APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 23, no. 2 (2022), 345-361
doi:10.4995/agt.2022.16332

The Zariski topology on the graded primary spectrum of a graded module over a graded commutative ring

Saif Salam and Khaldoun Al-Zoubi

Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan (smsalam19@sci.just.edu.jo, kfzoubi@just.edu.jo)

Communicated by P. Das

Abstract

Let R be a G-graded ring and M be a G-graded R-module. We define the graded primary spectrum of M, denoted by $\mathcal{PS}_G(M)$, to be the set of all graded primary submodules Q of M such that $(Gr_M(Q) :_R M) = Gr((Q :_R M))$. In this paper, we define a topology on $\mathcal{PS}_G(M)$ having the Zariski topology on the graded prime spectrum $Spec_G(M)$ as a subspace topology, and investigate several topological properties of this topological space.

2020 MSC: *13A02; 16W50.*

Keywords: *graded primary submodules; graded primary spectrum; Zariski topology.*

1. Introduction and Preliminaries

Let G be a multiplicative group with identity e and R be a commutative ring with identity. Then R is called a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g. If $r \in R$, then r can be written uniquely as $\sum_{g\in G} r_g$, where r_g is the component of r in R_g. The set of all homogeneous elements of R is denoted

Received 20 September 2021 – Accepted 29 May 2022

by $h(R)$, i.e. $h(R) = \bigcup_{g\in G} R_g$. Let R be a G-graded ring and I be an ideal of R. Then I is called G-graded ideal of R if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. By $I \lhd_G R$, we mean that I is a G-graded ideal of R, (see [13]). The graded radical of I is the set of all $a = \sum_{g\in G} a_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $a_g^{n_g} \in I$. By $Gr(I)$ (resp. $\sqrt{I}$) we mean the graded radical (resp. the radical) of I, (see [18]). The graded prime spectrum $Spec_G(R)$ of a graded ring R consists of all graded prime ideals of R. It is known that $Spec_G(R)$ is a topological space whose closed sets are $V_G^R(I) = \{p \in Spec_G(R) \mid I \subseteq p\}$ for each graded ideal I of R (see, for example, [14, 16, 18]).

Let R be a G-graded ring and M a left R-module. Then M is said to be a G-graded R-module if $M = \bigoplus_{g\in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. If $x \in M$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in M_g. The set of all homogeneous elements of M is denoted by $h(M)$, i.e. $h(M) = \bigcup_{g\in G} M_g$. Let $M = \bigoplus_{g\in G} M_g$ be a G-graded R-module. A submodule N of M is called a G-graded R-submodule of M if $N = \bigoplus_{g\in G}(N \bigcap M_g)$. By $N \leq_G M$ (resp. $N <_G M$) we mean that N is a graded submodule (resp. a proper graded submodule) of M, (see [13]). If $N \leq_G M$, then $(N :_R M) = \{r \in R \mid rM \subseteq N\}$ is a graded ideal of R, (see [3, Lemma 2.1]). A proper graded submodule P of M is called a graded prime submodule of M if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $m \in P$ or $r \in (P :_R M)$. It is easily seen that, if P is a graded prime submodule of M, then $(P :_R M)$ is a graded prime ideal of R (see [3, Proposition2.7]). The graded prime spectrum of M, denoted by $Spec_G(M)$, is the set of all graded prime submodules of M. A proper graded submodule Q of M is called a graded primary submodule of M, if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in Q$, then either $m \in Q$ or $r \in Gr((Q :_R M))$. Graded prime submodules and Graded primary submodules of graded modules have been studied by various authors (see, for example [1, 2, 3, 4, 5, 15]). The graded radical of a proper graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$, (see[5]).

A graded R-module M is called a multiplication graded R-module if any $N \leq_G M$ has the form IM for some $I \lhd_G R$. If N is a graded submodule of a multiplication graded module M, then $N = (N :_R M)M$, (see [15]). A graded submodule N of a graded module M is called graded maximal submodule of M if $N \neq M$ and there is no graded submodule L of M such that $N \subset L \subset M$. A graded ring R is called graded integral domain, if whenever $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded principal ideal domain R is a graded integral domain in which every graded ideal of R is generated by a homogeneous element. One can easily see that, if R is a graded principal ideal domain, then every non-zero graded prime ideal of R is graded maximal.

Let M be a G-graded R-module and let $\zeta^*(M) = \{V_G^*(N) \mid N \leq_G M\}$ where $V_G^*(N) = \{P \in Spec_G(M) \mid N \subseteq P\}$ for any $N \leq_G M$. Then M is called a G-top module if the set $\zeta^*(M)$ is closed under finite union. In this case, $\zeta^*(M)$ generates a topology on $Spec_G(M)$ and this topology is called the quasi Zariski topology on $Spec_G(M)$. In contrast with $\zeta^*(M)$, $\zeta(M) = \{V_G(N) \mid N \leq_G M\}$ where $V_G(N) = \{P \in Spec_G(M) \mid (N :_R M) \subseteq (P :_R M)\}$ for any $N \leq_G M$ always generates a topology on $Spec_G(M)$. Let M be a G-graded R-module. Then the map $\varphi : Spec_G(M) \to Spec_G(R/Ann(M))$ by $\varphi(P) = (P :_R M)/Ann(M)$ is called the natural map on $Spec_G(M)$. For more details concerning the topologies on $Spec_G(M)$ and the natural map on $Spec_G(M)$, one can look in [7, 14].

In this paper, we call the set of all graded primary submodules Q of a graded module M satisfying the condition $(Gr_M(Q) :_R M) = Gr((Q :_R M))$ the graded primary spectrum of M and denote it by $\mathcal{PS}_G(M)$. It is easy to see that $Spec_G(M) \subseteq \mathcal{PS}_G(M)$. The converse inclusion is not always true. For example, if F is a G-graded field and M is a G-graded F-module, then $Spec_G(M) = \mathcal{PS}_G(M) = \{N \mid N <_G M\}$. But if we take the ring of integers $R = \mathbb{Z}$, $G = \mathbb{Z}_2$ and $M = \mathbb{Z} \times \mathbb{Z}$, then R is a G-graded ring by $R_0 = R$ and $R_1 = \{0\}$. Also, M is a G-graded R-module by $M_0 = \mathbb{Z}\times\{0\}$ and $M_1 = \{0\}\times\mathbb{Z}$. By some computations, we can see that $N = \mathbb{Z}\times 4\mathbb{Z} \in \mathcal{PS}_G(M)$. However, $N \notin Spec_G(M)$, since $2 \in h(R)$ and $(2,0) \in h(M)$ such that $2(0,2) \in N$ but $(2,0) \notin N$ and $2 \notin (N :_R M)$. For a G-graded R-module M, it is clear that $Gr_M(Q) \neq M$ for any $Q \in \mathcal{PS}_G(M)$ as $Gr((Q :_R M)) \in Spec_G(R)$. We introduce the primary G-top module which is a generalization of the G-top module. For this, we define the variety of any $N \leq_G M$ by $\nu_G^*(N) = \{Q \in \mathcal{PS}_G(M) \mid N \subseteq Gr_M(Q)\}$ and we set $\Omega^*(M) = \{\nu_G^*(N) \mid N \leq_G M\}$. Then M is called a primary G-top module if $\Omega^*(M)$ is closed under finite union. When this the case, the topology generated by $\Omega^*(M)$ is called the quasi-Zariski topology on $\mathcal{PS}_G(M)$. In particular, every primary G-top module is a G-top module. Next, we define another variety of any $N \leq_G M$ by $\nu_G(N) = \{Q \in \mathcal{PS}_G(M) \mid (N :_R M) \subseteq (Gr_M(Q) :_R M)\}$. Then the collection $\Omega(M) = \{\nu_G(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}_G(M)$, which is called the Zariski topology on $\mathcal{PS}_G(M)$, or simply $\mathcal{PZ}_G$-topology. We give some properties of these topologies. We also relate some properties of the graded primary spectrum $\mathcal{PS}_G(M)$ and $Spec_G(R/Ann(M))$ by introducing the map $\rho : \mathcal{PS}_G(M) \to Spec_G(R/Ann(M))$ given by $\rho(Q) = (Gr_M(Q) :_R M)/Ann(M)$. It should be noted that $(Gr_M(Q) :_R M) \in Spec_G(R)$, since Q is a graded primary submodule of M and $(Gr_M(Q) :_R M) = Gr((Q :_R M))$. In the last two sections, we find a base for the Zariski topology on $\mathcal{PS}_G(M)$ and we make certain observations and obtain a few results involving some conditions under which $\mathcal{PS}_G(M)$ is compact, irreducible, T_0-space or spectral space.

