On set star-Lindelöf spaces

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Abstract

A space $X$ is said to be set star-Lindelöf if for each nonempty subset $A$ of $X$ and each collection $U$ of open sets in $X$ such that $A \subseteq \bigcup U$, there is a countable subset $V$ of $U$ such that $A \subseteq \text{St}(\bigcup V, U)$. The class of set star-Lindelöf spaces lie between the class of Lindelöf spaces and the class of star-Lindelöf spaces. In this paper, we investigate the relationship between set star-Lindelöf spaces and other related spaces by providing some suitable examples and study the topological properties of set star-Lindelöf spaces.

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1. Introduction and Preliminaries

Arhangel’skii [1] defined a cardinal number $sL(X)$ of $X$: the minimal infinite cardinality $\tau$ such that for every subset $A \subseteq X$ and every open cover $U$ of $X$, there is a subfamily $V \subseteq U$ such that $|V| \leq \tau$ and $A \subseteq \bigcup V$. If $sL(X) = \omega$, then the space $X$ is called $sL$indelöf space. Following this idea, Kočinac and Konca [7] introduced and studied the new types of selective covering properties called set-covering properties (for a similar studies, see [4, 14, 15, 16, 17]). A space $X$ is said to have the set-Menger [7] property if for each nonempty subset $A$ of $X$ and each sequence $(U_n : n \in \mathbb{N})$ of collections of open sets in $X$ such that for each $n \in \mathbb{N}$, $A \subseteq \bigcup U_n$, there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n$ is a finite subset of $U_n$ and $A \subseteq \bigcup_{n \in \mathbb{N}} \bigcup V_n$. The author [13] noticed

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that the set-Menger property is nothing but another view of Menger covering property. Recently, the author [12] defined and studied set starcompact and set strongly starcompact spaces (also see [8]).

In this paper, we consider the classes of set star-Lindelöf spaces and set strongly star-Lindelöf spaces already introduced in [9] and recently studied in [4]. Note that in fact in the class of $T_1$ spaces, set strongly star-Lindelöfness is equivalent to the property having countable extent [[4], Proposition 3.1].

If $A$ is a subset of a space $X$ and $\mathcal{U}$ is a collection of subsets of $X$, then $St(A, \mathcal{U})=\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We usually write $St(x, \mathcal{U}) = St(\{x\}, \mathcal{U})$.

Throughout the paper, by “a space” we mean “a topological space”, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Q}$ denotes the set of natural numbers, set of real numbers, and set of rational numbers, respectively, the cardinality of a set is denoted by $|A|$. Let $\omega$ denote the first infinite cardinal, $\omega_1$ the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. An open cover $\mathcal{U}$ of a subset $A \subseteq X$ means elements of $\mathcal{U}$ open in $X$ such that $A \subseteq \bigcup \mathcal{U} = \bigcup\{U : U \in \mathcal{U}\}$.

We first recall the classical notions of spaces that are used in this paper.

**Definition 1.1** ([5]). A space $X$ is said to be

1. starcompact if for each open cover $\mathcal{U}$ of $X$, there is a finite subset $V$ of $\mathcal{U}$ such that $X = St(\bigcup V, \mathcal{U})$.
2. strongly starcompact if for each open cover $\mathcal{U}$ of $X$, there is a finite subset $F$ of $X$ such that $X = St(F, \mathcal{U})$.

**Definition 1.2** ([12, 8]). A space $X$ is said to be

1. set starcompact if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $A \subseteq \bigcup \mathcal{U}$, there is a finite subset $V$ of $\mathcal{U}$ such that $A \subseteq St(\bigcup V, \mathcal{U})$.
2. set strongly starcompact if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $A \subseteq \bigcup \mathcal{U}$, there is a finite subset $F$ of $A$ such that $A \subseteq St(F, \mathcal{U})$.

