

Autumn School Mathematical Modeling – a cooperation between ZFM and TUDa

Escuela de Otoño de Modelización Matemática – cooperación ZFM - TUDa

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Abstract

Since 2000 the Zentrum für Mathematik in Bensheim (ZFM) and the Technical University Darmstadt (TUDa) have organized an annual autumn school for mathematically highly talented pupils in their last year of school. For one week, the participants work in groups on application problems, which first need to be transformed into mathematics and then be solved by mathematical analyzation and numerical approximation and implementation on a computer. Finally, the results are presented. The problems are very realistic. Only minor details may be skipped, to make the problem easier to handle and to focus on the main issue at hand, but only if they do not affect the basic result. For some of the problems there exist similar, but much simpler, counterparts that can be solved in a shorter time period even in a usual school setting. Solving those simple problems opens the outlook to the hard problems if they are mentioned before. This paper examines two applications that led to mathematical problems from analysis and geometry.

Desde el año 2000, el Zentrum für Mathematik de Bensheim (ZFM) y la Universidad Darmstadt (TUDa) organizan anualmente una escuela de otoño para alumnos con gran talento matemático que cursan el último año de estudios. Durante una semana, los participantes trabajan en grupos sobre problemas de aplicación, que primero hay que matematizar y luego resolver mediante análisis matemático y aproximación numérica. Finalmente, se presentan los resultados. Los problemas son muy realistas. Sólo se pueden omitir detalles menores, para facilitar el manejo del problema y centrarse en la cuestión principal, solo si no afectan al resultado básico. Para algunos de los problemas existen versiones más sencillas, que pueden resolverse en un periodo de tiempo más corto, incluso en un entorno escolar habitual. La resolución de esos problemas sencillos abre la perspectiva a los problemas difíciles si se introducen antes. En este trabajo se examinan dos aplicaciones que llevaron a problemas matemáticos problemas de análisis y geometría.

Keywords: Area of polygons, volume of polytopes, ski-jumping, optimization, differential equations.
Palabras clave: Área de polígonos, volumen de politopos, salto de esquí, optimización, ecuaciones diferenciales.

1. Introduction

The selection of the pupils:

Each year in spring, the ZFM organizes a competition for pupils in their eleventh year of school, who selected mathematics as their major subject. The best 40 of about 1500 participants are then invited to spend a week of their holidays in the following autumn to do mathematics. The pupils are thus extremely talented and highly motivated.

The basic conditions of the autumn school:

The autumn school lasts for one week from Sunday evening (arrival) to Friday afternoon. During this week, eight completely different problems are solved by eight different groups. Each group is accompanied by one or two teacher students of mathematics. The students do not help the pupils solve the problem itself. They don't know the solution, and if so they would not tell. Otherwise, the students' presence would interfere with the groups' discussion and brainstorming. The students are in charge of ensuring a good group dynamic, continuous result saving, feedback in discussions, time management, proof reading of the final report and give feed-back after the test presentations. The pupils do the modeling, solve the mathematical problems, do all the programming, they give the final talk and write the report. Each problem is introduced by a mathematical expert, who can answer questions the groups might have throughout the week. The experts provide only minimal assistance, to prevent the pupils from losing time with a completely wrong approach to the solution. Sometimes it is also necessary to draw the attention of the groups in a new direction, but at the end the groups find their own solution.

The autumn school takes place under nearly professional conditions in a congress center near a forest and far away from town and other distractions.

All of Friday is used for the presentations, where 20 minutes are planned for each talk and 10 minutes for the discussion. The autumn school ends in the afternoon with a final feed-back round.

The conditions are thus very close to optimal with respect to the selection of the pupils, the time, the location and the support provided. Therefore, it is clear that the problems of the autumn school need an adaption so that they can be used in a classroom.

Such an adaption could consist of providing hints and assistance for each step or using a somewhat similar but much simpler problem. Sometimes those simplifications can lead to the creation of problems which the pupils do not even recognize as real applications. In this case, it is of special importance to show the original more difficult problem, before the pupils work on the simplified version. But sometimes even the simplified problems describe interesting applications. In this paper, we will show two problems that were used in the autumn school and that can also be used in an adapted version in school. But we want to encourage teachers, to also study the complex version of the problems, so that they can give an outlook in class.

The selection of the problems:

In order to motivate the pupils, it is an advantage if the pupils have a strong relation to the problem by experience (e.g.: computing a volume) or interest (e.g.: sports or ecology). If the interdisciplinary context comes from other subjects in school, the pupils do not need too much help of other experts which simplifies the start of the project. The pupils can find a lot of missing information on the internet. However, a complete solution of the problems should not be available online.

In each autumn school we also try to find a set of eight problems that cover as many different fields of application as possible, to show the importance of mathematics for science, or by the words of Immanuel Kant: *Ich behaupte aber, daß in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist.*

(Immanuel Kant (1786)) (There is no real science without mathematics!)

We also try to cover many different mathematical research areas like: geometry, analysis, stochastics, linear, non-linear and discrete optimization, numerical analysis, graph theory, game theory and so on, to show that mathematics is a living science and not only a cultural heritage.

In the opinion of many pupils, mathematics is history and not a modern science. This is a drawback of one of its greatest advantages. Mathematical results lead to eternal knowledge that never needs to be changed. (To be honest, there is no proof for the last statement. It is only the experience of mankind since the beginning of historiography.) Some of the results may become less important but they remain valid and are often still helpful and necessary to understand the modern new results. School mathematics is therefore filled with historical results that have been found centuries ago. This is completely different in most other sciences, where in some fields results are published online and are often already outdated when they are finally printed on paper.

2. Filling level of the tank of an excavator

Construction machines, like crawlers or excavators, can drive and work on uneven ground and they often do. There are sensors to register the level difference between front and back and between left and right. According to the notation in flight mechanics, they measure the pitch angle α and the roll angle β of the excavator. If a coordinate system is used, where the x -axis is the direction from back to front of the excavator in horizontal position, the y -axis is directed horizontally to the left and the z -axis is pointing upwards, then roll angle $\beta \neq 0$ means a rotation around the x -axis, if the right and left front wheels have different level.

