Intrinsically Selective Mass Scaling with Hierarchic Structural Element Formulations

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Abstract: *Hierarchic shear deformable structural element formulations possess the advantage of being intrinsically free from transverse shear locking, that is they avoid transverse shear locking a priori through reparametrization of the kinematic variables. This reparametrization results in shear deformable beam, plate and shell formulations with distinct transverse shear degrees of freedom. The basic idea of selective mass scaling within explicit dynamic analyses is to scale down the highest frequencies in order to increase the critical time step size, while keeping the low frequency modes mostly unaffected. In most concepts, this comes at the cost of non-diagonal mass matrices. In this contribution, we present first investigations on selective mass scaling for hierarchic formulations. Since hierarchic structural formulations possess distinct transverse shear degrees of freedom, they offer the intrinsic ability for selective scaling of the high frequency shear modes, while keeping the bending dominated low frequency modes mostly unaffected. The proposed instrinsically selective mass scaling concept achieves high accuracy, which is typical for selective mass scaling schemes, but in contrast to existing concepts it retains the simplicity of a conventianl mass scaling method and preserves the diagonal structure of a lumped mass matrix. As model problem, we study frequency spectra of different isogeometric Timoshenko beam formulations for a simply supported beam. We discuss the effects of transverse shear parametrization, locking and mass lumping on the accuracy of results.*

1 INTRODUCTION

Finite element solution schemes in the context of structural dynamics can be classified as explicit and implicit methods. Explicit algorithms are particularly popular for highly nonlinear and non-smooth problems, since they do not require any iterative solution of the balance equations on the global level. In specific applications, like car crash or deep drawing simulations, they may be more robust than implicit methods. But due to the conditional stability of explicit methods, the admissible time step size is limited. The so-called critical time step Δt_{crit} crucially depends on the highest frequency $\omega_{\max}$ of the discrete system

$$\Delta t_{\mathrm{crit}} = \frac{2}{\omega_{\max}}. \tag{1}$$

Several approaches to reduce computational cost are available and it is common to use a combination of various approaches simultaneously. First, locking-free and accurate finite element formulations can be used to achieve satisfactory results for coarse meshes, since mesh refinement indirectly increases computational cost for time integration. Second, adaptive mesh control on the basis of error estimators can be used, which is a standard approach for deep drawing simulations. Third, different time steps may be used in areas with different mesh density, which

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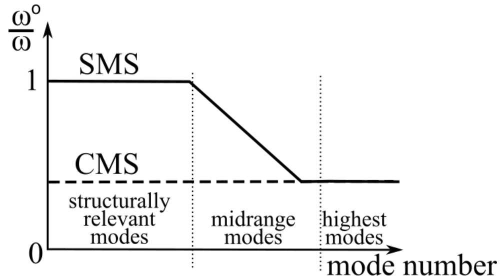


Figure 1: Schematic diagram of the ratio of the scaled eigenfrequencies to the original frequencies of a system for CMS and SMS.

is known as subcycling or asynchronous time integration. Fourth, reduced order modeling may increase efficiency. Fifth, mass scaling may increase the stable critical time step size and thus reduces the number of time steps and the total computational cost.

In the present contribution, we focus on innovative versions and a straight-forward combination of the first and the fifth approach. The outline is as follows. A short overview on established mass scaling concepts is presented in Section 2, before the novel, intrinsically selective mass scaling concept for hierarchic formulations is presented in Section 3. In Section 4, we study frequency spectra of different isogeometric Timoshenko beam formulations for a simply supported beam. We discuss effects of transverse shear parametrization, locking and mass lumping on the accuracy of results. Section 5 concludes our findings and provides an outlook on future work.

2 MASS SCALING

In the research field of mass scaling, it can be distinguished between conventional mass scaling (CMS) and selective mass scaling (SMS). All mass scaling techniques add artificial inertia to the global mass matrix. CMS adds inertia only on the diagonal entries, thus preserving the diagonal structure of the lumped mass matrix (LMM). When applied to translational inertia, as in case of continuum or solid shell element formulations with solely displacement degrees of freedom, translational inertia of the structure is increased. Uniform mass scaling for all elements significantly modifies the linear momentum of the entire structure and thus also affects the lowest, structurally relevant modes, see Figure 1. Therefore, application of CMS is usually limited to a small number of short and stiff elements that limit the critical time step size Δt_{crit}.