Throughout this paper, G is a multiplicative group, R is a commutative G-graded ring with identity and M is a G-graded R-module. We assume that $Spec_G(M)$ and $\mathcal{PS}_G(M)$ are non-empty.

2. The Zariski topology on $\mathcal{PS}_G(M)$

In this section, we introduce different varieties for graded submodules of graded modules. Using the properties of these varieties, we define the quasi Zariski topology and the $\mathcal{PZ}_G$-topology on $\mathcal{PS}_G(M)$. We also give some relationships between $\mathcal{PS}_G(M)$, $Spec_G(R/Ann(M))$ and $Spec_G(M)$.

Theorem 2.1. *Let M be a G-graded R-module. For any G-graded submodule N of M, we define the variety of N by $\nu^*_G(N) = \{Q \in \mathcal{PS}_G(M) \mid N \subseteq Gr_M(Q)\}$. Then the following hold:*

(1) $\nu^*_G(0) = \mathcal{PS}_G(M)$ and $\nu^*_G(M) = \varnothing$.
(2) If $N, N' \leq_G M$ and $N \subseteq N'$, then $\nu^*_G(N') \subseteq \nu^*_G(N)$.
(3) $\bigcap\limits_{i\in I} \nu^*_G(N_i) = \nu^*_G(\sum\limits_{i\in I} N_i)$ for any indexing set I and any family of graded submodules $\{N_i\}_{i\in I}$.
(4) $\nu^*_G(N) \cup \nu^*_G(N') \subseteq \nu^*(N \cap N')$ for any $N, N' \leq_G M$.
(5) $\nu^*_G(N) = \nu^*_G(Gr_M(N))$ for any $N \leq_G M$.

Proof. (1) and (2) are obvious.
(3) Since $N_i \subseteq \sum\limits_{i\in I} N_i$ for all $i \in I$, then by (2) we have $\nu^*_G(\sum\limits_{i\in I} N_i) \subseteq \nu^*_G(N_i)$ for all $i \in I$. Therefore $\nu^*_G(\sum\limits_{i\in I} N_i) \subseteq \bigcap\limits_{i\in I} \nu^*_G(N_i)$. Conversely, let $Q \in \bigcap\limits_{i\in I} \nu^*_G(N_i)$. Then $N_i \subseteq Gr_M(Q)$ for all $i \in I$, which implies that $\sum\limits_{i\in I} N_i \subseteq Gr_M(Q)$. Hence $Q \in \nu^*_G(\sum\limits_{i\in I} N_i)$.
(4) Since $N \cap N' \subseteq N$ and $N \cap N' \subseteq N'$, then by (2) we have $\nu^*_G(N) \subseteq \nu^*_G(N \cap N')$ and $\nu^*_G(N') \subseteq \nu^*_G(N \cap N')$. Therefore $\nu^*_G(N) \cup \nu^*_G(N') \subseteq \nu^*_G(N \cap N')$.
(5) As $N \subseteq Gr_M(N)$, we obtain $\nu^*_G(Gr_M(N)) \subseteq \nu^*_G(N)$. Conversely, let $Q \in \nu^*_G(N)$. Then $N \subseteq Gr_M(Q)$. So $Gr_M(N) \subseteq Gr_M(Gr_M(Q)) = Gr_M(Q)$. Thus $Q \in \nu^*_G(Gr_M(N))$. □

Note that the reverse inclusion in Theorem 2.1 (4) is not always true. Take $R = \mathbb{Z}, G = \mathbb{Z}_2, M = \mathbb{Z} \times \mathbb{Z}, N = 4\mathbb{Z} \times \{0\}$ and $N' = \{0\} \times 4\mathbb{Z}$. Then R is a G-graded ring by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Also M is a G-graded R-module by $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$. Moreover, $N, N' \leq_G M$. Now, $\nu^*_G(N \cap N') = \nu^*_G(\{(0,0)\}) = \mathcal{PS}_G(M)$. Let $P = \{(0,0)\}$. Then $P \in Spec_G(M) \subseteq \mathcal{PS}_G(M)$. It follows that $P \in \nu^*_G(N \cap N')$ and $Gr_M(P) = P$. But $N \nsubseteq P$ and $N' \nsubseteq P$. Thus $P \notin \nu^*_G(N) \cup \nu^*_G(N')$.

By Theorem 2.1 (1), (3), and (4), the collection $\Omega^*(M) = \{\nu^*_G(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}_G(M)$ if and only if $\Omega^*(M)$ is closed under finite union. When this is the case, we call M a primary G-top module and we call the generated topology the quasi Zariski topology on $\mathcal{PS}_G(M)$, or $\mathcal{PZ}^q_G$-topology for short. It is clear that, every G-graded simple

R-module is a primary G-top module. In the following theorem, we show that every multiplication graded R-module is a primary G-top module.

Theorem 2.2. *If M is a multiplication graded R-module, then M is a primary G-top module.*

Proof. Let $N, N' \leq_G M$. It is sufficient to show that $\nu^*_G(N \cap N') \subseteq \nu^*_G(N) \cup \nu^*_G(N')$. So let $Q \in \nu^*_G(N \cap N')$. Then $N \cap N' \subseteq Gr_M(Q)$. It follows that $(N :_R M) \cap (N' :_R M) = (N \cap N' :_R M) \subseteq (Gr_M(Q) :_R M) = Gr((Q :_R M)) \in Spec_G(R)$ as Q is a graded primary submodule. Therefore $(N :_R M) \subseteq (Gr_M(Q) :_R M)$ or $(N' :_R M) \subseteq (Gr_M(Q) :_R M)$. Since M is a multiplication graded R-module, then $N = (N :_R M)M \subseteq (Gr_M(Q) :_R M)M = Gr_M(Q)$ or $N' = (N' :_R M)M \subseteq (Gr_M(Q) :_R M)M = Gr_M(Q)$. Thus $Q \in \nu^*_G(N) \cup \nu^*_G(N')$. $\square$

Now we define another variety for a graded submodule N of a G-graded R-module M. We set $\nu_G(N) = \{Q \in PS_G(M) \mid (N :_R M) \subseteq (Gr_M(Q) :_R M)\}$. We state some properties of this variety in the following theorem to construct the Zariski topology on $\mathcal{PS}_G(M)$.

Theorem 2.3. *Let M be a G-graded R-module and let $N, N', N_i \leq_G M$ for any $i \in I$, where I is an indexing set. Then the following hold:*

(1) $\nu_G(0) = \mathcal{PS}_G(M)$ *and* $\nu_G(M) = \varnothing$.
(2) $\bigcap\limits_{i\in I} \nu_G(N_i) = \nu_G(\sum\limits_{i\in I}(N_i :_R M)M)$.
(3) $\nu_G(N) \cup \nu_G(N') = \nu_G(N \cap N')$.
(4) *If* $N \subseteq N'$, *then* $\nu_G(N') \subseteq \nu_G(N)$.