**Definition 1.3.** A space $X$ is said to be

1. star-Lindelöf [5] if for each open cover $\mathcal{U}$ of $X$, there is a countable subset $V$ of $\mathcal{U}$ such that $X = St(\bigcup V, \mathcal{U})$.
2. strongly star-Lindelöf [5] if for each open cover $\mathcal{U}$ of $X$, there is a countable subset $F$ of $X$ such that $X = St(F, \mathcal{U})$.

Note that the star-Lindelöf spaces have a different name such as 1-star-Lindelöf and $1\frac{1}{2}$-star-Lindelöf in different papers (see [5, 10]) and the strongly star-Lindelöf space is also called star countable in [10, 21]. It is clear that, every strongly star-Lindelöf space is star-Lindelöf.

Recall that a collection $A \subseteq P(\omega)$ is said to be almost disjoint if each set $A \in A$ is infinite and the sets $A \cap B$ are finite for all distinct elements $A, B \in A$. For an almost disjoint family $A$, put $\psi(A) = A \cup \omega$ and topologize $\psi(A)$ as follows: for each element $A \in A$ and each finite set $F \subseteq \omega$, $\{A\} \cup (A \setminus F)$ is a basic open neighborhood of $A$ and the natural numbers are isolated. The
spaces of this type are called Isbell-Mrówka $\psi$-spaces \cite{2, 11} or $\psi(A)$ space. For other terms and symbols, we follow \cite{6}.

The following result was proved in \cite{8}.

**Theorem 1.4** (\cite{8}). *Every countably compact space is set strongly starcompact.*

Note that in the class of Hausdorff spaces strongly starcompactness, set strongly starcompactness and countable compactness are equivalent \cite[Proposition 2.2]{4}.

## 2. SET STAR-LINDELÖF AND RELATED SPACES

In this section, we give some examples showing the relationship among set star-Lindelöf spaces, set strongly star-Lindelöf spaces, and other related spaces. First we define our main definition.

**Definition 2.1.** A space $X$ is said to be

1. set star-Lindelöf if for each nonempty subset $A$ of $X$ and each collection $U$ of open sets in $X$ such that $A \subseteq \bigcup U$, there is a countable subset $V$ of $U$ such that $A \subseteq \text{St}(\bigcup V, U)$.

2. set strongly star-Lindelöf if for each nonempty subset $A$ of $X$ and each collection $U$ of open sets in $X$ such that $A \subseteq \bigcup U$, there is a countable subset $F$ of $A$ such that $A \subseteq \text{St}(F, U)$.

Note that in the class of $T_1$ spaces the set strongly star-Lindelöfness is equivalent to the property to have a countable extent \cite[Proposition 3.1]{4}. Note that there is a misprint in the statement of the definition of relatively* set star strongly-compact in \cite{4}: the authors write that set $F$ is a finite subset of $A$ but the original definition asks that $F$ is contained in $A$ and Bonanzinga and Maesano use exactly this last fact during all the paper.

We have the following diagram from the definitions and \cite[Proposition 3.1]{4}.

However, the following examples show that the converse of these implications are not true.

\[
\begin{array}{ccc}
\text{set strongly starcompact} & \rightarrow & \text{set starcompact} \\
\downarrow & & \downarrow \\
\text{Lindelof} & \rightarrow & \text{countable extent} \leftrightarrow^T \text{set strongly star} \rightarrow \text{Lindelof} & \rightarrow & \text{set star} \rightarrow \text{Lindelof} \\
\downarrow & & \downarrow \\
\text{strongly star} \rightarrow \text{Lindelof} & \rightarrow & \text{star} \rightarrow \text{Lindelof}
\end{array}
\]

**Example 2.2.** (i) The discrete space $\omega$ has countable extent but it is not set starcompact space.

(ii) The space $[0, \omega_1)$ has countable extent but it is not Lindelöf.

(iii) Let $Y$ be a discrete space with cardinality $\mathfrak{c}$. Let $X = Y \cup \{y^*\}$, where $y^* \notin Y$ topologized as follows: each $y \in Y$ is an isolated point and a set $U$
containing \( y^* \) is open if and only if \( X \setminus U \) is countable. Then \( X \) has countable extent but it is not countably compact.