A pitch angle $\alpha \neq 0$ means, that after an optional rolling there is an additional rotation around the y -axis, depending on the slope in direction of travel.

There is a small cabin for the driver, but the main space of construction machines are needed for the engine, the wheels or tracks and working tools. There is usually also a large tank that has no simple form but uses all free space between the rest of the construction. Thus, the tank is usually not cuboid but has a very complex form.

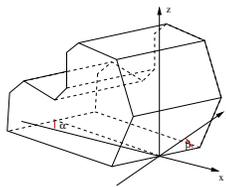


Figure 1 – Typical tank with pitch angle α and roll angle β .

The form is designed on a computer and is given by a list of triangles, where each triangle is given by three points which in turn are given by three coordinates each in the local coordinate system of the excavator. In addition to that, the normal vector is given for each triangle, orthogonal to the triangle and directed towards the outside of the tank.

Example: Assume a very simple tank which is given by the vertices

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad C = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This tetrahedron has 4 triangular faces and the data for this tank must list all these triangles together with the normal vector.

For triangle ABC we need the data

$$A, B, C, n = (1, 0, 0), (0, 1, 0), (0, 0, 1), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and similar information is required for each triangle.

The filling volume is measured either by a sensor at the drain in the bottom, that measures the pressure p of the liquid above the sensor, or by the position of the cutting point of the surface and a measuring tube. In case of $\alpha = \beta = 0$ the tank can be filled with fuel and the increasing pressure of the sensor can be registered and volume V of fuel and pressure $p(V)$ can be stored on a chip so that a microprocessor can later recall the fuel volume, according to the actual pressure at the sensor. The problem is that even with a constant filling volume, the pressure changes due to α and β , and therefore the processor needs either a table that contains the fuel volume in the tank for each triple $(\alpha; \beta; p)$, or a program that can compute the volume $V(\alpha, \beta, p)$ in real time, or a table that was generated in advance.

Before we look at how this problem was solved in the autumn school, we would like to show similar problems that can be handled in the classroom and that already show some of the difficulties and solution ideas.

Volume of a pitched beverage carton

Orange juice or any other drinks are often sold in cuboid beverage cartons. Many of them can be opened by unscrewing a cap. If some of the liquid has already been drunk and we want to estimate the remaining volume, we might pitch the container until the liquid appears at the outlet. We measure the pitch angle α and the rest is simple 2-D geometry.

The solution for special α can be calculated by geometrical construction and measuring the lengths of the edges of the constructed triangles.

A general formula that works with arbitrary rectangles and arbitrary α require linear and trigonometric functions.

The mathematical problem is: Given a rectangle $ABCD$ with the sides a and b , where we choose A as origin of the coordinate system. Given also the angle α between \overline{AD} and the x -axis, and given one point P of the actual surface of the liquid.

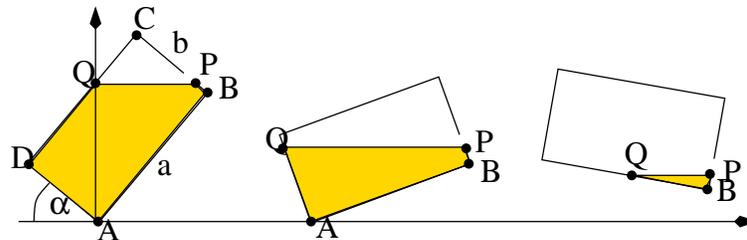


Figure 2 – Compute the area of the filled polygon.

The solution is quite simple. Depending on the angle α and the filling level, the filled polygon has the vertices P, Q, D, A and B (left picture), or P, Q and B for very small filling levels and $\alpha > 90^\circ$ (right picture), or P, Q, A and B (picture in the middle).

The volume of this area can be computed by cutting the region in different ways into elementary geometric objects, like triangles, rectangles, trapezia and rhomboids where we have formulas for calculating the area.

A more realistic variation is a fuel tank in the cellar of an old house, where the ground of the cellar sank due to the heavy weight, but more in the middle of the room and less at the walls.

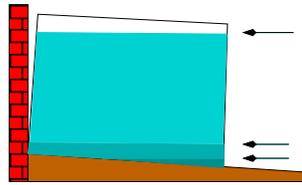


Figure 3 – Fuel tank in the cellar.

We assume that the fuel level can be seen at the tank side faced to the middle of the room. In the picture we see three arrows pointing on possible filling levels and the according volumes. Pitch angle α plays a very important role, especially in case of a nearly empty tank, as a warning can come too late and the house may run out of fuel.

The solution of this problem now requires domain decomposition into elementary geometrical domains like triangles and rectangles which can be done by hand. The rest is basic school geometry and is left to the reader.

On a higher level, we can also develop a formula for the area of general polygons like polygon $ADQPB$ in the picture. Such formulas help us answer even more interesting questions.

How can we estimate the amount of forest in Germany?

Many similar questions are very important to answer and they need a lot of geographic data. Many state organizations collect data and provide a database for a variety of applications, like drawing maps.

If we look at a map supplied by OpenStreetMap, we find not only streets and rivers, but also level lines and information about areas like forests, lakes, deserts, cities, industrial zones and agricultural land. If we zoom into the maps, we see that the boundaries of different regions at some level do not become more detailed, but are often given by polygons which can be defined by their vertices. We can assume that there is a convention to list the coordinates of the vertices starting by any one of them and going clockwise around the region.

(Remark: The Data in OpenStreetMap are often given by longitude, latitude and sometimes altitude. It can be another project, to transform these data into an orthogonal coordinate system and to find polygons in a plane. As an approximation we could neglect the altitude and replace it by sea level. As a single polygon usually is rather small, its vertices are nearly located on a plane, even though they lay on a sphere. We can find this plane by least square approximation of a plane to the vertices of the polygon and then find the projection of the data from OpenStreetMap on that plane. Here we assume, that this problem has already been solved.)

