APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 23, no. 2 (2022), 325-331
doi:10.4995/agt.2022.17006

Results about S_2-paracompactness

Ohud Alghamdi [a] and Lutfi Kalantan [b]

[a] Department of Mathematics, Faculty of Science and Arts in Almandaq, Al-Baha Univesity, P.O.Box 1988, Al-Baha 65581, Saudi Arabia (ofalghamdi@bu.edu.sa)
[b] Department of Mathematics, King Abdulaziz University, Saudi Arabia (lkalantan@kau.edu.sa and lnkalantan@hotmail.com)

Communicated by M. A. Sánchez-Granero

Abstract

We present new results regarding S_2-paracompactness, that we established in [1], and its relation with other properties such as S-normality, epinormality and L-paracompactness.

2020 MSC: *54C10; 54D20.*

Keywords: *separable; paracompact; S-paracompact; S_2-paracompact; L-paracompact; L_2-paracompact; S-normal; L-normal.*

1. Introduction

In this paper, we present some new results about S-paracompactness and S_2-paracompactness. First, we introduce the significant notations. An order pair will be denoted by $\langle x, y\rangle$. The sets of positive integers, rational numbers, irrational numbers and real numbers will be denoted by $\mathbb{N}, \mathbb{Q}, \mathbb{P}$ and $\mathbb{R}$, respectively. The closure and the interior of the subset A of a topological space X will be denoted respectively by $\overline{A}$ and $\text{int}(A)$. Throughout this paper, a T_1 normal space is called T_4 and a T_1 completely regular space is called Tychonoff space ($T_{3\frac{1}{2}}$). In the definitions of compactness, countable compactness, paracompactness, and local compactness we do not assume T_2. Moreover, in the definition of Lindelöfness we do not assume regularity. Also, the ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. We denote the first infinite ordinal by ω and the first uncountable ordinal by ω_1.

Received 13 January 2022 – Accepted 30 August 2022

Definition 1.1. A topological space X is called *S-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that for every separable subspace $A \subseteq X$ we have that $f|_A : A \longrightarrow f(A)$ is a homeomorphism. Moreover, If Y is T_2 paracompact, then X is S_2-paracompact [1].

2. S_2-paracompactness and other Topological Properties

2.1. S_2-paracompactness and L_2-paracompactness.

Recall from [5] that a topological space X is called *L-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that $f|_B : B \longrightarrow f(B)$ is a homeomorphism for all Lindelöf subspace B of X. In addition, if Y is T_2 paracompact, then X is L_2-paracompact.

Recall from [6] that a topological space X is called P-space if it is T_1 and every G_δ set is open. The countable complement topology defined on $\mathbb{R}$, $(\mathbb{R}, \mathcal{CC})$ (see [9, Example 20]), is an example of a space that is S_2-paracompact but not L_2-paracompact. It is S_2-paracompact because it is P-space, (see [1]), but not L_2-paracompact because it is Lindelöf and not paracompact space. In fact, it is not even L-paracompact.
We still do not have an answer for the following question:
Does there exist an L-paracompact space which is not S-paracompact?

Theorem 2.1. *If X is L-paracompact (resp. L_2-paracompact) such that for any separable subspace $A \subseteq X$ there exists a Lindelöf subspace $B \subseteq X$ such that $A \subseteq B$, then X is S-paracompact (resp. S_2-paracompact).*

Proof. Let X be L-paracompact such that for any separable subspace $A \subseteq X$ there exists a Lindelöf subspace $B \subseteq X$ such that $A \subseteq B$. Then, there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that $f|_B$: $B \longrightarrow f(B)$ is a homeomorphism for every Lindelöf subspace $B \subseteq X$. Let A be any separable subspace of X. Then, there exists a Lindelöf subspace B of X such that $A \subseteq B$. Then, $f|_A : A \longrightarrow f(A)$ is a homeomorphism. □

A similar proof as in Theorem 2.1 yields the following corollaries.