Bonanzinga [3] proved that every Isbell-Mrówka space is a Tychonoff strongly star-Lindelöf space with uncountable extent (hence, it is not set strongly star-Lindelöf). Note that in [3] strongly star-Lindelöf is called star-Lindelöf.

The following lemma was proved by Song [18].

**Lemma 2.3** ([18, Lemma 2.2]). A space \( X \) having a dense Lindelöf subspace is star-Lindelöf.

The following example shows that the Lemma 2.3 does not hold if we replace star-Lindelöf space by a set star-Lindelöf space.

**Example 2.4.** There exists a Tychonoff space \( X \) having a dense Lindelöf subspace such that \( X \) is not set star-Lindelöf.

**Proof.** Let \( D(\epsilon) = \{d_\alpha : \alpha < \epsilon\} \) be a discrete space of cardinality \( \epsilon \) and let \( Y = D(\epsilon) \cup \{d^*\} \) be one-point compactification of \( D(\epsilon) \). Let

\[
X = (Y \times [0,\omega)) \cup (D(\epsilon) \times \{\omega\})
\]

be the subspace of the product space \( Y \times [0,\omega] \). Then \( Y \times [0,\omega] \) is a dense Lindelöf subspace of \( X \) and by Lemma 2.3, \( X \) is star-Lindelöf.

In [4, Proposition 3.4] shows that if \( X \) is a space such that there exists a closed and discrete subspace \( D \) of \( X \) having uncountable cardinality and a disjoint family \( U = \{O_a : a \in D\} \) of open neighborhoods of points \( a \in D \), then \( X \) is not set star-Lindelöf. So, we conclude that \( X \) is not set star-Lindelöf. \( \square \)

Bonanzinga and Maesano [4, Example 3.5] constructed an example of a Tychonoff separable (hence set star-Lindelöf) non set strongly star-Lindelöf space.

**Remark 2.5.** (1) In [12], Singh gave an example of a Tychonoff set starcompact space \( X \) that is not set strongly starcompact.

(2) It is known that there are star-Lindelöf spaces that are not strongly star-Lindelöf (see [5, Example 3.2.3.2] and [5, Example 3.3.1]).

Now we give some conditions under which star-Lindelöfness coincides with set star-Lindelöfness and strongly star-Lindelöfness coincide with set strongly star-Lindelöfness.

Recall that a space \( X \) is paraLindelöf if every open cover \( \mathcal{U} \) of \( X \) has a locally countable open refinement.

Song and Xuan [19] proved the following result.

**Theorem 2.6** ([19, Theorem 2.24]). Every regular paraLindelöf star-Lindelöf spaces are Lindelöf.

We have the following theorem from Theorem 2.6 and the diagram.
**Theorem 2.7.** If $X$ is a regular paraLindelöf space, then the following statements are equivalent:

1. $X$ is Lindelöf;
2. $X$ is set strongly star-Lindelöf;
3. $e(X) = \omega$;
4. $X$ is set star-Lindelöf;
5. $X$ is strongly star-Lindelöf;
6. $X$ is star-Lindelöf.

A space is said to be metaLindelöf if every open cover of it has a point-countable open refinement.

Bonanzinga [3] proved the following result.

**Theorem 2.8** ([3]). Every strongly star-Lindelöf metaLindelöf spaces are Lindelöf.

We have the following theorem from Theorem 2.8 and the diagram.

**Theorem 2.9.** If $X$ is a metaLindelöf space, then the following statements are equivalent:

1. $X$ is Lindelöf;
2. $X$ is set strongly star-Lindelöf;
3. $e(X) = \omega$;
4. $X$ is strongly star-Lindelöf.

3. **Properties of set star-Lindelöf spaces**

In this section, we study the topological properties of set star-Lindelöf spaces.