As the data is updated every day, we need to have an algorithm that can compute the volume of these polygons very fast in order to answer important questions like how many square kilometers of Germany will be flooded, if the sea level rises by 1 meter. To do this we would need to have the 1-meter-contour-lines of Germany, which also form polygons.

The mathematical problem then is to compute the volume of polygons.

A simple method for doing so is to split the polygon into triangles. For one single polygon this is relatively easy.

But how can this be done automatically if there are thousands of polygons that are given

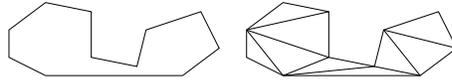


Figure 4 – Splitting the polygon into triangles.

by digital data. The information was often collected by a land survey where specific domains are defined as polygons, and where each polygon is given by a number n of its vertices. As a convention, we can start at any point of the boundary which gets the number 1. We then follow the border of the domain with the domain on the right hand side until we reach the starting point. Thereby, we store the coordinates of each point where we change direction on our way, define this point as the next numbered vertex and assign the next number. The polygon is then defined by its vertices which are numbered clockwise.

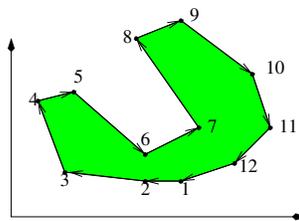


Figure 5 – Polygon defined by its vertices numbered clockwise.

How to compute the area of a polygon?

One principle of modeling is to reduce complex problems to a number of simple problems, and to approximate complex objects by a number of simple objects. As a result, we often see the approximation of curves by many small linear curves and the approximation of surfaces by piecewise linear functions, or even triangles.

Areas that are bounded by smooth curves can be approximated by polygons. An arbitrary polygon can of course be drawn and then divided into disjoint triangles, so that only the volume of these triangles need to be calculated. If this work has to be done very often, drawing is not a practical approach. For example, if we want to answer the question of how many square meters in a region are covered with forest, we want to take the data behind the maps and start a program that selects all forest data sets, which might be hundreds of polygons, each defined by the coordinates of many vertices.

Moreover, this program can use data files of a complete country with the format $(type, n, p)$ where $type$ can be forest, water, farmland, grass, \dots , $n \in \mathbb{N}$ is the number of vertices of the boundary and $p \in \mathbb{R}^{2, n}$ are the coordinates.

Instead of splitting the polygon into triangles, we can compute the area directly if we assume that all edges of the polygon are directed and given by two vertices, so that the area of the polygon is on the right hand side if we follow the edges in their direction.

In the figure below, we see 12 vertices P_1, \dots, P_{12} numbered clockwise.

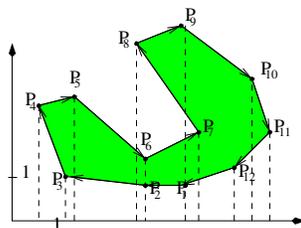


Figure 6 – (x_i, y_i) vertices coordinates.

The polygon is defined by the list

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	3.9	3.0	1.2	0.5	1.4	3.0	4.2	2.8	3.8	5.4	5.8	5.0
y_i	0.8	0.8	1.0	2.6	2.7	1.4	2.0	4.0	4.4	3.2	2.0	1.2

The area of the polygon P_1, \dots, P_{12} can be computed, by adding the areas between the vertices $P_4P_5, P_5P_6, P_6P_7, P_8P_9, P_9P_{10}, P_{10}P_{11}$ and the x -axis and by subtracting the areas between the vertices $P_{11}P_{12}, P_{12}P_1, P_1P_2, P_2P_3, P_3P_4, P_7P_8$ and the x -axis.

(The idea is not new for pupils, if they are not fixed to triangulization of the area. As a hint the scientific supervisor may ask, how the areas between the graphs of two functions $f(x)$ and $g(x)$ can be computed.)

Generally, we take each vertex

$$\overline{PQ} = \left(\left(\begin{matrix} p_x \\ p_y \end{matrix} \right), \left(\begin{matrix} q_x \\ q_y \end{matrix} \right) \right)$$

and compute the area of the trapezoid $P, Q, (q_x, 0)^T, (p_x, 0)^T$ with the formula

$$(q_x - p_x) \frac{q_y + p_y}{2}$$

If the polygon is completely above the x -axis, which is easy to achieve by choosing an appropriate coordinate system, then all directed edges pointing to the right with $q_x > p_x$ lead to a positive value and all the others to a negative one. When adding up all those values, the negative values compensate the areas of the trapezoids that are outside the domain. We thereby obtain a simple formula for the area of a polygon, if n vertices $P_i = (x_i, y_i)$ are given clockwise. We shift the y -coordinates, so that the polygon is completely above the x -axis. With $m := \min\{y_1, \dots, y_n\}$ we find

$$\begin{aligned} A(P_1, \dots, P_n) &= \\ &= \frac{1}{2} \left[(x_1 - x_n)(y_1 - m + y_n - m) + \sum_{i=1}^n (x_{i+1} - x_i)(y_{i+1} - m + y_i - m) \right] \\ &= \frac{1}{2} \left[(x_1 - x_n)(y_1 + y_n) + \sum_{i=1}^n (x_{i+1} - x_i)(y_{i+1} + y_i) \right] \end{aligned} \tag{1}$$

As $x_1 - x_n + \sum_{i=1}^n x_{i+1} - x_i = 0$ all terms with m cancel out, so that the formula is indeed independent of the position of the x -axis.