Proof. (1) and (4) are trivial.
(2) Let $Q \in \bigcap\limits_{i\in I} \nu_G(N_i)$. Then $(N_i :_R M) \subseteq (Gr_M(Q) :_R M)$ for all $i \in I$. So $(N_i :_R M)M \subseteq (Gr_M(Q) :_R M)M$ for all $i \in I$. This implies that $\sum\limits_{i\in I}(N_i :_R M)M \subseteq (Gr_M(Q) :_R M)M$. Therefore $(\sum\limits_{i\in I}(N_i :_R M)M : M) \subseteq ((Gr_M(Q) :_R M)M :_R M) = (Gr_M(Q) :_R M)$. Hence $Q \in \nu_G(\sum\limits_{i\in I}(N_i :_R M)M)$. For the reverse inclusion, let $Q \in \nu_G(\sum\limits_{i\in I}(N_i :_R M)M)$. Then $(\sum\limits_{i\in I}(N_i :_R M)M :_R M) \subseteq (Gr_M(Q) :_R M)$. But for any $i \in I$, we have $(N_i :_R M) = ((N_i :_R M)M :_R M) \subseteq (\sum\limits_{i\in I}(N_i :_R M)M :_R M) \subseteq (Gr_M(Q) :_R M)$. Thus $Q \in \bigcap\limits_{i\in I} \nu_G(N_i)$.
(3) For any $Q \in \mathcal{PS}_G(M)$, we have $Q \in \nu_G(N \cap N') \Leftrightarrow (N \cap N' :_R M) \subseteq (Gr_M(Q) :_R M) = Gr((Q :_R M)) \Leftrightarrow (N :_R M) \cap (N' :_R M) \subseteq Gr((Q :_R M)) \in Spec_G(R) \Leftrightarrow (N :_R M) \subseteq Gr((Q :_R M))$ or $(N' :_R M) \subseteq Gr((Q :_R M)) \Leftrightarrow Q \in \nu_G(N) \cup \nu_G(N')$. Hence $\nu_G(N \cap N') = \nu_G(N) \cup \nu_G(N')$. $\square$

In view of Theorem 2.3 (1), (2) and (3), the collection $\Omega(M) = \{\nu_G(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}_G(M)$, which is called the Zariski topology on $\mathcal{PS}_G(M)$, or $\mathcal{PZ}_G$-topology for short.

Now we state some relations between the varieties $V_G^*(N)$, $V_G(N)$, $\nu_G^*(N)$ and $\nu_G(N)$ for any graded submodule N of a G-graded R-module M. These relations will be used continuously throughout the rest of this paper.

Lemma 2.4. *Suppose that N and N' are graded submodules of a G-graded R-module M and that I is a G-graded ideal of R. Then the following hold:*

(1) $V_G(N) = \nu_G(N) \cap Spec_G(M)$.
(2) $V_G^*(N) = \nu_G^*(N) \cap Spec_G(M)$.
(3) If $Gr((N :_R M)) = Gr((N' :_R M))$, then $\nu_G(N) = \nu_G(N')$. The converse is also true if $N, N' \in \mathcal{PS}_G(M)$.
(4) $\nu_G(N) = \nu_G((N :_R M)M) = \nu_G^*((N :_R M)M) = \nu_G^*(Gr((N :_R M))M)$. In particular, $\nu_G^*(IM) = \nu_G(IM)$.

Proof. The proof is straightforward. $\square$

Corollary 2.5. *Every primary G-top module is a G-top module.*

Proof. Let M be a primary G-top module and $N, N' \leq_G M$. By Lemma 2.4 (2), we have $V_G^*(N) \cup V_G^*(N') = (Spec_G(M) \cap \nu_G^*(N)) \cup (Spec_G(M) \cap \nu_G^*(N')) = Spec_G(M) \cap (\nu_G^*(N) \cup \nu_G^*(N')) = Spec_G(M) \cap \nu_G^*(J) = V_G^*(J)$ for some graded submodule J of M and hence M is a G-top module. $\square$

By Corollary 2.5, if M is a primary G-top module, then $\zeta^*(M) = \{V_G^*(N) \mid N \leq_G M\}$ induces the quasi Zariski topology on $Spec_G(M)$ which will be, by Lemma 2.4 (2), a topological subspace of $\mathcal{PS}_G(M)$ equipped with $\mathcal{PZ}_G^q$-topology. Also by Lemma 2.4 (1), $Spec_G(M)$ with the Zariski topology is a topological subspace of $\mathcal{PS}_G(M)$ equipped with the $\mathcal{PZ}_G$-topology for any G-graded R-module M.

Consider φ and ρ as described in the introduction. Let M be a G-graded R-module. For $p \in Spec_G(R)$, we set $\mathcal{PS}_G^p(M) = \{Q \in \mathcal{PS}_G(M) \mid (Gr_M(Q) :_R M) = p\}$.

Proposition 2.6. *The following statements are equivalent for any G-graded R-module M:*

(1) If whenever $Q, Q' \in PS_G(M)$ with $\nu_G(Q) = \nu_G(Q')$, then $Q = Q'$.
(2) $|\mathcal{PS}_G^p(M)| \leq 1$ for every $p \in Spec_G(R)$.
(3) ρ is injective.

Proof. (1)⇒(2): Let $p \in Spec_G(R)$ and $Q, Q' \in \mathcal{PS}_G^p(M)$. Then $Q, Q' \in \mathcal{PS}_G(M)$ and $(Gr_M(Q) :_R M) = (Gr_M(Q') :_R M) = p$. By Lemma 2.4 (3), we have $\nu_G(Q) = \nu_G(Q')$. So, by the assumption (1), $Q = Q'$.
(2)⇒(3): Assume that $\rho(Q) = \rho(Q')$, where $Q, Q' \in \mathcal{PS}_G(M)$. Then $(Gr_M(Q) :_R M) = (Gr_M(Q') :_R M)$. Let $p = (Gr_M(Q) :_R M) \in Spec_G(R)$. Then we get $Q, Q' \in \mathcal{PS}_G^p(M)$ and by the hypothesis we obtain $Q = Q'$.
(3)⇒(1): Let $Q, Q' \in \mathcal{PS}_G(M)$ with $\nu_G(Q) = \nu_G(Q')$. Then, by Lemma 2.4 (3), we have $(Gr_M(Q) :_R M) = (Gr_M(Q') :_R M)$. So $\rho(Q) = \rho(Q')$. Since ρ is injective, then $Q = Q'$. $\square$

Corollary 2.7. *If $|\mathcal{PS}_G^p(M)| = 1$ for every $p \in Spec_G(R)$, then ρ is bijective.*

Proof. It is clear by Proposition 2.8. $\square$

Let M be a G-graded R-module. From now on, we will denote $R/Ann(M)$ by $\overline{R}$ and any graded ideal $I/Ann(M)$ of $\overline{R}$ by $\overline{I}$. In the following lemma, we recall some properties of the natural map φ of $Spec_G(M)$. These properties are important in the rest of this section.

Lemma 2.8 ([14, Proposition 3.13 and Proposition 3.15])**.** *Let M be a G-graded R-module. Then the following hold:*

(1) φ is continuous and $\varphi^{-1}(V_G^{\overline{R}}(\overline{I})) = V_G(IM)$ for every graded ideal I of R containing $Ann(M)$.

(2) If φ is surjective, then φ is both open and closed with $\varphi(V_G(N)) = {V_G}^{\overline{R}}(\overline{(N :_R M)})$ and $\varphi(Spec_G(M) - V_G(N)) = Spec_G(\overline{R}) - {V_G}^{\overline{R}}(\overline{(N :_R M)})$ for any $N \leq_G M$.

In the next two propositions, we give similar results for ρ.

Proposition 2.9. *Let M be a G-graded R-module. Then $\rho^{-1}(V_G^{\overline{R}}(\overline{I})) = \nu_G(IM)$, for every graded ideal I of R containing $Ann(M)$. Therefore ρ is continuous mapping.*

Proof. For any $Q \in \mathcal{PS}_G(M)$, we have $Q \in \rho^{-1}(V_G^{\overline{R}}(\overline{I})) \Leftrightarrow \rho(Q) \in V_G^{\overline{R}}(\overline{I}) \Leftrightarrow \overline{I} \subseteq \overline{(Gr_M(Q) :_R M)} \Leftrightarrow I \subseteq (Gr_M(Q) :_R M) \Leftrightarrow IM \subseteq (Gr_M(Q) :_R M)M \Leftrightarrow (IM :_R M) \subseteq ((Gr_M(Q) :_R M)M :_R M) = (Gr_M(Q) :_R M) \Leftrightarrow Q \in \nu_G(IM)$. Hence $\rho^{-1}(V_G^{\overline{R}}(\overline{I})) = \nu_G(IM)$. $\square$

Proposition 2.10. *Let M be a G-graded R-module. If ρ is surjective, then ρ is both open and closed; more precisely, for any $N \leq_G M$, $\rho(\nu_G(N)) = V_G^{\overline{R}}(\overline{(N :_R M)})$ and $\rho(\mathcal{PS}_G(M) - \nu_G(N)) = Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)})$.*

Proof. By Proposition 2.9, we have $\rho^{-1}(V_G^{\overline{R}}(\overline{I})) = \nu_G(IM)$ for every $I \lhd_G R$ containing $Ann(M)$. So $\rho^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)})) = \nu_G((N :_R M)M) = \nu_G(N)$ for any $N \leq_G M$. It follows that $V_G^{\overline{R}}(\overline{(N :_R M)}) = \rho(\rho^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)}))) = \rho(\nu_G(N))$ as ρ is surjective. For the second part, note that $\mathcal{PS}_G(M) - \nu_G(N) = \mathcal{PS}_G(M) - \rho^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)})) = \rho^{-1}(Spec_G(\overline{R})) - \rho^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)})) = \rho^{-1}(Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)}))$. This implies that $\rho(\mathcal{PS}_G(M) - \nu_G(N)) = \rho(\rho^{-1}(Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)}))) = Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)})$. $\square$

Corollary 2.11. *Let M be a G-graded R-module. Then ρ is bijective if and only if ρ is a homeomorphism.*

The following theorem is a result for Lemma 2.8, Proposition 2.9 and Proposition 2.10.