**Theorem 3.1.** If $X$ is a set star-Lindelöf space, then every open and closed subset of $X$ is set star-Lindelöf.

**Proof.** Let $X$ be a set star-Lindelöf space and $A \subseteq X$ be an open and closed set. Let $B$ be any subset of $A$ and $U$ be a collection of open sets in $(A, \tau_A)$ such that $\text{Cl}_A(B) \subseteq \bigcup U$. Since $A$ is open, then $U$ is a collection of open sets in $X$. Since $A$ is closed, $\text{Cl}_A(B) = \text{Cl}_X(B)$. Applying the set star-Lindelöfness property of $X$, there exists a countable subset $V$ of $U$ such that $B \subseteq \text{St}(\bigcup V, U)$. Hence $A$ is a set star-Lindelöf. \qed

Consider the Alexandorff duplicate $A(X) = X \times \{0, 1\}$ of a space $X$. The basic neighborhood of a point $\langle x, 0 \rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup (U \times \{1\} \setminus \{(x, 1)\})$, where $U$ is a neighborhood of $x$ in $X$ and each point $\langle x, 1 \rangle \in X \times \{1\}$ is an isolated point.

**Theorem 3.2.** If $X$ is a $T_1$-space and $A(X)$ is a set star-Lindelöf space. Then $e(X) < \omega_1$.

**Proof.** Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset $B$ of $X$ such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of...
A(X) and every point of $B \times \{1\}$ is an isolated point. Thus $A(X)$ is not set star-Lindelöf by Theorem 3.1. □

**Theorem 3.3.** Let $X$ be a space such that the Alexandroff duplicate $A(X)$ of $X$ is set star-Lindelöf. Then $X$ is a set star-Lindelöf space.

**Proof.** Let $B$ be any nonempty subset of $X$ and $U$ be an open cover of $\overline{B}$. Let $C = B \times \{0\}$ and

$$A(U) = \{U \times \{0, 1\} : U \in U\}.$$  

Then $A(U)$ is an open cover of $\overline{B}$. Since $A(X)$ is set star-Lindelöf, there is a countable subset $A(V)$ of $A(U)$ such that $C \subseteq \text{St}(\bigcup A(V), A(U))$. Let

$$V = \{U \in U : U \times \{0, 1\} \in A(V)\}.$$  

Then $V$ is a countable subset of $U$. Now we have to show that

$$B \subseteq \text{St}(\bigcup V, U).$$  

Let $x \in B$. Then $(x, 0) \in \text{St}(\bigcup A(V), A(U))$. Choose $U \times \{0, 1\} \in A(U)$ such that $(x, 0) \in U \times \{0, 1\}$ and $U \times \{0, 1\} \cap (\bigcup A(V)) \neq \emptyset$, which implies $U \cap (\bigcup V) \neq \emptyset$ and $x \in U$. Therefore $x \in \text{St}(\bigcup V, U)$, which shows that $X$ is set star-Lindelöf space. □

On the images of set star-Lindelöf spaces, we have the following result.

**Theorem 3.4.** A continuous image of set star-Lindelöf space is set star-Lindelöf.

**Proof.** Let $X$ be a set star-Lindelöf space and $f : X \to Y$ is a continuous mapping from $X$ onto $Y$. Let $B$ be any subset of $Y$ and $V$ be an open cover of $\overline{B}$. Let $A = f^{-1}(B)$. Since $f$ is continuous, $U = \{f^{-1}(V) : V \in V\}$ is the collection of open sets in $X$ with $A = f^{-1}(B) \subseteq f^{-1}(B) \subseteq f^{-1}(\bigcup V) = \bigcup U$. As $X$ is set star-Lindelöf, there exists a countable subset $U'$ of $U$ such that

$$A \subseteq \text{St}(\bigcup U', U).$$  

Let $V' = \{V : f^{-1}(V) \in U'\}$. Then $V'$ is a countable subset of $V$ and $B = f(A) \subseteq f(\text{St}(\bigcup U', U)) \subseteq \text{St}(\bigcup f(\{f^{-1}(V) : V \in V'\}), V) = \text{St}(\bigcup V', V)$. Thus $Y$ is set star-Lindelöf space. □

Next, we turn to consider preimages of set strongly star-Lindelöf and set star-Lindelöf spaces. We need a new concept called nearly set star-Lindelöf spaces. A space $X$ is said to be nearly set star-Lindelöf in $X$ if for each subset $Y$ of $X$ and each open cover $U$ of $X$, there is a countable subset $V$ of $U$ such that $Y \subseteq \text{St}(\bigcup V, U)$. For the strong version of this property (see [4]).