In the context of solid finite elements, the basic idea of selective mass scaling (SMS) is to add artificial contributions to both diagonal and off-diagonal entries of the mass matrix in order to preserve translational inertia. This results in a significant reduction of the highest eigenfrequencies, which are often irrelevant for structural response but limit the critical time step size. Manipulation of the low frequencies, which are essential for structural response, is reduced to a minimum. The qualitative picture of the desired ratio between scaled eigenfrequencies ω° and the unscaled eigenfrequencies ω, typically obtained with a LMM, is shown in Figure 1, comparing results obtained with SMS and CMS. The concept of SMS can provide a very good compromise between accuracy and critical time step size, but comes at the cost of non-diagonal mass matrices and thus the need to solve a linear system of equations at each time step. There exist a number of algebraic and variational SMS schemes, see for instance [1, 2, 3], but all of them are designed for continuum or solid shell elements with displacement degrees of freedom.

In case of structural element formulations with rotational degrees of freedom, the aforementioned SMS schemes are not extendable to rotational inertia in a straightforward manner. In fact, naively extending the SMS concept by Olovsson et al. [1] to the rotational part of the mass matrix is not capable of reducing the highest frequency and thus no benefit can be achieved

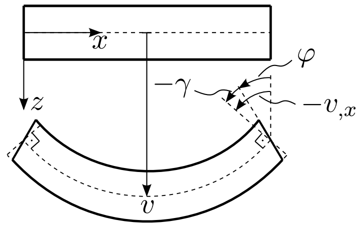


Figure 2: Geometrically linear kinematics of a planar, straight Timoshenko beam.

w.r.t. to increasing the critical time step size. But since in shear deformable structural element formulations the translational and the rotational part of the mass matrix can be computed separately, CMS of the rotational part may lead to some sort of semi-selective mass scaling. CMS of the rotational inertia is used in commercial explicit codes in dayly practice, as can be seen for instance in [4] in the context of isogeometric shell analysis in LS-DYNA. However, since the entire rotational inertia is increased by this concept, also rigid body rotations and the bending dominated modes are severely affected. In case of structural element formulations based on so-called first order shear defomation theory, that is Timoshenko or Reissner-Mindlin theory, the highest frequencies of the discretized system are typically related to transverse shear modes. The direct and isolated access of these high frequency shear modes by a simple and efficient concept is described in the following section.

3 HIERARCHIC REPARAMETRIZATION AND INTRINSICALLY SELECTIVE MASS SCALING

Hierarchic shear deformable structural element formulations are intrinsically free from transverse shear locking, that is they avoid transverse shear locking a priori through reparametrization of the kinematic variables. Although there already exist shear deformable beam, plate and shell formulations, we restrict ourselves to planar straight shear deformable Timoshenko beams with linearized kinematics within this study. In this chapter, different parametrizations of the Timoshenko beam model are briefly summarized and discussed. For further details we refer to [5] or [6]. Figure 2 shows the kinematics of a planar, straight Timoshenko beam, where v describes the mid-line displacement in z-direction, φ and γ are the total and the shear rotation of the beam's cross section. As a general rule in subsequent derivations, equal order interpolation of all involved primary fields is assumed.

The standard formulation of the Timoshenko model (T-st) introduces the total vertical displacement v and the total cross-sectional rotation φ as primary variables. The shear rotation γ and the curvature κ can be expressed as

$$\gamma = v_{,x} + \varphi \qquad \text{and} \qquad \kappa = \varphi_{,x}, \tag{2}$$

where $(\bullet)_{,x} = \frac{\mathrm{d}(\bullet)}{\mathrm{d}x}$ describes the derivative with respect to the spatial x-coordinate. When the discrete primary parameters v^h and φ^h are discretized via any equal order interpolation, pure bending with $\gamma^h = 0$, cannot be fulfilled. The imbalance of the shape functions $v^h_{,x}$ and φ^h leads to the well-known phenomenon of transverse shear locking.