Corollary 2.2. *If X is S-paracompact (resp. S_2-paracompact) such that for any Lindelöf subspace $B \subseteq X$, there exists a separable subspace A with $B \subseteq A$. Then, X is L-paracompact (resp. L_2-paracompact).*

Recall from [7] that a space X is called C-paracompact if there exist a paracompact space Y and a bijection $f : X \longrightarrow Y$ such that $f|_K : K \longrightarrow f(K)$ is a homeomorphism for every compact subspace $K \subseteq X$. Moreover, If Y is T_2 paracompact, we say that X is C_2-paracompact.

Corollary 2.3. *If X is S-paracompact (resp. S_2-paracompact) such that for any compact subspace $B \subseteq X$, there exists a separable subspace A of X with $A \subseteq B$. Then, X is C-paracompact (resp. C_2-paracompact).*

Application of Corollary 2.3:
Take $(\mathbb{R}, \mathcal{CC})$, the countable complement topology on $\mathbb{R}$. Since $A \subset \mathbb{R}$ is compact if and only if A is finite, we can say that every compact subspace is contained in a separable subspace of $\mathbb{R}$. Thus, $(\mathbb{R}, \mathcal{CC})$ is C_2-paracompact.

Recall from [3, 4.4.F] that a space X is *locally separable* if each element of X has a separable open neighborhood.

Theorem 2.4. *Every S-paracompact (resp. S_2-paracompact) hereditarily locally separable is L-paracompact (resp. L_2-paracompact).*

Proof. Let X be S-paracompact (resp. S_2-paracompact) and hereditarily locally separable and let B be any Lindelöf subspace of X. Then, B is a locally separable Lindelöf subspace of X. Pick U_x to be a separable open neighborhood of each $x \in B$. Then, $\{U_x\}_{x\in B}$ is an open cover of B. Let $\mathcal{U}$ be a countable open subcover of $\{U_x\}_{x\in B}$ and let D_x be a countable dense subset of each $U_x \in \mathcal{U}$. Then, $D = \bigcup D_x$ is a countable dense subset of B, implying that B is separable. Therefore, since every Lindelöf subspace of X is separable and X is S-paracompact (resp. S_2-paracompact), then X is L-paracompact (resp. L_2-paracompact). □

Problem 2.5. *Does there exist a topological space that is L-paracompact (resp. L_2-paracompact) but not locally separable or not S-paracompact (resp. S_2-paracompact)?*

Note that local separability is essential in Theorem 2.4. For example, $(\mathbb{R}, \mathcal{CC})$ is S-paracompact not locally separable. Observe that $(\mathbb{R}, \mathcal{CC})$ is not L-paracompact.

Theorem 2.6. *Let X be a topological space such that the only separable or Lindelöf subspaces are the countable ones. Then, X is S-paracompact (resp. S_2-paracompact) if and only if X is L-paracompact (resp. L_2-paracompact).*

Proof. Let X be any topological space such that the only separable or Lindelöf subspaces are the countable ones. Suppose that X is S-paracompact (resp. S_2-paracompact). If B is any Lindelöf subspace of X, then B is countable, implying that B is separable. Hence, X is L-paracompact (resp. L_2-paracompact). Conversely, suppose that X is L-paracompact (resp. L_2-paracompact) and A is any separable subspace of X. Then, A is countable, implying that A is Lindelöf. Hence, X is S-paracompact (resp. S_2-paracompact). □

Application of Theorem 2.6:
Consider ω_1 with its usual ordered topology. Let A be any uncountable subset of ω_1. Then A is not bounded. Hence, $\{[0, \alpha] : \alpha < \omega_1\}$ is an open cover of A that has no countable subcover, which implies that A is not Lindelöf. Since ω_1 satisfies the condition in Theorem 2.6, then ω_1 is L_2-paracompact because it is S_2-paracompact, (see [1]).