A polygon with vertices ordered clockwise can also be given by the n directed edges

$$(P_1, P_2), \dots, (P_{n-1}, P_n), (P_n, P_1),$$

and these edges can also be given in any order. If a polygon is given by n directed edges $(P_i, Q_i) = ((P_{ix}, P_{iy})^T, (Q_{ix}, Q_{iy})^T)$ in the form $((P_1, Q_1), \dots, (P_n, Q_n))$, where all the edges form a closed loop, the formula (1) becomes

$$A((P_1, Q_1), \dots, (P_n, Q_n)) = \frac{1}{2} \sum_{i=1}^n (Q_{ix} - P_{ix})(Q_{iy} + P_{iy}) \quad (2)$$

This makes computation of the area of the domain extremely simple. The formula can also be used if the polygon is not given by ordered vertices but by all the n edges, as long as each edge is given in the right direction. This is important when we think of the 3D-problem of the tank, where the volume is bounded by triangles that cannot be ordered in a similar way, but that can be oriented.

Remark: The formula can also be obtained by summing up the area of the triangles $\triangle OP_i Q_i$, where the areas are negative if $\vec{P_i Q_i}$ is pointing counter-clockwise from the origin O .

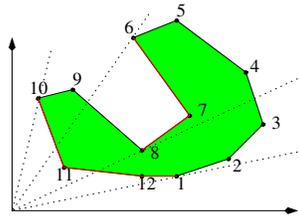


Figure 7 – Computation of the area of the domain.

For the polygon in the figure the areas of the triangles $OP_1P_2, OP_2P_3, OP_3P_4, OP_5P_6, OP_8P_9$ and OP_9P_{10} , are negative and the areas of the triangles $OP_6P_7, OP_7P_8, OP_{10}P_{11}, OP_{11}P_{12}, OP_{12}P_1$ are positive.

This means a polygon with n vertices $P_1, \dots, P_n \in \mathbb{R}^2$ has the volume

$$A(P_1, \dots, P_n) = \frac{1}{2} \left\| \vec{OP}_n \times \vec{OP}_1 + \sum_{i=1}^{n-1} \vec{OP}_i \times \vec{OP}_{i+1} \right\| \quad (3)$$

with any origin $O \in \mathbb{R}^2$. Thereby we use the formula to compute the area of a triangle via cross product. In case of $O = P_1$ this becomes

$$A(P_1, \dots, P_n) = \frac{1}{2} \left\| \sum_{i=2}^{n-1} \vec{P_1 P_i} \times \vec{P_1 P_{i+1}} \right\|. \quad (4)$$

The last formula also applies to all polygons in \mathbb{R}^3 where all vertices $P_i \in \mathbb{R}^3$ are in a plane.

At the autumn school, the pupils deal with the 3-D domain of a tank, which is given in a special form. The facets of the boundary of the tank are given by a list of triangles $\triangle(P_1, P_2, P_3)$ where each point is given by three coordinates. The three points define a plane and to indicate the outside of a tank, the outer normal vector n pointing outside the tank was given for each triangle.

At that point a typical hint for mathematical modeling can be helpful: Start with a simpler problem or more specific start with the 2D case.

Once the 2D problem is solved the supervisor can encourage the pupils to transfer the solution technique to 3D.

Following the solution strategy of the 2D-problem, we can compute the volume of a polytope by computing the volume between the x, y -plane and each facet PQR of the polytope.

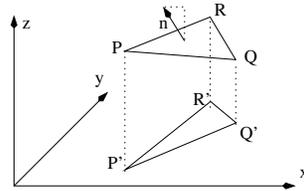


Figure 8 – Projection of a triangle on the x - y -plane.

In case of a facet given by m vertices P_1, \dots, P_m and a normal vector n that is pointing outside the domain the volume is given by the area of the facet, the z component of the normal vector and the mean z component of the vertices by

$$V(P_1, \dots, P_m, x, y) = A(P_1, \dots, P_m)(n^T e_3) \frac{P_{1z} + \dots + P_{mz}}{n} . \tag{5}$$

Similar to the 2D-case, we find that this volume becomes negative if the domain is completely above the x, y -plane and $n^T e_3 = n_3$ is negative and positive otherwise. Thus we get the volume of the polytope by summing up all these positive or negative volumes.

As above, any coordinate system can be used, and for simplicity we can choose the surface of the fuel in the tank as x, y -plane. If the tank is given by its facets, which are all given by the vertices and a normal vector pointing outside, we first need to transform all vectors into the new coordinates if pitch angle α and roll angle β of the bagger are given. We assume that the original x -axis is the direction from the back to the front of the vehicle and the y -axis the direction from right to left. Each vector v thereby becomes

$$\tilde{v} = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix} v =: Tv . \tag{6}$$

The filling level is now given either by the coordinates of the drain D of the tank and the pressure p , or by the coordinates of an arbitrary point P of the surface. In both cases we can compute the z -coordinate of the surface S_z in the new coordinate system by using

$$S_z = (TP)^T e_3 = (TD)^T e_3 + p/\rho , , \tag{7}$$

where ρ is the density of the fuel. We now make another transformation

$$\tilde{v} = Tv - S_z e_3 , \tag{8}$$

so that the surface of the fuel is the \tilde{x}, \tilde{y} -plane. We now have to compute the volume of the filled polytope and therefore we have to find all facets of the filled polytope. These facets can be found easily by examination of the facets of the tank. We distinguish between four cases:

1. All \tilde{z} -components of the vertices of a facet are non-negative. This indicates that the facet lies completely above the surface. Those facets are not facets of the filled polytope and they are just skipped.

2. All \tilde{z} -components of the vertices of a facet are non-positive. This indicates that the facet lies completely below the surface. Those facets are also facets of the filled polytope.

3. Some facets of the tank are cut by the surface. If the facets are triangles, we have two cases, shown in the figure.

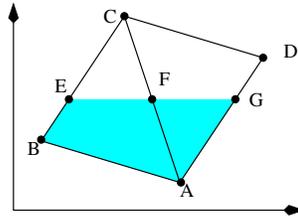


Figure 9 – Tank cut into two triangles.