**Theorem 3.5.** If $f : X \to Y$ is an open and perfect continuous mapping and $Y$ is a set star-Lindelöf space, then $X$ is nearly set star-Lindelöf.
Proof. Let $A \subseteq X$ be any nonempty set and $\mathcal{U}$ be an open cover of $X$. Then $B = f(A)$ is a subset of $Y$. Let $y \in \overline{B}$. Then $f^{-1}\{y\}$ is a compact subset of $X$, so there is a finite subset $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}\{y\} \subseteq \bigcup \mathcal{U}_y$. Let $U_y = \bigcup \mathcal{U}_y$. Then $V_y = Y \setminus f(X \setminus U_y)$ is a neighborhood of $y$, since $f$ is closed. Then $\mathcal{V} = \{V_y : y \in \overline{B}\}$ is an open cover of $\overline{B}$. Since $Y$ is set star-Lindelöf, there exists a countable subset $\mathcal{V}'$ of $\mathcal{V}$ such that $B \subseteq \text{St}(\bigcup \mathcal{V}', \mathcal{U})$.

Without loss of generality, we may assume that $\mathcal{V}' = \{V_{y_i} : i \in N' \subseteq N\}$. Let $\mathcal{W} = \bigcup_{i \in N'} \mathcal{U}_{y_i}$. Since $f^{-1}(V_{y_i}) \subseteq \bigcup\{U : U \in \mathcal{U}_{y_i}\}$ for each $i \in N'$. Then $\mathcal{W}$ is a countable subset of $\mathcal{U}$ and

$$f^{-1}(\bigcup \mathcal{V}') = \bigcup \mathcal{W}.$$ 

Next, we show that $A \subseteq \text{St}(\bigcup \mathcal{W}, \mathcal{U})$.

Let $x \in A$. Then there exists a $y \in B$ such that $f(x) \in V_y$ and $V_y \cap (\bigcup \mathcal{V}') \neq \emptyset$.

Since $x \in f^{-1}(V_y) \subseteq \bigcup\{U : U \in \mathcal{U}_y\}$, we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f(U)$. Thus $U \cap f^{-1}(\bigcup \mathcal{V}') \neq \emptyset$. Hence $x \in \text{St}(f^{-1}(\bigcup \mathcal{V}'), \mathcal{U})$. Therefore $x \in \text{St}(\bigcup \mathcal{W}, \mathcal{U})$, which shows that $A \subseteq \text{St}(\bigcup \mathcal{W}, \mathcal{U})$. Thus $X$ is nearly set star-Lindelöf. □

It is known that the product of star-Lindelöf space and compact space is a star-Lindelöf (see [5]).

**Problem 3.6.** Does the product of set star-Lindelöf space and a compact space is set star-Lindelöf?

The following example shows that the product of two countably compact (hence, set star-Lindelöf) spaces need not be set star-Lindelöf.

**Example 3.7.** There exist two countably compact spaces $X$ and $Y$ such that $X \times Y$ is not set star-Lindelöf.

**Proof.** Let $D(\omega)$ be a discrete space of the cardinality $\omega$. We can define $X = \bigcup_{\alpha < \omega_1} E_{\alpha}$ and $Y = \bigcup_{\alpha < \omega_1} F_{\alpha}$, where $E_{\alpha}$ and $F_{\alpha}$ are the subsets of $\beta(D(\omega))$ which are defined inductively to satisfy the following three conditions:

1. $E_{\alpha} \cap F_{\beta} = D(\omega)$ if $\alpha \neq \beta$;
2. $|E_{\alpha}| \leq \omega$ and $|F_{\alpha}| \leq \omega$;
3. every infinite subset of $E_{\alpha}$ (resp., $F_{\alpha}$) has an accumulation point in $E_{\alpha+1}$ (resp, $F_{\alpha+1}$).
Those sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta(D(c))$ has the cardinality $2^c$ (see [20]). Then $X \times Y$ is not set star-Lindelöf, since the diagonal $\{\langle d, d \rangle : d \in D(c)\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $c$.

van Douwen-Reed-Roscoe-Tree [5, Example 3.3.3] gave an example of a countably compact $X$ (hence, set star-Lindelöf) and a Lindelöf space $Y$ such that $X \times Y$ is not strongly star-Lindelöf. Now we use this example to show that $X \times Y$ is not set star-Lindelöf.

**Example 3.8.** There exists a countably compact space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not set star-Lindelöf.

**Proof.** Let $X = [0, \omega_1)$ with the usual order topology. Let $Y = [0, \omega_1]$ with the following topology. Each point $\alpha < \omega_1$ is isolated and a set $U$ containing $\omega_1$ is open if and only if $Y \setminus U$ is countable. Then, $X$ is countably compact and $Y$ is Lindelöf. It is enough to show that $X \times Y$ is not star-Lindelöf.

For each $\alpha < \omega_1$, $U_\alpha = X \times \{\alpha\}$ is open in $X \times Y$. For each $\beta < \omega_1$, $V_\beta = [0, \beta] \times (0, \omega_1]$ is open in $X \times Y$. Let $U = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\beta : \beta < \omega_1\}$. Then $U$ is an open cover of $X \times Y$. Let $V$ be any countable subset of $U$. Since $V$ is countable, there exists $\alpha' < \omega_1$ such that $U_\alpha \notin V$ for each $\alpha > \alpha'$. Also, there exists $\alpha'' < \omega_1$ such that $V_\beta \notin V$ for each $\beta > \alpha''$. Let $\beta = \sup\{\alpha', \alpha''\}$. Then $U_\beta \cap \left(\bigcup \mathcal{V}\right) = \emptyset$ and $U_\beta$ is the only element containing $\langle \beta, \beta \rangle$. Thus $\langle \beta, \beta \rangle \notin St(\bigcup \mathcal{V}, U)$, which shows that $X$ is not star-Lindelöf.

van Douwen-Reed-Roscoe-Tree [5, Example 3.3.6] gave an example of Hausdorff regular Lindelöf spaces $X$ and $Y$ such that $X \times Y$ is star-Lindelöf. Now we use this example and show that the product of two Lindelöf spaces is not set star-Lindelöf.

**Example 3.9.** There exists a Hausdorff regular Lindelöf spaces $X$ and $Y$ such that $X \times Y$ is not set star-Lindelöf.

**Proof.** Let $X = \mathbb{R} \setminus \mathbb{Q}$ have the induced metric topology. Let $Y = \mathbb{R}$ with each point of $\mathbb{R} \setminus \mathbb{Q}$ have the induced metric topology. Hence both spaces $X$ and $Y$ are Hausdorff regular Lindelöf spaces and first countable, so $X \times Y$ is Hausdorff regular and first countable. Now we show that $X \times Y$ is not set star-Lindelöf. Let $A = \{(x, x) \in X \times Y : x \in X\}$. Then $A$ is an uncountable closed and discrete set (see [5, Example 3.3.6]). For $(x, x) \in A$, $U_x = X \times \{x\}$ is the open subset of $X \times Y$. Then $U = \{U_x : (x, x) \in A\}$ is an open cover of $A$. Let $V$ be any countable subset of $U$. Then there exists $(a, a) \in A$ such that $(a, a) \notin \bigcup \mathcal{V}$ and thus $(\bigcup \mathcal{V}) \cap U_a = \emptyset$. But $U_a$ is the only element of $U$ containing $(a, a)$. Thus $(a, a) \notin St(\bigcup \mathcal{V}, U)$, which completes the proof.
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