Following the idea from [7], [8] and [9], the shear rotation γ may be introduced directly as primary variable instead of the total rotation φ. Accordingly, φ has to be expressed in terms of the two primary variables, that is

$$\varphi = -v_{,x} + \gamma. \tag{3}$$

formulation	displacement	total rotation	shear rotation	curvature
T-st (v-φ)	v	$\varphi = \varphi$	$\gamma = v_{,x} + \varphi$	$\kappa = \varphi_{,x}$
T-hr (v-γ)	v	$\varphi = -v_{,x} + \gamma$	$\gamma = \gamma$	$\kappa = -v_{,xx} + \gamma_{,x}$
T-hd (v-v_{s})	v	$\varphi = -v_{,x} + v_{\mathrm{s},x}$	$\gamma = v_{\mathrm{s},x}$	$\kappa = -v_{,xx} + v_{\mathrm{s},xx}$

Table 1: Comparison of different Timoshenko beam formulations with different parametrizations of the kinematic variables.

The combination of Equations (2) and (3) yields the modified kinematics

$$\gamma = \gamma \qquad \text{and} \qquad \kappa = \varphi_{,x} = -v_{,xx} + \gamma_{,x}. \tag{4}$$

From Equation (4), the hierarchic structure of the kinematics is visible, since the formulation includes the Euler-Bernoulli beam model for vanishing shear strain γ. Since γ represents the shear rotation, which is superimposed on the rotated cross section according to the Euler-Bernoulli model, this formulation is denoted as Timoshenko beam formulation with hierarchic rotation (T-hr).

An alternative reparametrization is introduced in Timoshenko beam formulations with hierarchic displacements (T-hd). In contrast to the previously presented hierarchic split of the total rotation, the basic idea is the hierarchic split of the displacements into parts resulting from bending and shear, i. e.

$$v = v_{\mathrm{b}} + v_{\mathrm{s}}. \tag{5}$$

This idea is not new, in fact closed form solutions for static and dynamic problems can be found for instance in [10], finite element formulations to solve dynamic problems are presented in [11], among others. Based on Equation (5), a single-variable isogeometric formulation for shear deformable beams is presented in [12].

The following derivations are based on the notation of [5] and a practical modification of the initial concept presented in [13]. Starting from Equation (5), the reparametrized rotation can be written as

$$\varphi = -v_{,x} + \gamma = -(v_{\mathrm{b}} + v_{\mathrm{s}})_{,x} + v_{\mathrm{s},x} = -v_{\mathrm{b},x}. \tag{6}$$

In general, three different T-hd formulations can be derived by using two out of three displacment parameters v, v_{b} and v_{s}. The present study is restricted to the parametrization utilizing v and v_{s} as primary paramters, which is probably the most practical one, see also [13]. The Timoshenko beam formulations used herein are summarized in Table 1. For detailed interpretations and result w.r.t. locking in the context of static analyses, we refer to [5, 13, 6]. Some remarkable features of hierarchic formulations are:

1. The variational index is equal to two, thus consistency requires at least quadratic C^1-continuous shape functions. Thus, the smooth discretization schemes of isogeometric analysis [14] are well-suited for discretization.

2. Both T-hr and T-hd are free from transverse shear locking as the thin limit constraint $\gamma = 0$ can be trivially satisfied by the related degree of freedom being zero. T-hd has fully balanced kinematics, whereas in T-hr, the imbalance is shifted from γ to κ.

3. In the case of beams, shear is completely decoupled from bending.

4. Both formulations possess distinct shear degrees of freedom.

Starting from this point of departure, we introduce the novel idea of a selective and effective mass scaling strategy in the context of hierarchic structural element formulations. As introduced in Section 2, the idea of SMS adresses the effective reduction of the highest frequencies, while keeping the more relevant low frequency modes mostly unaffected. In case of shear deformable structural element formulations based on Timoshenko or Reissner-Mindlin theory, the highest frequencies of the discretized system are typically related to transverse shear modes. A deeper look at features 3 and 4 leads to the following hypotheses for a novel mass scaling strategy in the context of hierarchic formulations:

- The shear frequencies and corresponding modes can be directly accessed by the distinct shear degrees of freedom.
- Decoupling bending and shear facilitates selective scaling of high shear frequencies.
- Both aspects lead to a mass scaling strategy being as effective as a SMS strategy, while retaining the simplicity of a CMS scheme.