3.1. Only one vertex A of $\triangle ACD$ is below the surface. The two edges starting from vertex A intersect with the surface, and the two intersections F and G together with A define a triangle facet of the filled polytope.

3.2. Two vertices A and B of $\triangle ABC$ below the surface, the third C above. The edges BC and CA intersect with the surface at E respectively F . The facet $ABEF$ is a facet of the filled polytope.

4. The last facet of the filled polytope is the facet of the surface and this facet usually is a polygon with many vertices, but this facet is actually not needed to compute the volume, if we use the surface as the $\tilde{x} - \tilde{y}$ -plane of the coordinate system, because then the volume between the plane and the surface facet is zero.

Given a complete list of facets of a polytope describing the region of a tank, density ρ of the liquid and coordinates of the drain D . Given also pitch angle α , roll angle β and pressure p then the volume $V(\alpha, \beta, p)$ is obtained in four steps.

1. Compute S_z by formula (7)
2. Transform all data (vertices and normal vectors of the tank) using formula (8) $\tilde{v} = Tv - S_z e_3$.
3. Find intersections of edges of triangles with the surface. Let P and Q be the vertices of such an edge, Then the z -component of the two coordinate vectors p and q need to have different signs and the coordinate vector of the intersection point R is then given by

$$r = p - \frac{p_3}{q_3 - p_3}(q - p)$$

This is ordinary school geometry.

4. Split those triangles into polygons above the surface and polygons below the surface and select only those facets, that are now completely below the surface.
5. Compute the volumes between all those facets and the $x - y$ -plane, using formula (4) and (5).

Most of the steps are easy to implement, but finding the intersection points with the surface, making the correct splitting and select the right facets requires some experience in computer programming. At the autumn school, there are always many pupils with sufficient experience and the groups are put together accordingly to ensure that they can handle all the implementation problems.

At the beginning the pupils usually first have some problems how to start programming and once they have a first idea they often become too enthusiastic and tend to forget thinking of better ways. Instead of spoiling the problem by giving the solution to the pupils, the supervisor may encourage the pupils to find a more elegant way to solve a problem. This task can also be assigned to those pupils, that are not busy with the implementation.

3. How to win a gold medal in ski jumping

A ski jump can be divided into 4 important phases:

In-run: The strategy is to squat as low as possible to minimize air resistance and to gain maximum speed.

Take-off: The strategy is to jump with maximum power which requires physical training and correct timing as late as possible but as long as the feet still have contact with the take-off platform, which in turn requires mental training.

Flying: This is the most interesting phase, which mainly decides the flight distance and therefore whether the athlete will win the competition or end up somewhere in the middle of the ranking. The strategies for this phase have changed a lot during the past and the best strategy has not yet been found. It should be optimized in this project using mathematics. The problem is to find the best position that reduces air resistance (drag) and increases the ascending force (lift).

Landing: The idea is to stop the flying phase as late as possible, but to still perform a secure landing, preferably in Telemark style. This requires a lot of training, as the flying posture are completely different to the landing posture and a fast and secure transition from one flying to the landing posture is difficult.

As everybody can imagine, ski jumping is dangerous and requires a lot of exercise. A single jump only lasts a few seconds in the air, but requires long time to prepare (ascending and mental concentration). Thus, only very few jumps can be made on a single training day.

If the result of a single jump was not satisfactory, it is very difficult for the trainer to tell the exact reason. This is especially true if a new flying style should be developed that might be better than the styles that were used before. A new style can only be judged if the style was performed perfectly by the athlete. The problem is that there is not time enough to learn all possible styles to perfection. Therefore, the trainer wants to know in advance which styles during the flying phase would lead to the best results so that the athlete needs to train only these styles to perfection. However, the influence of small posture changes on the distance is still difficult to predict.

The training therefore starts with exploring different positions in a wind channel, where one hour of training in the air stream leads to flight experiences that compares to 450 jumps and that can in addition be optimally analyzed by the assisting trainer.

The posture of the athlete during the flight has a complex influence on aerodynamics but boils down to two parameters from flight mechanics that describe the air resistance F_D or drag and the ascending force F_L or lift. Even small changes like straddled fingers can have a severe impact on these two forces and this can be experimentally explored in the wind channel.

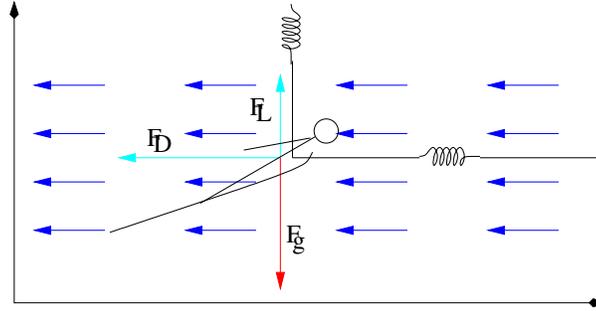


Figure 10 – Measuring lift and drag of an athlete in the air stream of a wind channel.

Here, the athlete is hanging in the air stream on elastic strings and the horizontal and vertical force to hold him in place are measured. (photographs can be found in wiew1 and a video in Audi).

For a given velocity v of the air stream in the channel and air density ρ the horizontal force is

$$F_D = \frac{\rho C_D}{2} A v^2$$

The vertical force to hold the athlete in place must compensate the gravity force $F_g = mg$ reduced by the lift

$$F_L = \frac{\rho C_L A}{2} v^2$$

so in the channel we measure $mg - F_L$ and can thus compute F_L .

The posture of the jumper relative to the incoming air influences drag and lift and determines the coefficients C_D and C_L . In the wind tunnel, all postures can be tested without any risk and the according aerodynamic parameters can be determined exactly. If the drag increases the athlete is pushed by the air flow towards the air extraction tube and must be held by elastic ropes that measure the horizontal force. If the lift increases, this reduces the vertical force that is needed to keep the athlete in his position in the air stream of the wind tunnel. Postures with little drag and high lift are interesting.