Theorem 2.12. *Let M be a G-graded R-module. Consider the following statements:*

(1) $Spec_G(M)$ is connected.

(2) $\mathcal{PS}_G(M)$ *is connected.*
(3) $Spec_G(\overline{R})$ *is connected.*

(i) *If* ρ *is surjective, then (1)⇒(2)⇔(3).*
(ii) *If* φ *is surjective, then all the three statements are equivalent.*

Proof. (i) (1)⇒(2): Assume that $Spec_G(M)$ is connected. If $\mathcal{PS}_G(M)$ is disconnected, then there is a U clopen in $\mathcal{PS}_G(M)$ such that $U \neq \varnothing$ and $U \neq \mathcal{PS}_G(M)$. Since U is clopen in $\mathcal{PS}_G(M)$, then $U = \mathcal{PS}_G(M) - \nu_G(N_1) = \nu_G(N_2)$ for some $N_1, N_2 \leq_G M$. By Proposition 2.10, we have $\rho(U)$ is clopen in $Spec_G(\overline{R})$. But φ is continuous. So $\varphi^{-1}(\rho(U))$ is clopen in $\mathcal{PS}_G(M)$, and so $\varphi^{-1}(\rho(U)) = \varnothing$ or $\varphi^{-1}(\rho(U)) = Spec_G(M)$. If $\varphi^{-1}(\rho(U)) = \varnothing$, then $\varphi^{-1}(\rho(\nu_G(N_2))) = \varnothing$. Thus $\varphi^{-1}(V_G^{\overline{R}}(\overline{(N_2 :_R M)})) = \varnothing$. It follows that $V_G(N_2) = \varnothing$, which means that $(N_2 :_R M) \nsubseteq (P :_R M)$ for any $P \in Spec_G(M)$. As $U = \nu_G(N_2) \neq \varnothing$, then $\exists Q \in \mathcal{PS}_G(M)$ such that $(N_2 :_R M) \subseteq (Gr_M(Q) :_R M)$. Since $Gr_M(Q) \neq M$, then $\exists P' \in Spec_G(M)$ such that $Q \subseteq P'$. Therefore $(Gr_M(Q) :_R M) \subseteq (Gr_M(P') :_R M) = (P' :_R M)$. Hence $(N_2 :_R M) \subseteq (P' :_R M)$ which is a contradiction. Now, if $\varphi^{-1}(\rho(U)) = Spec_G(M)$, then $Spec_G(M) = \varphi^{-1}(\rho(\nu_G(N_2))) = V_G(N_2) = Spec_G(M) \cap \nu_G(N_2)$. It follows that $Spec_G(M) \subseteq \nu_G(N_2) = U = \mathcal{PS}_G(M) - \nu_G(N_1)$. As $U = \mathcal{PS}_G(M) - \nu_G(N_1) \neq \mathcal{PS}_G(M)$, then $\exists Q \in \mathcal{PS}_G(M) \cap \nu_G(N_1)$. Therefore $(N_1 :_R M) \subseteq (Gr_M(Q) :_R M)$ and $\exists P \in Spec_G(M)$ such that $Q \subseteq P$. It follows that $(N_1 :_R M) \subseteq (Gr_M(Q) :_R M) \subseteq (Gr_M(P) :_R M)$. Then $P \in \nu_G(N_1)$. But $P \in Spec_G(M) \subseteq \mathcal{PS}_G(M) - \nu_G(N_1)$. Therefore $P \notin \nu_G(N_1)$ which is a contradiction. Consequently, $\mathcal{PS}_G(M)$ is a connected space.
(2)⇒(3): Since ρ is continuous surjective map and $\mathcal{PS}_G(M)$ is connected, then $Spec_G(\overline{R})$ is connected.
(3)⇒(2): Assume by way of contradiction that $\mathcal{PS}_G(M)$ is disconnected. Then there is a U clopen in $\mathcal{PS}_G(M)$ such that $U \neq \varnothing$ and $U \neq \mathcal{PS}_G(M)$. Since ρ is surjective, then $\rho(U)$ is clopen in $Spec_G(\overline{R})$, and so $\rho(U) = \varnothing$ or $\rho(U) = Spec_G(\overline{R})$ as $Spec_G(\overline{R})$ is connected. Also, since U is open in $\mathcal{PS}_G(M)$, then $U = \mathcal{PS}_G(M) - \nu_G(N)$ for some $N \leq_G M$. Now, if $\rho(U) = Spec_G(\overline{R})$, then $Spec_G(\overline{R}) = \rho(\mathcal{PS}_G(M) - \nu_G(N)) = Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)})$ by Proposition 2.10. Thus $V_G^{\overline{R}}(\overline{(N :_R M)}) = \varnothing$ which implies that $\varnothing = \rho^{-1}(\varnothing) = \rho^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)})) = \nu_G(N)$. Therefore $\nu_G(N) = \varnothing$ and hence $U = \mathcal{PS}_G(M)$, a contradiction. Also if $\rho(U) = \varnothing$, then $U \subseteq \rho^{-1}(\rho(U)) = \varnothing$. It follows that $U = \varnothing$ which is also a contradiction. Therefore $\mathcal{PS}_G(M)$ is connected.

(ii) If φ is surjective, then it is clear that ρ is surjective and hence (1)⇒(2)⇔(3) by (i). So it is enough to show that (3)⇒(1). We prove it in a similar way to proof of (3)⇒(2) using the properties of φ. So again, we assume by way of contradiction that $Spec_G(M)$ is disconnected, which means that there is a U clopen in $Spec_G(M)$ such that $U \neq \varnothing$ and $U \neq Spec_G(M)$. As $Spec_G(\overline{R})$ is connected and φ is surjective, then $\varphi(U) = \varnothing$ or $\varphi(U) = Spec_G(\overline{R})$. Also the open set U can be written as $U = Spec_G(M) - V_G(N)$ for some $N \leq_G M$. It is clear that if $\varphi(U) = \varnothing$, then $U = \varnothing$ and we have a contradiction. If $\varphi(U) = Spec_G(\overline{R})$, then $Spec_G(\overline{R}) = \varphi(Spec_G(M) - V_G(N)) = Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)})$, and so $V_G^{\overline{R}}(\overline{(N :_R M)}) = \varnothing$. Thus $V_G(N) = \varphi^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)})) = \varphi^{-1}(\varnothing) = \varnothing$. It follows that $U = Spec_G(M)$ which is a contradiction. Therefore $Spec_G(M)$ is a connected space and this completes the proof. $\square$

Let M and S be two G-graded R-modules. Recall that an R-module homomorphism $f : M \to S$ is called a G-graded R-module homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$, see [13]. Let $f : M \to M'$ be a G-graded module epimorphism between the two graded modules M and M'. If $N' \leq_G M'$, then $(N' :_R M') = (f^{-1}(N') :_R M)$. Also it is easy to check that, $(N :_R M) = (f(N) :_R M')$ and $f(Gr_M(N)) = Gr_{M'}(f(N))$ for any $N \leq_G M$ containing the kernel of f. We will denote the kernel of f by $kerf$.