Starting point for the following derivations is d'Alembert's principle, specified for Timoshenko beam theory

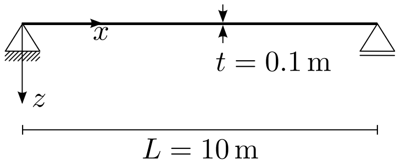
$$\delta W = \int_0^L \left(\delta\gamma GA\gamma + \delta\kappa EI\kappa\right)\,\mathrm{d}x + \int_0^L \left(\delta v\rho A\ddot{v} + \delta\varphi\rho I\ddot{\varphi}\right)\,\mathrm{d}x - \delta W^{\mathrm{ext}} \quad = 0, \tag{7}$$

where E und G denote Young's modulus and shear modulus. The cross-sectional area, the second moment of inertia and the density are denoted by A, I und ρ, respectively. The consistent mass matrix (CMM) $\mathbf{M}_{\mathrm{C}}$ is computed as

$$\mathbf{M}_{\mathrm{C}} = \int_0^L \mathbf{N}^{\mathrm{T}} \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \mathbf{N}\,\mathrm{d}x, \tag{8}$$

with $\mathbf{N}$ being the matrix of shape functions w.r.t. displacement v and total rotation φ. The matrix $\mathbf{N}$ depends on the kinematic description of the chosen beam formulation, that is T-st, T-hr or T-hd, summarized in Table 1. Since for explicit dynamic simulations LMMs are desirable for efficiency reasons, they are of high interest. Cottrell et al. [15] showed for isogeometric analysis that a LMM obtained by row-sum lumping is only second order accurate, independent of the polynomial order. Nevertheless, row-sum lumping for isogeometric elements is still the state of the art in commercial software like LS-DYNA, see for instance [4]. Since the present contribution focuses on mass scaling and not mass lumping, we further consider traditional row-sum lumping and subsequent CMS of the rotational inertia by a scaling parameter α. In case of T-st the rotational part is related to the total rotation φ of the beam's cross section. Thus, any value $\alpha > 1$ for the scaling parameter leads to artificial rotational inertia and thus angular momentum is not preserved. How significantly the bending modes of the T-st formulation are influenced by $\alpha > 1$ is studied in the next section. For the T-hr formulation the rotational entries in the LMM are associated with the shear rotation γ, representing only the shear part of the total rotation φ, but not the rigid body part. Thus, it is expected that a scaling parameter $\alpha > 1$ mainly influences the shear modes, while keeping the bending modes significantly less affected.

For the second hierarchic Timoshenko beam formulation T-hd standard row-sum lumping leads to singular LMMs. In fact, the rotational entries vanish, since the parametrization of the rotation φ is fully balanced, as can be seen in Table 1. In the lumping process each contribution from v is canceled by the corresponding contribution from v_{s}. This issue is not addressed herein, but deserves further consideration in future work, for instance by a detailed study of alternative lumping schemes, as presented for instance in [16] or [17]. The development of accurate mass lumping schemes in general is still an open research topic in the context of isogeometric analysis.



$E = 210\,\mathrm{GPa}$
$\nu = 0.3$
$\rho = 7800\,\mathrm{kg/m^3}$

Figure 3: Simply supported beam, problem setup.