In the following tabular we listed parameter combinations of $C_D A \rho_0 / 2$ and $C_L A \rho_0 / 2$ with area A and standard air density ρ_0 of a typical top level athlete with a weight of 75 kilograms (including all equipment).

Style	1	2	3	4	5	6	7	8	9	10
$C_L A \rho_0 / 2$ in $\frac{kg}{m}$.620	.644	.666	.686	.702	.716	.729	.741	.760	.780
$C_D A \rho_0 / 2$ in $\frac{kg}{m}$.468	.485	.502	.519	.536	.556	.583	.623	.713	.802
C_L / C_D	1.325	1.328	1.329	1.322	1.310	1.288	1.251	1.188	1.067	0.973

(In fact these data do not come from a single athlete, but were obtained by motion capturing of several top-athletes (Hans-H Gasser, 2008)

Each data set corresponds to a posture of the athlete in the wind channel which can precisely be photographed and recorded by the trainer team. The search for the best posture now becomes the search for the best data set. Each set is uniquely defined by its style value so that only this single parameter needs to be optimized. But this parameter may change during the flight as the athlete can take different postures after lift-off, during flight and before landing. Therefore, an optimal parameter function $style(t)$ is sought.

But how does the jumping style influence the jumping distance? You cannot predict this by a simple formula and it takes a long time to try it out because this requires to perfectly train

the style before you can measure the jumping distance, but you can simulate the equations of motion once you have the aerodynamic parameters of a given style.

For this simulation we use Newton's central law of motion:

force equals mass of the body times acceleration

or acceleration divided by mass equals the sum of all acting forces.

From this law, one can calculate and predict the motion of bodies for different special cases. Unfortunately, only in some very special cases an analytical solution formula can be found that could be differentiated and optimized. But in the general case, at least approximations of the movements can be calculated with arbitrary accuracy.

To approach this problem, we start with much simpler problems that can be treated in school.

Vertical drop in vacuum:

Let $p(t) \in \mathbb{R}$ be the height of a vertically falling body and $\dot{p}(t) = v(t)$ the velocity. Then we obtain by Newton's law:

$$m\ddot{p}(t) = -mg = F_g \quad (\text{force of gravity}) .$$

Together with the initial conditions $p(t_0) = p_0$ and $v(t_0) = v_0$ we obtain

$$\begin{aligned} \ddot{p}(t) &= -g \\ v(t) &= v(t_0) + \int_{t_0}^t \dot{v}(\tau) d\tau = v(t_0) + \int_{t_0}^t \ddot{p}(\tau) d\tau \\ &= v_0 + \int_{t_0}^t -g d\tau = v_0 - (t - t_0)g \\ p(t) &= p(t_0) + \int_{t_0}^t \dot{p}(\tau) d\tau = p_0 + \int_{t_0}^t v_0 - (\tau - t_0)g d\tau \\ &= p_0 + (t - t_0)v_0 - \frac{(t - t_0)^2}{2}g . \end{aligned}$$

In school, usually only the very special case $t_0 = 0$, $v_0 = 0$, $p_0 = h_0$ is treated.

Everything happens in one dimension (vertical movement) and we obtain

$$p(t) = h(t) = h_0 - t^2/2g$$

and

$$v(t) = -tg .$$

The time to fall a distance s is calculated from

$$s = h_0 - p(t) = h_0 - h(t) = t^2/2g .$$

In the worst case, the three formulas are memorized and called physics.

The general principle to derive such formulas from just one general physical law by just using mathematics, unfortunately often falls by the wayside.

General throwing curve in a vacuum

If you set up the equations of motion in space and consider that in a vacuum, i.e. without any air forces, we only observe acceleration in vertical direction, then the state variables are the position of the body $p(t) \in \mathbb{R}^3$ and the velocity $v(t) \in \mathbb{R}^3$. The equations of motion become

$$m\ddot{p}(t) = F_g = mg \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -mge_3 ,$$

$$p(t_0) = p_0 \quad ; \quad \dot{p}(t_0) = v_0$$

and we obtain the same formulas for the vertical z component as before while the horizontal x and y components change unaccelerated at constant initial velocity. This results in the parabolic curves, which is only an approximation to reality in presence of air. The approximation is, however, quite good if air forces can be neglected as for shot put or basketball throwing.

General throwing curve (with atmosphere)

Taking the air into account, there are other forces besides gravity, for example air resistance.

The drag increases quadratically with the speed $\|v\|^2$, works in direction $-\frac{v}{\|v\|}$ and depends on the air density ρ and the drag coefficient C_D of the body and the cross sectional area A with

$$F_D = -\frac{C_D A \rho}{2} \|v\| v .$$

We assume a flying object with no rotation and therefore no Magnus effect. But asymmetrically shaped flying objects still have a lift effect which can be negative and works perpendicular to F_D with

$$F_L = \frac{C_L A \rho}{2} \|v\| v_{\perp} .$$

Usually the pupils find these formulas on the internet. They may need some assistants to decide which forces need to be considered and which forces can be neglected.

A simple solution formula can no longer be given in this case. But Newton's law still applies. You only have to consider the new influencing forces

$$\begin{aligned} \dot{v}(t) = \ddot{p}(t) &= \frac{F_g + F_w + F_a}{m} \\ &= -ge_3 - \frac{C_D A \rho}{2m} \|v\| v + \frac{C_L A \rho}{2m} \|v\| v_{\perp} . \end{aligned}$$

Once the pupils have set up the differential equations and start to look for exact solution techniques which do not exist for this type of problems, the scientific supervisor must introduce the idea of approximation which is new for the pupils.

But with this hint the classical method attributed to Euler (1707-1783) is reinvented every year by the participants of the modeling autumn school whenever it is necessary to solve differential equations.