Lemma 2.13. *Let M and M' be G-graded R-modules. Let $f : M \to M'$ be a G-graded module epimorphism. Then the following hold:*

(1) If $Q' \in \mathcal{PS}_G(M')$, then $f^{-1}(Q') \in \mathcal{PS}_G(M)$.
(2) If $Q \in \mathcal{PS}_G(M)$ and $kerf \subseteq Q$, then $f(Q) \in \mathcal{PS}_G(M')$.

Proof. (1) It is easy to verify that $f^{-1}(Q')$ is graded primary submodule of M and it remains to show that $(Gr_M(f^{-1}(Q')) :_R M) = Gr((f^{-1}(Q') :_R M))$. Since $kerf \subseteq f^{-1}(Q') \subseteq Gr_M(f^{-1}(Q'))$ and $(Gr_{M'}(Q') :_R M') = Gr((Q' :_R M'))$, we obtain $(Gr_M(f^{-1}(Q')) :_R M) = (f(Gr_M(f^{-1}(Q'))) :_R M') = (Gr_{M'}(f(f^{-1}(Q'))) :_R M') = (Gr_{M'}(Q') :_R M') = Gr((Q' :_R M')) = Gr((f^{-1}(Q') :_R M))$ as required.
(2) First note that $f(Q)$ is a graded proper submodule of M', since Q is a graded proper submodule of M containing $kerf$. Let $rm' \in f(Q)$ for $r \in h(R)$ and $m' \in h(M')$. As f is a graded module epimorphism and $m' \in h(M')$, we get $\exists m \in h(M)$ such that $f(m) = m'$, which implies that $f(rm) \in f(Q)$. Thus $\exists t \in Q$ such that $rm - t \in kerf \subseteq Q$. So $rm \in Q$, and so $m \in Q$ or $r \in \sqrt{(Q :_R M)} = \sqrt{(f(Q) :_R M')}$. Hence $f(Q)$ is a graded primary submodule of M'. Moreover, $(Gr_{M'}(f(Q)) :_R M') = (f(Gr_M(Q)) :_R M') = (Gr_M(Q) :_R M) = Gr((Q :_R M)) = Gr((f(Q) :_R M'))$. Therefore $f(Q) \in \mathcal{PS}_G(M')$. $\square$

Theorem 2.14. *Let M and M' be G-graded R-modules and $f : M \to M'$ be a graded module epimorphism. Then the mapping $\pi : \mathcal{PS}_G(M') \to \mathcal{PS}_G(M)$ by $\pi(Q') = f^{-1}(Q')$ is an injective continuous map. Moreover, if π is surjective map, then $\mathcal{PS}_G(M)$ is homeomorphic to $\mathcal{PS}_G(M')$.*

Proof. By Lemma 2.13, π is well-defined. Also, the injectivity of π is obvious. Now for any $O \in \mathcal{PS}_G(M')$ and any closed set $\nu_G(N)$ in $\mathcal{PS}_G(M)$, where $N \leq_G M$, we have $O \in \pi^{-1}(\nu_G(N)) = \pi^{-1}(\nu_G^*(Gr((N :_R M))M)) \Leftrightarrow Gr((N :_R M))M \subseteq Gr_M(f^{-1}(O)) \Leftrightarrow Gr((N :_R M)) \subseteq (Gr_M(f^{-1}(O)) :_R M) = Gr((f^{-1}(O) :_R M)) = Gr((O :_R M')) = (Gr_{M'}(O) :_R M') \Leftrightarrow Gr((N :_R M))M' \subseteq Gr_{M'}(O) \Leftrightarrow O \in \nu_G^*(Gr((N :_R M))M') = \nu_G(Gr((N :_R M))M')$. Therefore $\pi^{-1}(\nu_G(N)) = \nu_G(Gr((N :_R M))M')$ and hence π is continuous.

For the last statement, we assume that π is surjective and it is enough to show that π is closed. So let $\nu_G(N')$ be a closed set in $\mathcal{PS}_G(M')$, where $N' \leq_G M'$. As we have seen, $\pi^{-1}(\nu_G(N)) = \nu_G(Gr((N :_R M))M')$ for any $N \leq_G M$. It follows that $\pi^{-1}(\nu_G(f^{-1}(N'))) = \nu_G(Gr((f^{-1}(N') :_R M))M') = \nu_G(Gr((N' :_R M'))M') = \nu_G(N')$ and hence $\pi^{-1}(\nu_G(f^{-1}(N'))) = \nu_G(N')$. Thus $\pi(\nu_G(N')) = \nu_G(f^{-1}(N'))$ as π is surjective. Therefore π is closed, and so $\mathcal{PS}_G(M)$ is homeomorphic to $\mathcal{PS}_G(M')$. □

Corollary 2.15. *Let M and M' be G-graded R-modules. Let $f : M \to M'$ be a G-graded module isomorphism. Then $\mathcal{PS}_G(M)$ is homeomorphic to $\mathcal{PS}_G(M')$.*

Proof. By Lemma 2.13 (2) and Theorem 2.14. □

3. A base for the Zariski topology on $\mathcal{PS}_G(M)$

Let M be a G-graded R-module. In [14, Theorem 2.3], it has been proved that for each $r \in h(R)$, the set $D_r = Spec_G(R) - V_G^R(rR)$ is open in $Spec_G(R)$ and the family $\{D_r \mid r \in h(R)\}$ is a base for the Zariski topology on $Spec_G(R)$. In addition, each D_r is compact and thus $D_1 = Spec_G(R)$ is compact. In this section, we set $S_r = \mathcal{PS}_G(M) - \nu_G(rM)$ for each $r \in h(R)$ and prove that $S = \{S_r \mid r \in h(R)\}$ forms a base for the Zariski-topology on $\mathcal{PS}_G(M)$. Also, we show that each S_r is compact and hence $\mathcal{PS}_G(M)$ is compact.

Proposition 3.1. *For any G-graded R-module M, the set $S = \{S_r \mid r \in h(R)\}$ forms a base for the Zariski topology on $\mathcal{PS}_G(M)$.*

Proof. Let $U = \mathcal{PS}_G(M) - \nu_G(N)$ be an open set in $\mathcal{PS}_G(M)$, where $N \leq_G M$. Let $Q \in U$ and it is enough to find an element $r \in h(R)$ such that $Q \in S_r \subseteq U$. Since $Q \in U$, then $(N :_R M) \nsubseteq (Gr_M(Q) :_R M)$, and so there exists $x \in R$ and $g \in G$ such that $x_g \in (N :_R M) - (Gr_M(Q) :_R M)$. Take $r = x_g \in h(R)$. Therefore $(rM :_R M) \nsubseteq (Gr_M(Q) :_R M)$ and thus $Q \in S_r$. Now for any $Q' \in S_r$, we have $(rM :_R M) \nsubseteq (Gr_M(Q') :_R M)$, which implies that $(N :_R M) \nsubseteq (Gr_M(Q') :_R M)$. Thus $Q' \in U$ and hence $Q \in S_r \subseteq U$ which completes the proof. □

Lemma 3.2 ([14, Theorem 2.3 (2)])**.** *Let R be a G-graded ring. Then $D_r \cap D_t = D_{rt}$ for any $r, t \in h(R)$.*

Let R be a G-graded ring. As usual, the nilradical of R and the set of all units of R will be denoted by $N(R)$ and $U(R)$, respectively.

Proposition 3.3. *Let M be a G-graded R-module and $r \in h(R)$. Then,*

(1) $\rho^{-1}(D_{\overline{r}}) = S_r$
(2) $\rho(S_r) \subseteq D_{\overline{r}}$. *If ρ is surjective, then the equality holds.*
(3) $S_r \cap S_t = S_{rt}$, *for any $r, t \in h(R)$.*
(4) If $r \in N(R)$, then $S_r = \varnothing$.
(5) If $r \in U(R)$, then $S_r = \mathcal{PS}_G(M)$.