4 NUMERICAL EXAMPLE

As model problem, we study frequency spectra of various isogeometric Timoshenko beam formulations for the case of a simply supported beam, as shown in Figure 3. In all cases, the beam is discretized by 50 elements using quadratic, C^1-continuous B-splines, constructed from an open knot vector. The studied combinations of beam formulations and technologies to tackle transverse shear locking are listed as follows:

- **T-st**: standard Timoshenko beam formulation with displacment v and total rotation φ as primary variables.
- **T-st-low**: as T-st, but the shape functions for φ are one order lower to overcome transverse shear locking, as presented in [18] and [19] for isogeometric elements.
- **T-st-SRI**: As T-st, with selective reduced integration for the shear strain contributions to the stiffness matrix in order to remove transverse shear locking, as shown in [20].
- **T-hr**: Hierarchic Timoshenko beam formulation with hierarchic rotation with displacement v and shear rotation γ as primary fields.
- **T-hr-low**: As T-hr, but the shape functions for γ are one order lower to overcome the imbalance in the kinematic equations, which can be seen in Table 1.
- **T-hd**: Hierarchic Timoshenko beam formulation with hierarchic displacement, with total displacement v and shear displacement v_{s} as primary fields.

First, we study the accuracy of frequency spectra obtained by CMM and LMM with respect to analytical solutions from Cazzani et al. [21]. Figure 4 shows the results for the three T-st formulations, where in the top row of diagrams the frequency spectra are plotted and the bottom row of diagrams displays the ratio of numerical to analytical frequencies. As expected, the CMM provides more accurate results than the LMM in all cases. The T-st formulation suffers from shear locking, as can be seen from the very high frequencies in the bending dominated branch. The selective reduced integrated version T-st-SRI is locking-free, but exhibits low accuracy in the bending dominated branch of the spectrum. The most accurate results of all three standard formulations are obtained by T-st-low.

Figure 5 shows the results for the hierarchic formulations. The imbalance in the kinematic equations of the T-hr formulation leads to slightly increased frequencies in the right part of the bending dominated branch. This is surprising, since this element formulation has proven to be free from transverse shear locking for the static case. But the shifted imbalance from γ to κ theoretically leads to a locking effect in the very thick regime, although we have not observed any relevance of such a locking effect in static analyses. For this problem setup with this relatively fine discretization, the actual element slenderness ratio is $L_{\mathrm{e}}/t = 2$. For more slender elements, this effect vanishes also in the frequency spectra. Theoretically, T-hr-low and T-hd should achieve the same accuracy, which is clearly visible for the spectra obtained by a CMM. But, as already mentioned, the expression $\varphi = -v_{,x} + v_{\mathrm{s},x}$ is perfectly balanced in case of