A family $\{A_s\}_{s\in S}$ of subsets of a space X is called point-finite if for each $x \in X$, the set $\{s \in S : x \in A_s\}$ is finite, (see [3]).
Recall from [9] that a space X is *metacompact* if every open cover of X has a point-finite open refinement.

Theorem 2.7. *Any hereditarly metacompact L-paracompact (resp. L_2-paracompact) is S-paracompact (resp. S_2-paracompact).*

Proof. Let X be L-paracompact (resp. L_2-paracompact) hereditarly metacompact and let A be any separable subspace of X. Then, A is a separable metacompact subspace of X. Suppose that A is not Lindelöf. Then, there exists an open cover of A, say $\mathcal{W} = \{W_\alpha : \alpha \in \Lambda\}$, which has no countable subcover. Let $\mathcal{U}$ be a point-finite open refinement of $\mathcal{W}$. Then, $\mathcal{U}$ is uncountable by our assumption. Let D be the countable dense subset of A. Hence, $D \cap U \neq \varnothing$ for all $U \in \mathcal{U}$ implying that there exists $d \in D$ contained in uncountable members of $\mathcal{U}$ which contradicts the fact that $\mathcal{U}$ is a point-finite family. Hence, A is Lindelöf, implying that X is S-paracompact (resp. S_2-paracompact). □

Problem 2.8. *Does there exist a topological space which is S-paracompact (resp. S_2-paracompact) but not hereditarly metacompact or L-paracompact (resp. L_2-paracompact)?*

2.2. **S_2-paracompactness and Epinormality.**

Definition 2.9. A topological space (X,τ) is *epinormal* if there exists a coarser topology, say $\mathcal{V}$, such that $(X, \mathcal{V})$ is T_4, (see [2]).

Since every epinormal space is Hausdorff as it is proved in [2], then the countable complement topology on $\mathbb{R}$, $(\mathbb{R}, \mathcal{CC})$, is an example of S_2-paracompact that is not epinormal. On the other hand, the following example shows that there exists an epinormal space which is not S_2-paracompact.

Example 2.10. Let $A = \{\langle x, 0\rangle : 0 < x \leq 1\}$ and $B = \{\langle x, 1\rangle : 0 \leq x < 1\}$.

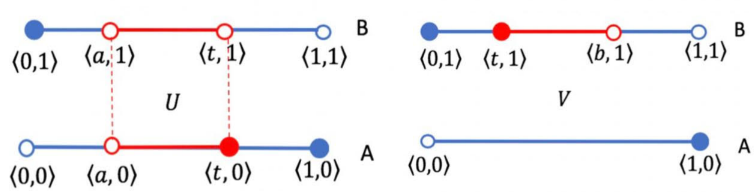

Figure 1. This figure illustrates the neighborhood system of Strong Parallel Line Topology (X,σ).

The *strong parallel line* topology σ on $X = A \cup B$ is the unique topology generated by the following neighborhood system:
For each $\langle t, 0\rangle \in A$, let $\mathfrak{B}(\langle t, 0\rangle) = \{U : U = \{\langle x, 0\rangle : 0 \leq a < x \leq t\} \cup \{\langle x, 1\rangle : a < x < t\}\}$, and for each $\langle t, 1\rangle \in B$, let $\mathfrak{B}(\langle t, 1\rangle) = \{V : V = \{\langle x, 1\rangle : t \leq x < b \leq 1\}\}$, (see [9, Example 96]).
Since (X,σ) is separable and not paracompact space because it is a Hausdorff and not regular topological space, then (X,σ) cannot be S_2-paracompact.
Define τ on X to be the unique topology that is generated by the following neighborhood system:
Every element in A has the same local base as σ and for each element $\langle t, 1\rangle \in B$, let $\mathfrak{B}(\langle t, 1\rangle) = \{V : V = \{\langle x, 0\rangle : t < x < b \leq 1\} \cup \{\langle x, 1\rangle : t \leq x < b\}\}$. The topology (X,τ) is named *weak parallel line,* (see [9, Example 96]).