Proof. (1) $\rho^{-1}(D_{\overline{r}}) = \rho^{-1}(Spec_G(\overline{R}) - V_G^{\overline{R}}(\overline{r}\overline{R})) = \mathcal{PS}_G(M) - \rho^{-1}(V_G^{\overline{R}}(\overline{r}\overline{R})) = \mathcal{PS}_G(M) - \nu_G(rM) = S_r$ by Proposition 2.9.
(2) Trivial.
(3) For any $r, t \in h(R)$, we have $\overline{r}, \overline{t} \in h(\overline{R})$ and hence $D_{\overline{r}} \cap D_{\overline{t}} = D_{\overline{rt}}$ by Lemma 3.2. It follows that $S_r \cap S_t = \rho^{-1}(D_{\overline{r}}) \cap \rho^{-1}(D_{\overline{t}}) = \rho^{-1}(D_{\overline{rt}}) = S_{rt}$.
(4) Assume that $r \in N(R)$. It follows that $D_r = \varnothing$ by [18, Proposition 3.6 (2)], and thus $D_{\overline{r}} = \varnothing$. Therefore $S_r = \rho^{-1}(D_{\overline{r}}) = \varnothing$ by (1).
(5) Assume that $r \in U(R)$. By [18, Proposition 3.6 (3)], we have $D_r = Spec_G(R)$ and hence $D_{\overline{r}} = Spec_G(\overline{R})$. By (1), we obtain $S_r = \rho^{-1}(D_{\overline{r}}) = \rho^{-1}(Spec_G(\overline{R})) = \mathcal{PS}_G(M)$. □

In part (a) of the next example, we see that if F is a G-graded field and M is a G-graded F-module, then the Zariski topology on $\mathcal{PS}_G(M)$ is the trivial topology. However, a G-graded ring R for which for a G-graded R-module M, the Zariski topology on $\mathcal{PS}_G(M)$ is the indiscrete topology is not necessarily a G-graded field and this will be discussed in part (b).

Example 3.4. (a) Let F be a G-graded field and M be a G-graded F-module. Then any non-zero homogeneous element of F is unit. By Proposition 3.3 (5), we have $S_r = \mathcal{PS}_G(M)$ for any non-zero homogeneous element r of F. Also $S_0 = \mathcal{PS}_G(M) - \nu_G(0) = \varnothing$ and hence $S = \{S_r \mid r \in h(F)\} = \{\mathcal{PS}_G(M), \varnothing\}$. Therefore, the Zariski topology on $\mathcal{PS}_G(M)$ is the trivial topology on $\mathcal{PS}_G(M)$.
(b) Let $R = \mathbb{Z}_8$ as a $\mathbb{Z}_2$-graded $\mathbb{Z}_8$ module by $R_0 = \mathbb{Z}_8$ and $R_1 = \{0\}$. Note that $1, 3, 5, 7 \in h(\mathbb{Z}_8) \cap U(\mathbb{Z}_8)$. So $S_1 = S_3 = S_5 = S_7 = \mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_8)$ by Proposition 3.3 (5). Also $S_0 = S_2 = S_4 = S_6 = \varnothing$ by Proposition 3.3 (4), since $0, 2, 4, 6 \in N(\mathbb{Z}_8) \cap h(\mathbb{Z}_8)$. Now $S = \{S_r \mid r \in h(R)\} = \{\varnothing, \mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_8)\}$ and hence the Zariski topology on $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_8)$ is the trivial topology. But $\mathbb{Z}_8$ is not $\mathbb{Z}_2$-graded field.

Theorem 3.5. *Let M be a G-graded R-module. If ρ is surjective, then the open set S_r in $\mathcal{PS}_G(M)$ for each $r \in h(R)$ is compact; in particular, the space $\mathcal{PS}_G(M)$ is compact.*

Proof. Let $r \in h(R)$ and $\zeta = \{S_t \mid t \in \Delta\}$ be a basic open cover for S_r, where Δ is a subset of $h(R)$. Then $S_r \subseteq \bigcup\limits_{t\in\Delta} S_t$ and thus $D_{\overline{r}} = \rho(S_r) \subseteq \bigcup\limits_{t\in\Delta} \rho(S_t) = \bigcup\limits_{t\in\Delta} D_{\overline{t}}$ by Proposition 3.3 (2). Then $\overline{\zeta} = \{D_{\overline{t}} \mid t \in \Delta\}$ is a basic open cover for the compact set $D_{\overline{r}}$ and hence it has a finite subcover $\overline{\overline{\zeta}} = \{D_{\overline{t_i}} \mid i = 1, ..., n\}$, where $t_i \in \Delta$ for any $i = 1, ..., n$. This means that $D_{\overline{r}} \subseteq \bigcup\limits_{i=1}^{n} D_{\overline{t_i}}$ and it follows that $S_r = \rho^{-1}(D_{\overline{r}}) \subseteq \bigcup\limits_{i=1}^{n} \rho^{-1}(D_{\overline{t_i}}) = \bigcup\limits_{i=1}^{n} S_{\overline{t_i}}$ by Proposition 3.3 (1). Therefore $\underline{\zeta} = \{S_{\overline{t_i}} \mid i = 1, ..., n\} \subseteq \zeta$ is a finite subcover for S_r. For the other part of the theorem, since $\mathcal{PS}_G(M) = S_1$, then $\mathcal{PS}_G(M)$ is compact. □

Theorem 3.6. *Let M be a G-graded R-module. If ρ is surjective, then the compact open sets of $\mathcal{PS}_G(M)$ are closed under finite intersection and form an open base.*

Proof. Let C_1, C_2 be quasi compact open sets of $\mathcal{PS}_G(M)$ and $\zeta = \{S_r \mid r \in \Delta\}$ be a basic open cover for $C_1 \cap C_2$, where Δ is a subset of $h(R)$. Since S is a base for the Zariski topology on $\mathcal{PS}_G(M)$, then the compact open sets C_1, C_2 can be written as a finite union of elements of S. So let $C_1 = \bigcup\limits_{i=1}^{n} S_{t_i}$ and $C_2 = \bigcup\limits_{j=1}^{m} S_{z_j}$. By Proposition 3.3 (3), we have $C_1 \cap C_2 = \bigcup\limits_{i,j}(S_{t_i} \cap S_{z_j}) = \bigcup\limits_{i,j} S_{t_i z_j} \subseteq \bigcup\limits_{r\in\Delta} S_r$. Note that for any i, j we have $t_i z_j \in h(R)$ as $t_i, z_j \in h(R)$. So without loss of generality we can assume that $C_1 \cap C_2 = \bigcup\limits_{k=1}^{L} S_{h_k}$ where $h_k \in h(R)$, for $k = 1, ..., L$. Then $S_{h_k} \subseteq \bigcup\limits_{r\in\Delta} S_r$ for each k. Now each S_{h_k} is compact by Theorem 3.5 and it follows that $S_{h_k} \subseteq \bigcup\limits_{i=1}^{d_k} S_{r_{k,i}}$, where $d_k \geq 1$ depends on k and $r_{k,i} \in \Delta$, for any $k = 1, ..., L$ and $i = 1, ..., d_k$. Therefore $C_1 \cap C_2 = \bigcup\limits_{k=1}^{L} S_{h_k} \subseteq \bigcup\limits_{k=1}^{L}\bigcup\limits_{i=1}^{d_k} S_{r_{k,i}}$ and thus $\overline{\zeta} = \{S_{r_{k,i}} \mid k = 1, ..., L, i = 1, ..., d_k\}$ is a finite subcover for $C_1 \cap C_2$. The other part of the theorem is trivially true. □

4. Irreducibility in $\mathcal{PS}_G(M)$

Let M be a G-graded R-module and Y be a subset of $\mathcal{PS}_G(M)$. We will denote the closure of Y in $\mathcal{PS}_G(M)$ by $Cl(Y)$ and the intersection $\bigcap\limits_{Q\in Y} Gr_M(Q)$ by $\eta(Y)$. If Z is a subset of $Spec_G(R)$ or $Spec_G(M)$, then the intersection of all members of Z will be expressed by $\gamma(Z)$.

