Shape Optimization for Thermal Insulation Problems

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Abstract: *In this work we consider two domains: an external domain whose geometry varies, and an internal fixed one. From the thermal insulation viewpoint, we are considering a body to be insulated, enveloped in a layer of insulator, and we want to find the best shape for the thermal insulator, in terms of heat dispersion. Mathematically, our problem is described by an elliptic partial differential equation with Dirichlet-Robin boundary conditions.*

1 INTRODUCTION

One of the major challenges for environmental improvement is represented by thermal insulation. Problems related to insulation are well-known and widely studied in mathematical physics. Nevertheless, mathematics involved is still hard especially when one looks at shape optimization issues [1, 2], and sometimes the answers are counterintuitive [3]. In this work we focus on the case of an internal domain of circular shape (the body to be insulated), enveloped in a layer of thermal insulator whose geometry varies. Our aim is to explore different shapes for the external domain, in order to find configurations which produce low values in terms of heat dispersion.

The work is organized as follows: In Sect. 2 we formulate our problem, explaining the peculiar behavior of the heat dispersion looking at the case of two concentric circles. In Sect. 3 we move to the numerical part, describing the numerical resolution and the results obtained, ending with final remarks and future perspectives contained in Sect. 4.

2 THE PROBLEM

Let us consider a domain Ω embedded into a domain D. The formulation of the problem we deal with is the following:

$$\begin{cases} \text{Find } D^* \in \mathcal{D} \text{ such that} \\ F_\beta(D^*, \Omega) := \min\limits_{D \in \mathcal{D}} F_\beta(D, \Omega) \\ \text{where } F_\beta(D,\Omega) := \beta \int_{\partial D} u \, dx \\ \text{and } u \text{ is solution of} \\ (PDE) \begin{cases} \Delta u = 0, & \text{in } D \setminus \Omega, \\ \dfrac{\partial u}{\partial n} + \beta u = 0, & \text{on } \partial D, \\ u = 1, & \text{on } \partial \Omega, \end{cases} \end{cases} \tag{1}$$

where $u \in H'(D \cup \Omega)$ represents the temperature, $\mathcal{D}$ is the set of admissible domains, n the exterior normal vector, and $\beta > 0$ a fixed parameter depending on the physical characteristics

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of the insulating material. We fix the domain Ω as a unit circle, i.e., $\Omega := B_1(0)$, and $\mathcal{D}$ as the class of polygons, in which the domain D varies. From the thermal insulation viewpoint, the compact connected set Ω represents a conductor of constant temperature fixed to 1, which is thermally insulated by surrounding it with a layer of thermal insulator, denoted by the open set $D \setminus \Omega$, with $\Omega \subset \bar{D}$. The goal of the present work is to find configurations for the domain D which give sufficiently low values for the heat dispersion functional defined as

$$F_\beta(D, \Omega) := \beta \int_{\partial D} u \, dx, \tag{2}$$

comparing the results with the case of two concentric circles, for which we are able to compute the heat dispersion functional analytically. In fact, let us consider Ω and D as two circles of radius r and R, respectively, with $0 < r < R$. The set of solutions to the Laplace's equation in the case of a circular crown is

$$A \log \sqrt{x^2 + y^2} + C = u(x, y), \tag{3}$$

where A and C are two constants. Using for u the expression (3), the functional (2) can be written as

$$\begin{aligned} \beta \int_{\partial D} u &= \beta \int_{\partial D} A \log \sqrt{x^2 + y^2} + C & (4) \\ &= \beta \int_0^{2\pi} (A \log \left(\sqrt{R^2 \cos^2(t) + R^2 \sin^2(t)} \right) + C) R dt & (5) \\ &= \beta 2\pi R (A \log(R) + C). & (6) \end{aligned}$$

Using the boundary conditions of the partial differential equation (PDE) in (1), the constants A and C can be computed solving the following system of two equations:

$$\begin{cases} A \log r + C = 1 \\ \beta(A \log R + C) + A \frac{1}{R} = 0. \end{cases} \tag{7}$$

In that way, we get

$$A = -\frac{1}{\log(\frac{R}{r}) + \frac{1}{\beta R}}, \qquad C = 1 - A \log r = 1 + \frac{1}{\log(\frac{R}{r}) + \frac{1}{\beta R}} \log r \,. \tag{8}$$

Notice that for $r = 1$, the constant C is always equal to one, independently from the admissible values for R and β.

