is in the centre of attention), and then we are rather in the field of what we called dressed-up word problems. Other researchers may have another view on the term modelling, namely to use it also for problems that we described as dressed-up word problems. Then, of course, one may call also these activities modelling problems including corresponding tasks in examinations and assessments. If prospective teachers are supposed to include modelling (not just in the sense of dressed-up word problems) in their classroom teaching, they should have opportunities during their education at university to gain experiences in this field. The Teacher Education Programme for Secondary Mathematics Teachers (BA/MA) only includes a very limited number of ECTS points in mathematics and its didactics (at the University of Vienna and presumably at other locations as well, that number had to be further reduced in comparison to the old programme), resulting in "there is no designated room for modelling within the curriculum. It is our intention to provide students with an opportunity to encounter the topic of modelling (experiencing it themselves and using it in classroom teaching) during the seminar, which students can choose as an elective one during their master's studies. Both of those experiences are important and made possible in the seminar: On the one hand "What is modelling and what does it feel like to work on a modelling problem as a team?" (there is enough time for this, after all, over the course of an entire semester), on the other hand "How can this topic be integrated into classroom teaching and how can teachers mentor modelling processes of their students?". An anonymous evaluation of our seminar is conducted on a regular basis and we receive personal oral feedback after the modelling days at the schools. Both clearly show that practically all students especially appreciate the practical experience of this modelling day at a school (for their future profession):

1. Supervising a group of modelling students
2. Structure of a day like this
3. What can one reasonably expect from students?
4. Which emphases can be placed?
5. What should be considered when developing problems and handling final presentations?
etc.

References

-  [Blum W. & Niss M. \(1991\).](#)
Applied mathematical problem solving, modelling, applications, and links to other subjects – State, trends and issues in mathematics instruction.
Educational Studies in Mathematics (22), p. 37–68.
-  [Blum W. & Leiss D \(2005\)](#)
Modellieren im Unterricht mit der “Tanken”-Aufgabe.
Mathematiklehren, issue 128, p. 18–21.
-  [Böer H., Kliemann S., Puscher R., Segelken S., Schmidt W., Trapp M. Vernay R. \(2008\).](#)
Mathe live, Erweiterungskurs 9.
Stuttgart: Klett.
-  [Bracke M., Göttlich S. & Götz T. \(2013\).](#)
Modellierungsproblem Dart spielen.
In: Borromeo Ferri R., Greefrath G., Kaiser G. (Ed.),
Mathematisches Modellieren für Schule und Hochschule (p. 147–162).
Wiesbaden: Springer Fachmedien.
-  [Bracke M. & Humenberger H. \(2012\).](#)
Steigerung der Effizienz einer Kläranlage –eine erprobte Modellierungsaufgabe.
Mathematische Semesterberichte 59, 2, 261–288 (DOI:10.1007/s00591-012-0099-0)
-  [Bruder R. \(2000\).](#)
*Eine akzentuierte Aufgabenauswahl und Vermitteln heuristischer Erfahrung:
Wege zu einem anspruchsvollen Mathematikunterricht für alle.*
In: Flade, L. & Herget, W. (Hrsg.), Mathematik lernen und lernen nach
TIMSS –Anregungen für die Sekundarstufe (p. 69–78).
Berlin: Volk und Wissen.
-  [Herget W. \(2013\).](#)
Ufgaben öffnen!
In: Plackner E.-M. & Wörner D. (Ed.), Aufgaben öffnen.
Materialien für den Mathematikunterricht (p. 17–38).
Hildesheim: Franzbecker.
-  [Holzäpfel L. & LeißD. \(2014\).](#)
Modellieren in der Sekundarstufe.
In H. Linneweber-Lammerskitten (Ed.), Fachdidaktik Mathematik: Grundbildung
und Kompetenzaufbau im Unterricht der Sek. I und II (p. 159–178).
Seelze: Kallmeyer.
-  [Kaiser G., Bracke M., Göttlich, S. & Kaland Ch. \(2013\).](#)
Authentic complex modelling problems in mathematics education.
In: Damlamian, A. et al. (Ed.), Educational interfaces between mathematics and
industry. Report on an ICMI-ICIAM-study. The 20th ICMI study (p. 287–297).
Cham: Springer.
-  [Kaiser G. & Schwarz B. \(2010\).](#)
Authentic Modelling Problems in Mathematics Education–Examples and Experiences.
Journal für Mathematik-Didaktik (31), p. 51–76.

-  Köhler R. (1998).
TIMSS und die Folgen: Was kann man in der Praxis ändern?
Blum, W. & Neubrand, M. (Ed.), TIMSS und der Mathematikunterricht.
Informationen, Analysen, Konsequenzen (p. 40–45).
Hannover: Schroedel.
-  Maaß K. & Mischo Ch. (2012).
Fördert mathematisches Modellieren die Motivation in Mathematik?
Befunde einer Interventionsstudie bei HauptschülerInnen.
mathematica didactica (35), p. 25–49.
-  Malle G., Koth M., Woschitz H., Malle S., Salzger B. & Ulovec, A. (2010)
Mathematik verstehen 6
Vienna: öbv.
-  Niss M. & Blum W. (2020).
The Teaching and Learning of Mathematical Modelling
London: Routledge.
-  Schupp H. (1988).
*Anwendungsorientierter Mathematikunterricht in der Sekundarstufe zwischen
Theorie und neuen Impulsen.*
Der Mathematikunterricht 34, 6, p. 5–16.
-  Vom Hofe R., Kleine M., Blum W. & Pekrun, R. (2006).
*The effect of mental models (“Grundvorstellungen”) for the development
of mathematical competencies.*
In: Bosch, M. (Ed.), European Research in Mathematics Education (p. 142–151).
-  Vos P. (2011).
What is “Authentic” in the Teaching and Learning of Mathematical Modelling?
Kaiser, G. et al. (eds): Trends in Teaching and Learning of Mathematical
Modelling, pp. 713–722.
Springer.