The explicit Euler method

Given is a differential equation with initial conditions

$$\dot{z}(t) = f(t, z(t)), \quad z(t_0) = z_0 .$$

Using the definition of the derivative and the limit value one gets

$$\begin{aligned} \dot{z}(t) &:= \lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} , \\ &\approx \frac{z(t+h) - z(t)}{h} \text{ if } h \text{ small enough.} \\ \Rightarrow f(t, z(t)) &\approx \frac{z(t+h) - z(t)}{h} \\ \Rightarrow z(t+h) &\approx z(t) + hf(t, z(t)) \text{ if } h \text{ small enough.} \end{aligned}$$

Thus, starting with $z(t_0) = z_0$ we can compute $z(t_0 + h)$, $z(t_0 + 2h)$, \dots , $z(t_0 + kh)$.

The vector of the state variables $z(t)$ includes all time dependent variables you need to know, to specify the temporal change.

How to win the gold medal in ski flying

We now return to the problem of the autumn school.

We use the simplifying assumption that the competition will be canceled for safety reasons if there is a significant side wind so that the entire movement takes place in only 2 dimensions. In this case the states are the position $p(t) \in \mathbb{R}^2$ and the instantaneous velocity $v(t) = \dot{p}(t) \in \mathbb{R}^2$. The initial conditions are $p(0) = p_0$ and $v(0) = v_0$.

The air resistance increases quadratically with the speed, which the jumper has relative to the air and the force is directed to this relative speed r . With wind direction w and jumping speed v the relative air flow is $r = w - v$ and we get the drag force $F_D = \frac{C_D A \rho}{2} (w - v) \|w - v\|$

The lift also depends quadratically on the speed, but is perpendicular to the relative airflow

$$F_L = \frac{C_L A \rho}{2} (w - v)_\perp \|w - v\| .$$

How to get the right lift direction?

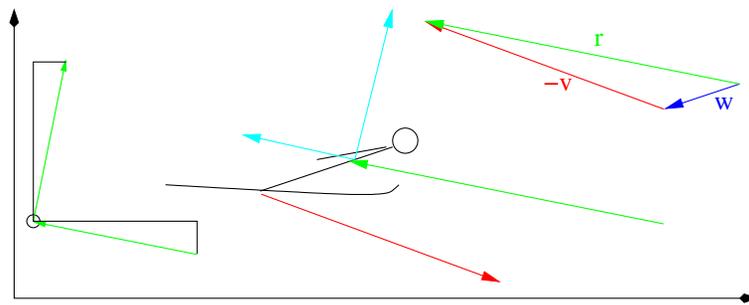


Figure 11 – How to win the gold medal in ski flying.

The trajectory can rise slightly at the beginning due to the jump from the platform and it falls at the end. However, even with a strong tail wind, the ski flyer always has relative headwind during the flight due to his high horizontal speed. That means the x component r_x of the virtual headwind r always is negative if we assume that the x axis points horizontally in the direction of the landing area.

From the relative headwind

$$r = w - v = \begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

you get the direction of the lift by swapping the x and y components and for one component additionally the sign (see sketch).

Possible are

$$r_\perp = \begin{pmatrix} r_y \\ -r_x \end{pmatrix} \text{ and } \bar{r}_\perp = \begin{pmatrix} -r_y \\ r_x \end{pmatrix} .$$

Since r_x is always negative, r_\perp guarantees a positive 2nd component and thus positive lift.

For a given ski jumping hill (the place where the next gold medal will be awarded is known long in advance) and specifications of the possible wind conditions, each skier wants to know the best jumping technique which can then be practiced. It could be an advantage to divide the jump into different phases and for example decide to jump with little air resistance but little

lift at the beginning to gain width with high speed (e.g.: V-style holding the arms close to the body) and avoid a premature landing by increased lift at the expense of speed towards the end of the jump (using the arms as additional wings with some distance parallel to the body).

With a given wind $w = (w_x, w_y)^T$ we then get the classical equations of motion

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = v(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}$$

$$m\dot{v}(t) = m \begin{pmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \end{pmatrix} = \text{sum of forces}$$

for position $p = (x, y)^T$, velocity $v = (v_x, v_y)^T$ and the three forces

$$\text{Gravity: } F_g = mg \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ vertical}$$

$$\text{Virtual wind direction: } r = \begin{pmatrix} r_x(t) \\ r_y(t) \end{pmatrix} = \begin{pmatrix} w_x - v_x(t) \\ w_y - v_y(t) \end{pmatrix}$$

$$\text{Drag force: } F_D = \frac{C_D A \rho}{2} |r(t)| \begin{pmatrix} r_x(t) \\ r_y(t) \end{pmatrix}$$

$$\text{Lift force: } F_L = \frac{C_L A \rho}{2} |r(t)| \begin{pmatrix} r_y(t) \\ -r_x(t) \end{pmatrix} \text{ perpendicular to } r(t)$$

One step of the explicit Euler method then reads:

$$\begin{aligned} r_x &:= w_x - v_x(t) \\ r_y &:= w_y - v_y(t) \\ nr &:= \sqrt{r_x^2 + r_y^2} \\ x(t + \Delta t) &= x(t) + \Delta t v_x(t) \\ y(t + \Delta t) &= y(t) + \Delta t v_y(t) \\ v_x(t + \Delta t) &= v_x(t) + \Delta t \left[\frac{C_D A \rho}{2m} r_x nr + \frac{C_L A \rho}{2m} r_y nr \right] \\ v_y(t + \Delta t) &= v_y(t) + \Delta t \left[\frac{C_D A \rho}{2m} r_y nr - \frac{C_L A \rho}{2m} r_x nr - g \right]. \end{aligned}$$

If you choose a sufficiently small step size $\Delta t = 0.01s$ you can simulate 20 seconds by 2000 steps which is enough for a ski jump from takeoff to landing.

The optimal style for the Gross-Titlis jumping hill in Engelberg

For a given jumping site, we first need to know the basic construction parameters.