For fixed $\beta > 0$ and $r \in (0, R)$, the dispersion computed according to (6) only depends on R. In particular, it is an increasing function for $R < 1/\beta$, and decreasing for $R > 1/\beta$. Since must be $R > 1$, for $\beta > 1$ the dispersion is a decreasing function (the insulation increases adding insulator), whereas for $0 < \beta < 1$ the dispersion increases for $R < 1/\beta$ and decreases for $R > 1/\beta$. About the increasing phase, it may seem surprising that adding insulator increases the heat dispersion; however this is a well-known phenomenon, that from the physical viewpoint can be explained by the competing effects of the convection and the conduction resistances (see [4], Sect. 3.3.1-3.3.2).

The dependence of the dispersion function on the parameter β vanishes looking at its asymptotic behavior, as visible also in Fig. 1 (see [5] for details).

Considering a geometry which is different from the circle of radius R for the external domain D, we noticed that the qualitative behavior of the dispersion function by varying the parameter β is analogous. As an example, see the plots in Fig. 2 for a comparison between a circle and a regular octagon as external domain.

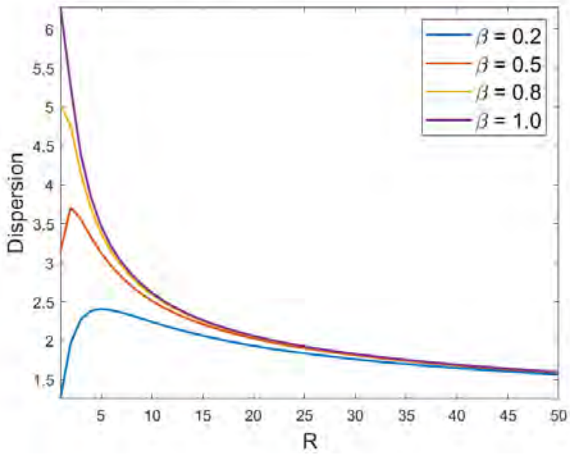


Figure 1: Plot of $(R, F_\beta(B_R(0), B_r(0)))$ for different values of β ($\beta = 0.2, 0.5, 0.8, 1$).