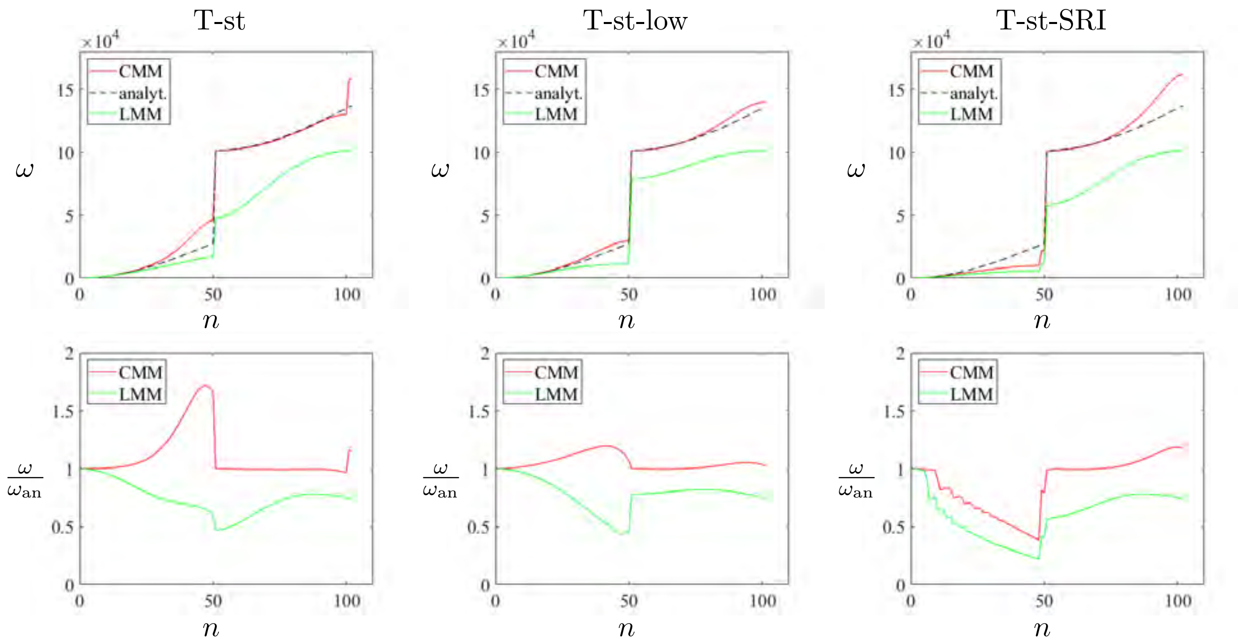


Figure 4: Simply supported beam: Standard formulations T-st, T-st-low and T-st-SRI, top: discrete spectra, bottom: ratio of numerical frequencies to analytical frequencies.

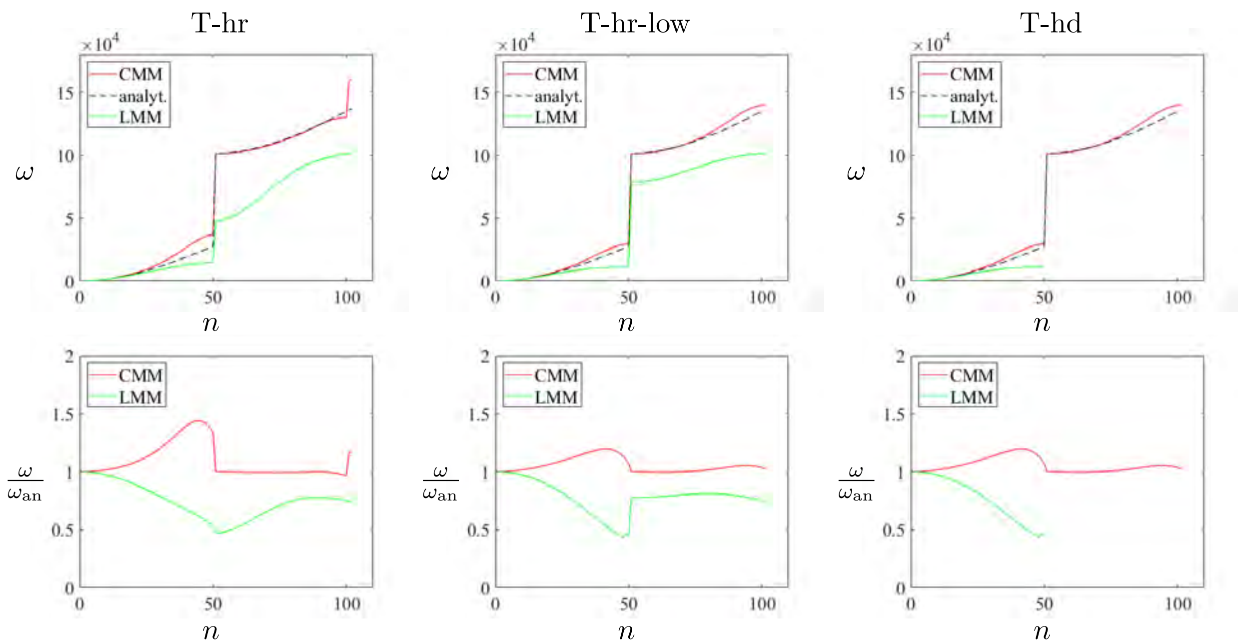


Figure 5: Simply supported beam: Hierarchic formulations T-hr, T-hr-low and T-hd, top: discrete spectra, bottom: ratio of numerical frequencies to analytical frequencies.