Proposition 4.1. *Let M be a G-graded R-module and $Y \subseteq \mathcal{PS}_G(M)$. Then $Cl(Y) = \nu_G(\eta(Y))$. Thus, Y is closed in $\mathcal{PS}_G(M)$ if and only if $\nu_G(\eta(Y)) = Y$.*

Proof. Let $\nu_G(N)$ be any closed set containing Y, where $N \leq_G M$. Note that $Y \subseteq \nu_G(\eta(Y))$, and so it is enough to show that $\nu_G(\eta(Y)) \subseteq \nu_G(N)$. So let $Q \in \nu_G(\eta(Y))$. Then $(\eta(Y) :_R M) \subseteq (Gr_M(Q) :_R M)$. Note that for any $Q' \in Y$, we have $(N :_R M) \subseteq (Gr_M(Q') :_R M)$ and hence $(N :_R M) \subseteq \bigcap\limits_{Q'\in Y}(Gr_M(Q') :_R M) = (\bigcap\limits_{Q'\in Y} Gr_M(Q') :_R M) = (\eta(Y) :_R M) \subseteq (Gr_M(Q) :_R M)$. Thus $Q \in \nu_G(N)$. Therefore $\nu_G(\eta(Y))$ is the smallest closed set containing Y and hence $Cl(Y) = \nu_G(\eta(Y))$. □

Recall that a topological space X is irreducible if any two non-empty open subsets of X intersect. Equivalently, X is irreducible if for any decomposition $X = F_1 \cup F_2$ with closed subsets F_i of X with $i = 1, 2$, we have $F_1 = X$ or $F_2 = X$. A subset X' of X is irreducible if it is an irreducible topological space with the induced topology. Let X be a topological space. Then a subset Y of X is irreducible if and only if its closure is irreducible. Also every singleton subset of X is irreducible, (see [6]).

Theorem 4.2. *Let M be a G-graded R-module. Then for each $Q \in \mathcal{PS}_G(M)$, the closed set $\nu_G(Q)$ is irreducible closed subset of $\mathcal{PS}_G(M)$. In particular, if $\{0\} \in \mathcal{PS}_G(M)$, then $\mathcal{PS}_G(M)$ is irreducible.*

Proof. For any $Q \in \mathcal{PS}_G(M)$, we have $Cl(\{Q\}) = \nu_G(\eta(\{Q\})) = \nu_G(Gr_M(Q)) = \nu_G(Q)$. Now $\{Q\}$ is irreducible in $\mathcal{PS}_G(M)$, then its closure $\nu_G(Q)$ is irreducible. The other part of the theorem follows from the equality $\nu_G(\{0\}) = \mathcal{PS}_G(M)$. □

In Theorem 4.2, if we drop the condition that $Q \in \mathcal{PS}_G(M)$, then $\nu_G(Q)$ might not be irreducible. Actually, $\mathcal{PS}_G(M)$ itself is not always irreducible. For this, if we take $R = \mathbb{Z}_6$ as $\mathbb{Z}_2$-graded $\mathbb{Z}_6$ module by $R_0 = R$ and $R_1 = \{0\}$. Then it can easily be checked that $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) = \{\{0,3\},\{0,2,4\}\}$, $\nu_{\mathbb{Z}_2}(3\mathbb{Z}_6) = \{\{0,3\}\}$ and $\nu_{\mathbb{Z}_2}(2\mathbb{Z}_6) = \{\{0,2,4\}\}$. Now $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) = \nu_{\mathbb{Z}_2}(3\mathbb{Z}_6) \cup \nu_{\mathbb{Z}_2}(2\mathbb{Z}_6)$. But $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) \neq \nu_{\mathbb{Z}_2}(3\mathbb{Z}_6)$ and $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) \neq \nu_{\mathbb{Z}_2}(2\mathbb{Z}_6)$. Therefore $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) = \nu_{\mathbb{Z}_2}(\{0\})$ is not irreducible.

Now, we need the following lemma to prove the next theorem.

Lemma 4.3 ([10, Lemma 4.9])**.** *A subset Y of $Spec_G(R)$ for any graded ring R is irreducible if and only if $\gamma(Y)$ is a graded prime ideal of R.*

Proof. $\Rightarrow$: Let Y be irreducible subset of $Spec_G(R)$ and $r_1, r_2 \in h(R)$ with $r_1 r_2 \in \gamma(Y)$. Then $r_1 r_2 \in p$ for any $p \in Y$. Let $U_1 = Y \cap (Spec_G(R) - V_G^R(r_1 R))$ and $U_2 = Y \cap (Spec_G(R) - V_G^R(r_2 R))$. If U_1, U_2 are non-empty sets, then $U_1 \cap U_2 \neq \varnothing$ as Y is irreducible and U_1, U_2 are open sets in Y. So $\exists p \in Y$ such that $r_1 R \nsubseteq p$ and $r_2 R \nsubseteq p$. It follows that $r_1 \notin p$ and $r_2 \notin p$ and hence $r_1 r_2 \notin p$ as $p \in Spec_G(R)$, a contradiction. Therefore $U_1 = \varnothing$ or $U_2 = \varnothing$. If $U_1 = \varnothing$, then $Y \subseteq V_G^R(r_1 R)$. This implies that $r_1 R \subseteq Q$ for any $Q \in Y$ and thus $r_1 R \subseteq \bigcap_{Q \in Y} Q = \gamma(Y)$. Therefore $r_1 \in \gamma(Y)$. Similarly, if $U_2 \neq \varnothing$, then $r_2 \in \gamma(Y)$. Hence $r_1 \in \gamma(Y)$ or $r_2 \in \gamma(Y)$.

$\Leftarrow$: Assume that $\gamma(Y)$ is a graded prime ideal of R, where $Y \subseteq Spec_G(R)$. Let $Y = F_1 \cup F_2$, where F_1, F_2 are closed sets in Y. Now $F_1 = V_G^R(I_1) \cap Y$ and $F_2 = V_G^R(I_2) \cap Y$ for some $I_1, I_2 \lhd_G R$. It follows that $Y = Y \cap V_G^R(I_1 \cap I_2)$ and hence $Y \subseteq V_G^R(I_1 \cap I_2)$, which implies that $I_1 \cap I_2 \subseteq p$ for any $p \in Y$. Thus $I_1 \cap I_2 \subseteq \gamma(Y)$. Since $\gamma(Y) \in Spec_G(R)$, then $I_1 \subseteq \gamma(Y)$ or $I_2 \subseteq \gamma(Y)$. If $I_1 \subseteq \gamma(Y)$, then $V_G^R(\gamma(Y)) \subseteq V_G^R(I_1)$ and thus $Y \subseteq V_G^R(I_1)$ as $Y \subseteq V_G^R(\gamma(Y))$. It follows that $F_1 = Y$. Similarly, if $I_2 \subseteq \gamma(Y)$, we obtain $F_2 = Y$. This proves that $F_1 = Y$ or $F_2 = Y$, hence, Y is irreducible. □

Theorem 4.4. *Let M be a G-graded R-module and $Y \subseteq \mathcal{PS}_G(M)$. Then:*

(1) If $\eta(Y)$ is graded primary submodule of M, then Y is irreducible.
(2) If Y is irreducible, then $\Upsilon = \{(Gr_M(Q) :_R M) \mid Q \in Y\}$ is an irreducible subset of $Spec_G(R)$, i.e., $\gamma(\Upsilon) = (\eta(Y) :_R M) \in Spec_G(R)$.

Proof. (1) Assume that $\eta(Y)$ is a graded primary submodule of M, then it is easy to see that $\eta(Y) \in \mathcal{PS}_G(M)$. By Theorem 4.2 and Proposition 4.1, we have $\nu_G(\eta(Y)) = Cl(Y)$ is irreducible and hence Y is irreducible.

(2) Assume that Y is irreducible. Then $\rho(Y) = Y'$ is irreducible subset of $Spec_G(\overline{R})$ as ρ is continuous by Proposition 2.9. Note that $\gamma(Y') = \gamma(\rho(Y)) = \bigcap_{Q\in Y} \overline{(Gr_M(Q) :_R M)} = \overline{(\bigcap_{Q\in Y} Gr_M(Q) :_R M)} = \overline{(\eta(Y) :_R M)}$ and hence $\gamma(Y') = \overline{(\eta(Y) :_R M)} \in Spec_G(\overline{R})$ by Lemma 4.3. It follows that $(\eta(Y) :_R M) \in Spec_G(R)$. Now $\gamma(\Upsilon) = \bigcap_{Q\in Y} (Gr_M(Q) :_R M) = (\bigcap_{Q\in Y} Gr_M(Q) :_R M) = (\eta(Y) :_R M) \in Spec_G(R)$ and thus Υ is irreducible subset of $Spec_G(R)$ by Lemma 4.3 again. □

Let X be a topological space and Y be a closed subset of X. An element $y \in Y$ is called a generic point if $Y = Cl(\{y\})$. An irreducible component of X is a maximal irreducible subset of X. The irreducible components of X are closed and they cover X, (see [9]).