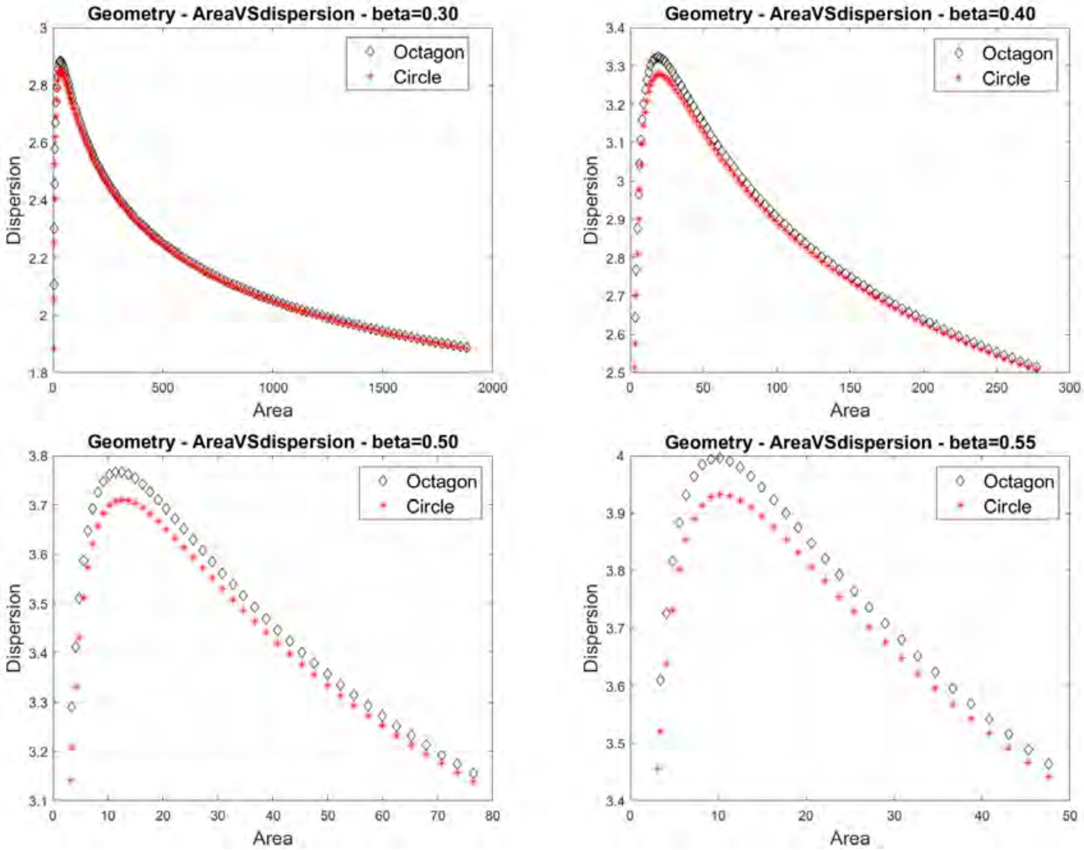


Figure 2: Plots of $(Area, F_\beta(D, \Omega))$ with $\Omega = B_1(0)$, D as a regular octagon (black $\diamond$) or D as a circle (red $*$), for different values of β ($\beta = 0.3, 0.4, 0.5, 0.55$).

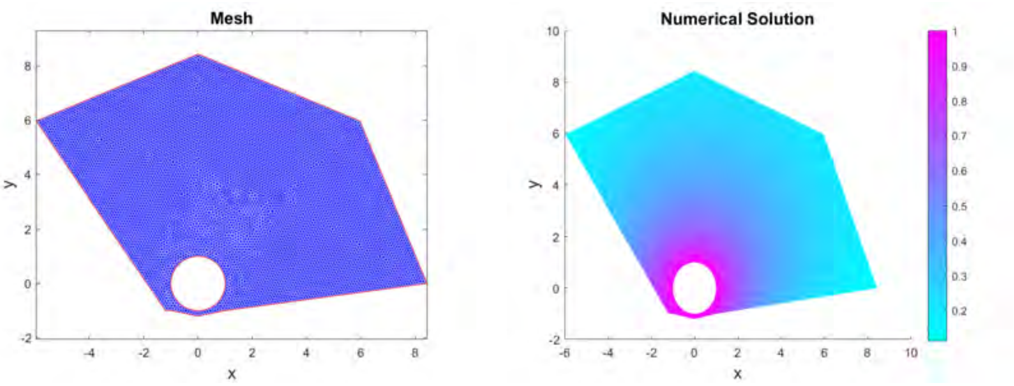
3 NUMERICAL EXPERIMENTS

We discretized the weak formulation of (PDE) in (1), that is

$$\int_D \nabla u \cdot \nabla \phi + \int_{\partial D} \beta u \phi = 0, \tag{9}$$