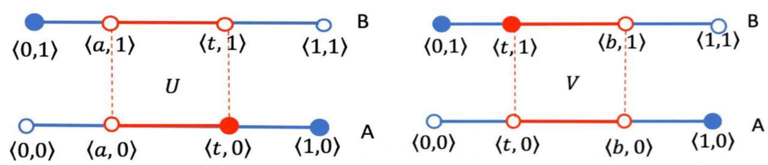

Figure 2. This figure illustrates the neighborhood system of Weak Parallel Line Topology (X,τ).

Define a relation $\preceq$ on X as follows:
For $\langle x, y\rangle$ and $\langle k, l\rangle \in X$, we write $\langle x, y\rangle \preceq \langle k, l\rangle$ if and only if either $x < k$ or $x = k$ and $y = 0 < l = 1$, or $x = k$ and $y = l$. Then, (X,τ) is a linearly ordered topological space (LOTS). Since any LOTS is T_4, we have (X,τ) is T_4 and τ is coarser than σ, hence, we get that (X,σ) is epinormal.

Theorem 2.11. *Any S_2-paracompact Fréchet space is epinormal.*

Proof. Let (X,τ) be S_2-paracompact Fréchet space. Without loss of generality, assume that (X,τ) is not normal. Let (Y, τ') be a T_2 paracompact space and let $f : X \longrightarrow Y$ be a bijective function such that $f\,|_A : A \longrightarrow f(A)$ is a homeomorphism for every separable subspace $A \subseteq X$. Then, f is continuous since X is Fréchet. Consider $\mathcal{V} = \{f^{-1}(U) : U \in \tau'\}$. Then, $\mathcal{V}$ is a topology on X and since any open set in $\mathcal{V}$ is open in τ by continuity of f, we get that $\mathcal{V}$ is coarser than τ. Observe that $f : (X, \mathcal{V}) \longrightarrow (Y, \tau')$ is a homeomorphism. Therefore, $(X, \mathcal{V})$ is T_2 paracompact and, hence, T_4. □

For the converse of Theorem 2.11, we have the left ray topological space defined on $\mathbb{R}$, $(\mathbb{R}, \mathcal{L})$, as an example of epinormal Fréchet space that is not S_2-paracompat since it is separable and not paracompact space.

2.3. S_2-paracompactness and S-normality.

Recall that a topological space X is *S-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each separable subspace A of X, (see [4]). From the definition of S-normality, it is clear that any S_2-paracompact is S-normal. However, we show in the following example that this relation is not reversible.

Example 2.12. An example of a T_4 topological space that is S-normal but not S_2-paracompact is the sigma product $\Sigma(0)$ as a subspace of 2^{ω_1}, where $2 = \{0, 1\}$ considered with the discrete topology. It is not S_2-paracompact since it cannot be condensed onto a T_2 paracompact space, (see [8]).

Theorem 2.13. *Let X be Fréchet and Lindelöf space such that any finite subspace of X is discrete. X is S-normal if and only if X is S_2-paracompact.*

Proof. Let Y be a normal space and let $f : X \longrightarrow Y$ be a bijective function such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each separable subspace A of X. Without loss of generality, let X have more than one element. Thus, Y is T_1 since any finite subspace of X is separable and discrete. By continuity of f and since X is Lindelöf, then Y is Lindelöf. Since Y is T_3 Lindelöf, then Y is T_2 paracompact. Thus, X is S_2-paracompact.
Conversely, assume that X is S_2 paracompact. Let Y be a T_2 paracompact space and let $f : X \longrightarrow Y$ be a bijective function such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each separable subspace A of X. Hence, since Y is T_2 paracompact, then Y is T_4. Therefore, X is S-normal. $\square$

Recall that from [3] that a topological space X is *locally metrizable* if there exists a metrizable open nighborhood for each $x \in X$..