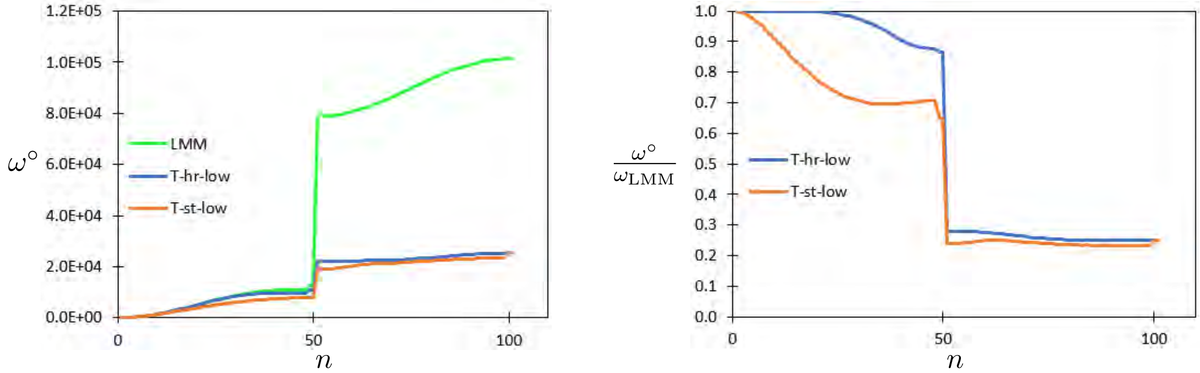


Figure 6: Simply supported beam: T-hr-low and T-st-low, left: scaled spectra in comparison to LMM (unscaled), right: ratio of scaled frequencies to unscaled frequencies for LMM.

T-hd, which leads to zero rotational mass in the LMM obtained by standard row-sum lumping. Alternative lumping strategies are subject of recent investigations, but not discussed herein. The results for T-hr-low are the most accurate ones from Figure 5 and they are practically identical to the results obtained by T-st-low. In fact, the maximum relative difference between both spectra obtained by LMM is 1.51%, while the highest frequency of $\omega_{\max} = 101435.07$ is identical.

Due to practically identical spectra, the two formulations T-st-low and T-hr-low are the optimal starting point for studying the accuracy of mass scaling and the corresponding accuracy of the scaled eigenfrequencies ω°. As stated in the previous section, in case of T-st or T-st-low the rotational part is related to the total rotation φ. Thus, any value for the scaling parameter $\alpha > 1$ leads to artificial rotational inertia and an influence of the bending dominated modes is expected. In contrast to T-st-low, for the T-hr-low formulation the rotational entries in the LMM are solely associated with the shear rotation γ.

Figure 6 compares the accuracy of the scaled eigenfrequencies ω° on the basis of LMMs and a simple CMS of the rotational masses by a scaling parameter α, as explained in Section 3. For both formulations α is chosen such that the maximum eigenfrequncy $\omega_{\max} = 101435.07$ is reduced by 75%. For T-st-low a scaling parameter of $\alpha = 28.52$ is needed to reduce the maximum frequency to $\omega^{\circ}_{\max} = 25358.84$, while the bending dominated frequencies are affected by up to 35% w.r.t. to the unscaled solution obtained by a LMM. In case of T-hr-low $\alpha = 16.0$ yields $\omega^{\circ}_{\max} = 25358.77$ and the highest deviation from LMM in the bending branch is only 14%. The largest deviation in the first quarter of the spectrum is only 1%. This highlights the significant influence of hierarchic parametrization on the selective scalability of the shear dominated frequencies and shows the high potential of the proposed *instrinsically selective mass scaling*, namely possessing high accuracy while preserving the diagonal structure of a lumped mass matrix.

5 CONCLUSIONS AND OUTLOOK

The concept of *instrinsically selective mass scaling* (ISMS) has been proposed. The key idea is to make use of the distinct shear degrees of freedom of shear deformable, hierarchic structural element formulations in the context of an efficient and effective mass scaling strategy. As a model problem, we studied frequency spectra of various isogeometric Timoshenko beam formulations. The results indicate that the proposed ISMS scheme is able to effectively reduce the highest shear frequencies, while keeping the low bending dominated frequencies mostly unaffected. This property is typical for SMS schemes, but, in contrast to standard SMS schemes, the ISMS scheme proposed herein is as simple as a CMS scheme and preserves the diagonal structure of LMMs.

Further developments address the extension to other smooth discretization schemes, to hierarchic shell formulations and to nonlinear transient analyses. In addition, the studies on optimal scaling parameters and developments of time step estimates and more accurate mass lumping schemes are of high interest.

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