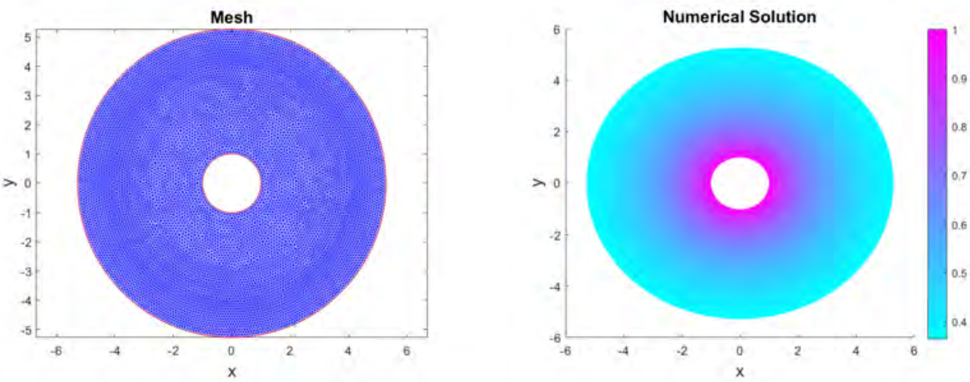
using Finite Element method implemented via Matlab software, in order to calculate the function u. We analyzed the results of the numerical simulations by distinguishing them in the cases summarized below (see [5] for more details):

- We considered circles and polygons with the same area as convex geometries for D. The computational experiments show that the circle seems to be in general the best choice for the external domain, even if, in a few cases, irregular polygons produce a smaller dispersion $F_\beta(D, \Omega)$ for values of $\beta < 1$. An example is depicted in Fig. 3 related to an irregular octagon, compared to the circle visible in Fig. 4 with the same area. However, such an example may be misleading: the example in Figs. 3-4 refers to a case in which the prescribed quantity of insulator represents a technically inadvisable option if we want to have little heat dispersion, since in this case using no insulator at all would be a much better choice (looking at Fig. 1 for $\beta = 0.2$, $R \simeq 5$ corresponds to the maximum possible heat dispersion for the case of concentric circles).



(a) (b)

Figure 3: Plots of (a) the mesh and (b) the numerical solution in the case $\beta = 0.2$, area $A = 87.49$. Dispersion $F_{0.2}(D, \Omega) = 2.32$.



(a) (b)

Figure 4: Plots of (a) the mesh and (b) the numerical solution in the case $\beta = 0.2$, area $A = 87.49$. Dispersion $F_{0.2}(D, \Omega) = 2.41$.

- Motivated by the previous considerations, that seem to advise against a fixed amount of insulator, we considered the problem of minimizing the heat dispersion functional (2) under the constraints $\text{Area}(D) \leq Amax$, $D \in \mathcal{D}$, considering different values of $Amax$. For the numerical minimization, we made use of the `patternsearch` MATLAB routine for global minimization, varying the starting point, and imposing geometrical non linear constraints on the outer polygons. We observed different behaviors of the heat dispersion for $\beta < 1$ and $\beta > 1$. In particular, for $\beta > 1$ all the amount of insulator material is used (saturation case), whereas for $\beta < 1$ the minimization process may lead to different solutions, depending on the starting point (see [5] for more details).

- Simulations by considering internal domains with shape different from the unit circle have been carried out confirming that the external circle is the domain which gets the lowest values of heat dispersion among all the convex domains considered (see [5]).

Further details and results will be shown during the talk and can also be found in [5].

4 CONCLUSIONS

In this work we have performed a systematic numerical analysis of a mathematical model for thermal insulation problems described by a shape optimization formulation. The results obtained seem to suggest that the most effective thermal insulation for a conductor of constant temperature is obtained by surrounding it with insulating material disposed according to a circular geometry, independently from the shape of the internal body. Counterexamples to that, obtained comparing different geometries which share the same area, do not seem to be of practical interest. Nevertheless, a similar computational approach, possibly with a more complex model, appears extremely useful for the understanding of the physical problem of thermal insulation. In that line, in the future we would like to further explore the thermal insulation problem in order to face in a more accurate way the real life needs coming from engineering applications.

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