Theorem 2.14. *If X is hereditary Lindelöf, S_2-paracompact and locally metrizable, then X is T_2 paracompact and, hence, T_4.*

Proof. Set U_x to be a metrizable open nieghborhood of each $x \in X$. Since X is Lindelöf, then there exists a countable set E such that $X \subseteq \bigcap_{x \in E} U_x$. Now, since X is hereditary Lindelöf, then U_x is Lindelöf as a subspace of X for every $x \in X$. Hence, U_x is separable being Lindelöf and metrizable for every $x \in X$. Since X is S_2-paracompact and separable, then X is T_2 paracompact and, hence, T_4. $\square$

Let (X,τ) be a topological space and let M be a proper nonempty subset of X. The discrete extension of the topological space (X,τ) is defined by the following neighborhood system:
For each $x \in X \setminus M$, let $\mathcal{B}(x) = \{\{x\}\}$ and for each $x \in M$, let $\mathcal{B}(x) = \{U \in \tau : x \in U\}$. We denote the discrete extension of X by X_M, (see [9]).

The following example shows that the discrete extension of S_2-paracompact need not to be S_2-paracompact.

Example 2.15. Consider $(\mathbb{R}, \mathcal{RS})$, the rational sequence topology on $\mathbb{R}$, (see [9, Example 65]). Since $\mathcal{RS}$ is separable and not paracompact, then it is not S_2-paracompact. Also, because it is a Tychonoff locally compact space, we can set $X = \mathbb{R} \cup \{p\}$ to be the one point compactification of it. X is T_2 compact, which implies that X is S_2-paracompact. Consider $X_{\mathbb{R}}$, the discrete extension of X. Since $\{p\}$ is closed and open subset in $X_{\mathbb{R}}$, then $\mathbb{R}$ is a closed subspace of $X_{\mathbb{R}}$. However, since $(\mathbb{R}, \mathcal{RS})$ is not normal, we conclude that $X_{\mathbb{R}}$ cannot be normal. Since $X_{\mathbb{R}}$ is T_2 and not normal space, then $X_{\mathbb{R}}$ is not paracompact. Since $X_{\mathbb{R}}$ is separable as $\mathbb{Q} \cup \{p\}$ is a countable dense subset of $X_{\mathbb{R}}$, then $X_{\mathbb{R}}$ is not S_2-paracompact.

References

[1] O. Alghamdi, L. Kalantan and W. Alagal, S-paracompactness and S_2-paracompactness, Filomat 33, no. 17 (2019), 5645–5650.

[2] S. Alzharani and L. Kalantan, Epinormality, J. Nonlinear Sci. Appl. 9 (2016), 5398–5402.

[3] R. Engelking, General Topology, PWN, Warszawa, 1977.

[4] L. Kalantan and M. Alhomieyed, S-normality, J. Math. Anal. 9, no. 5 (2018), 48–54.

[5] L. Kalantan, L-paracompactness and L_2-paracompactness, Hacet. J. Math. Stat. 48, no. 3 (2019), 779–784.

[6] A. K. Misra, A topological view of P-spaces, Gen. Topol. Appl. 2 (1972), 349–362.

[7] M. M. Saeed, L. Kalantan and H. Alzumi, C-paracompactness and C_2-paracompactness, Turk. J. Math. 43 (2019), 9–20.

[8] M. M. Saeed, L. Kalantan and H. Alzumi, Result about C_2-paracompactness, Eur. J. Pure Appl. Math. 14, no. 2 (2021), 351–357.

[9] L. Steen and J. A. Seebach, Counterexamples in Topology, Dover Publications, 1995.

[10] V. Zaitsev, On certain classes of topological spaces and their bicompactification, Dokl. Akad. Nauk SSSR 178 (1968), 778–779.