For the Gross-Titlis jumping hill in Engelberg, Switzerland these parameters could be:

Vertical distance between take-off and critical K-point $h = 59.77m$

Horizontal distance $d = 103.37m$

Landing zone tilt at K-point $\beta = 0.61261$ in radians

Take-off tilt $\alpha = 0.18326$ in radian measure

For typical wind conditions the length of the in-run is chosen so that a speed at the platform of about $95.15km/h$ is possible.

Assume an athlete with mass $m = 75kg$ and the coefficients of the tabular who plans to jumps on the Engelberg hill when there is no wind. Assume his physical condition allows to add an additional speed perpendicular to the ski jump table of $3.1m/s$ when pushing off the ground just before lift-off. We can then virtually simulate all flight styles.

Style	1	2	3	4	5	6	7	8	9	10
width in m	132.99	136.22	139.06	140.86	141.49	140.73	137.88	131.83	119.74	110.28

Style 3 would be suggested first by many flight mechanics because it has the best glide ratio of C_L/C_D . So without simulation we would probably start training this style first, but by simulation we find that the largest width (141.49 m) is obtained with style 5. This style could therefore be trained intensively without spendig time on style 3.

The influence of changing conditions can also be simulated. Assuming that the athlete use style 5, the following table shows some results for different wind conditions w where negative w indicates head wind.

w in m/s	-1	-0.5	0	0.5	1
width	150.34	145.80	142.49	133.47	137.38

(We can simulate any wind direction but here we assumed that the wind is directed parallel to the slope inclination, so that head wind is always combined with upwind. This is a typical situation, as near the ground wind is often blowing parallel to the surface. For safety reasons the vertical distance of the athletes from the ground is only a few meters even if the camera angle gives a different impression.) The simulation results can be used by the referees to correct the width of athletes and to reduce the in-run in case of head wind.

Although style 5 seems to be optimal we might try to combine two styles as an improvement. Then you have three parameters to select: The 2 strategies and the time t_c where the style changes.

After optimizing these 3 parameters it turns out that the best result is obtained starting with style 3 at the beginning and changing to style 7 after $t_c = 2.03$ seconds or about 50 meters of horizontal flight.

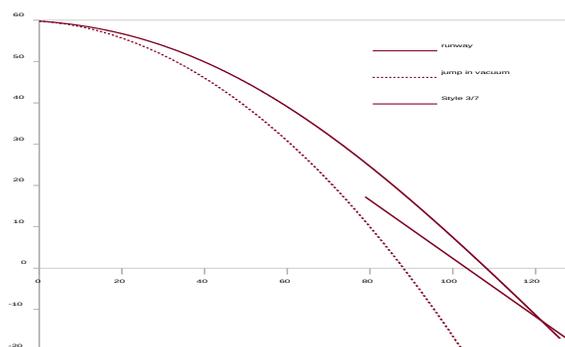


Figure 12 – Optimizing parameters.

Without head wind, this allows a width of 144.04 m which was the record achieved by Domen Prevc in 2016. In the simulation output you see the landings zone (straight line) intersecting the x -axis at the K-point, a quadratic curve showing a jump in vacuum without drag and lift (dotted line) and the curve of the athlete using an optimal combination of style 3 and 7. This curve intersects the landings zone at the landing point. (Note that the horicontal distance is

only about 120 meters, whereas the length of the Euclidian distance from starting point to landing point.)

We assume that a change of style can be performed from one moment to the other. This can be difficult if the two postures of both styles are completely different which is not what we expect. (For example the body posture may differ mainly in the arm position.)

Compared to 141.49 m for the best fixed style we are able to improve the result with a changing style by nearly 2.5 meters. This can decide about bronze, silver or gold.

Outlook

It is not clear if a combination of these two styles are best for all jumping sites, but usually most sites of the next olympic games are well known many years in advance, so that the athletes can prepare. So the simulation of the mathematical model of flight dynamics can help in the athletes to find their optimal styles for the next competition.

It should be mentioned that the coach team of the German national team in fact contacted us in 2001 with the request to optimize the jumping styles of their athletes. Unfortunately, they withdrew the request after Sven Hannawald won all 4 competitions of the Four Hills Tournament in 2001/2. Therefore, we did not optimize his style and maybe as a consequence he did not win the gold medal 2002 in Salt Lake City.”). As an excuse for the coach team we should emphasize, that changing a winning strategy always is risky until a new strategy is mastered perfectly.

A further improvement might be possible in theory if we allow 3 or more flight phases with a different style in each phase, or even by changing the style continuously during the whole flight. However the optimal solution of the mathematical problem finally needs to be executed by a human. This limits the number of styles that can optimally be trained and combined.

We can also make the wind model more flexible, to allow different wind velocities and wind directions in different phases of the flight. However a special strategy for special wind conditions does not help, as the actual wind conditions are not known to the athlete.

The optimization has to be done for each athlete. This means each athlete has to try his own positions in a wind channel and find his personal optimal combinations of lift and drag. This might lead to different looking optimal styles for different athletes.

The problem can also be reduced to only simulation of the flight without optimization of the style, by using typical values for lift and drag but with different wind and different speed after the inrun. We will then find, how the flight distance depends on these parameters. This can be used to compensate if jumping conditions are changing during the competition.

4. Conclusion

Applications in school books often seem to be very artificial. They can even scare pupils away from mathematics. If even teacher do not know the importance of mathematics and cannot give impressive examples, it is difficult to convince the pupils. It is therefore important to show realistic applications. This does not mean, that realistic problems need to be solved in class. Usually a reduction is necessary, that allows the treatment of the problem in limited time and even without help of a computer. Mathematical modeling also requires many different mathematical techniques, that are usually not present to the whole class. A modeling project in class should therefore be combined with a repetition of the needed techniques. This is also a strong hint for the pupils how to deal with the project without giving them the solution itself. In a summer school the project can also start with some kind of workshop, where the required techniques are taught. This, however, was not necessary for our selected pupils.

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