Theorem 4.5. *Let M be a G-graded R-module. Let $Y \subseteq \mathcal{PS}_G(M)$ and ρ be surjective. Then Y is an irreducible closed subset of $\mathcal{PS}_G(M)$ if and only if $Y = \nu_G(Q)$ for some $Q \in \mathcal{PS}_G(M)$. Hence every irreducible closed subset of $\mathcal{PS}_G(M)$ has a generic point.*

Proof. Assume that Y is an irreducible closed subset of $\mathcal{PS}_G(M)$. Then $Y = \nu_G(N)$ for some $N \leq_G M$. Also $(\eta(Y) :_R M) = (\eta(\nu_G(N)) :_R M) \in Spec_G(R)$ by Theorem 4.4. It follows that $\overline{(\eta(Y) :_R M)} \in Spec_G(\overline{R})$ and hence $\exists Q \in \mathcal{PS}_G(M)$ such that $(Gr_M(Q) :_R M) = (\eta(\nu_G(N)) :_R M)$ as ρ is surjective. So $Gr((Q :_R M)) = Gr((\eta(\nu_G(N)) :_R M))$ and so $\nu_G(Q) = \nu_G(\eta(\nu_G(N))) = Cl(\nu_G(N)) = \nu_G(N) = Y$ by Proposition 4.1 and Lemma 2.4 (3). Conversely, if $Y = \nu_G(Q)$ for some $Q \in \mathcal{PS}_G(M)$, then Y is irreducible by Theorem 4.2. □

Theorem 4.6. *Let M be a G-graded R-module and $Q \in \mathcal{PS}_G(M)$. If $\overline{(Gr_M(Q) :_R M)}$ is a minimal graded prime ideal of $\overline{R}$, then $\nu_G(Q)$ is irreducible component of $\mathcal{PS}_G(M)$. The converse is true if ρ is surjective.*

Proof. Note that $\nu_G(Q)$ is irreducible by Theorem 4.2 and it remains to show that it is a maximal irreducible. Let Y be irreducible subset of $\mathcal{PS}_G(M)$ with $\nu_G(Q) \subseteq Y$ and if we show that $Y = \nu_G(Q)$, then we are done. Since $Q \in \nu_G(Q) \subseteq Y$, then $Q \in Y$ and thus $\overline{(\eta(Y) :_R M)} \subseteq \overline{(Gr_M(Q) :_R M)}$. It follows that $\overline{(\eta(Y) :_R M)} = \overline{(Gr_M(Q) :_R M)}$ as $\overline{(Gr_M(Q) :_R M)}$ is a minimal graded prime ideal of $\overline{R}$ and $\overline{(\eta(Y) :_R M)} \in Spec_G(\overline{R})$ by Theorem 4.4 (2). Hence $V_G^{\overline{R}}(\overline{(\eta(Y) :_R M)}) = V_G^{\overline{R}}(\overline{(Gr_M(Q) :_R M)})$, which implies that $\nu_G(\eta(Y)) = \rho^{-1}(V_G^{\overline{R}}(\overline{(\eta(Y) :_R M)})) = \rho^{-1}(V_G^{\overline{R}}(\overline{(Gr_M(Q) :_R M)})) = \nu_G(Gr_M(Q)) = \nu_G(Q)$ by Proposition 2.9 and Lemma 2.4 (4). Since $Y \subseteq \nu_G(\eta(Y))$, then $Y \subseteq \nu_G(Q)$ and thus $Y = \nu_G(Q)$. For the converse, we assume that ρ is surjective. Since $Q \in \mathcal{PS}_G(M)$, then $\overline{(Gr_M(Q) :_R M)} \in Spec_G(\overline{R})$. Let $\overline{J} \in Spec_G(\overline{R})$ with $\overline{J} \subseteq \overline{(Gr_M(Q) :_R M)}$ and it is enough to show that $\overline{J} = \overline{(Gr_M(Q) :_R M)}$. Note that $\exists Q' \in \mathcal{PS}_G(M)$ such that $\rho(Q') = \overline{J}$ as ρ is surjective. So we have $J = (Gr_M(Q') :_R M)$. Now $(Gr_M(Q') :_R M) \subseteq (Gr_M(Q) :_R M)$ and thus

$\nu_G(Q) \subseteq \nu_G(Q')$. Since $\nu_G(Q)$ is irreducible component and $\nu_G(Q')$ is irreducible by Theorem 4.2, then $\nu_G(Q) = \nu_G(Q')$. By Lemma 2.4 (3), we get $(Gr_M(Q) :_R M) = (Gr_M(Q') :_R M) = J$ and thus $\overline{J} = \overline{(Gr_M(Q) :_R M)}$. $\square$

Corollary 4.7. *Let M be a G-graded R-module and $\mathcal{K} = \{Q \in \mathcal{PS}_G(M) \mid \overline{(Gr_M(Q) :_R M)}$ is a minimal graded prime ideal of $\overline{R}\}$. If ρ is surjective, then the following hold:*

(1) $T = \{\nu_G(Q) \mid Q \in \mathcal{K}\}$ is the set of all irreducible components of $\mathcal{PS}_G(M)$.

(2) $\mathcal{PS}_G(M) = \bigcup\limits_{Q\in\mathcal{K}} \nu_G(Q)$.

(3) $Spec_G(\overline{R}) = \bigcup\limits_{Q\in\mathcal{K}} V_G^{\overline{R}}(\overline{(Q :_R M)})$.

(4) $Spec_G(M) = \bigcup\limits_{Q\in\mathcal{K}} V_G(Q)$.

(5) If $\{0\} \in Spec_G(M)$, then the only irreducible component subset of $\mathcal{PS}_G(M)$ is $\mathcal{PS}_G(M)$ itself.

Proof. (1) follows from Theorem 4.5 and Theorem 4.6.
(2) Since any topological space is the union of its irreducible components, then $\mathcal{PS}_G(M) = \bigcup\limits_{Y\in T} Y = \bigcup\limits_{Q\in\mathcal{K}} \nu_G(Q)$.
(3) Since ρ is surjective, then $Spec_G(\overline{R}) = \rho(\mathcal{PS}_G(M)) = \rho(\bigcup\limits_{Q\in\mathcal{K}} \nu_G(Q)) = \bigcup\limits_{Q\in\mathcal{K}} \rho(\nu_G(Q)) = \bigcup\limits_{Q\in\mathcal{K}} V_G^{\overline{R}}(\overline{(Q :_R M)})$ by Proposition 2.10.
(4) $Spec_G(M) = \mathcal{PS}_G(M) \cap Spec_G(M) = (\bigcup\limits_{Q\in\mathcal{K}} \nu_G(Q)) \cap Spec_G(M) = \bigcup\limits_{Q\in\mathcal{K}} V_G(Q)$ by Lemma 2.4 (1).
(5) Assume that $\{0\} \in Spec_G(M)$. Then $\overline{(\{0\} :_R M)} \in Spec_G(\overline{R})$. For any $Q \in \mathcal{K}$, we have $\overline{(\{0\} :_R M)} \subseteq \overline{(Gr_M(Q) :_R M)}$ and hence $\overline{(\{0\} :_R M)} = \overline{(Gr_M(Q) :_R M)}$ as $\overline{(Gr_M(Q) :_R M)}$ is a minimal graded prime ideal of $\overline{R}$. Therefore $Gr((Q :_R M)) = Gr((\{0\} :_R M))$ and thus $\nu_G(Q) = \nu_G(\{0\}) = \mathcal{PS}_G(M)$ by Lemma 2.4 (3). By (1), the set of all irreducible components of $\mathcal{PS}_G(M)$ is $T = \{\nu_G(Q) \mid Q \in \mathcal{K}\} = \{\mathcal{PS}_G(M)\}$ which completes the proof. $\square$

Proposition 4.8. *Let R be a G-graded principal ideal domain and M be a multiplication graded R-module. Let $Y \subseteq \mathcal{PS}_G(M)$. If $\eta(Y)$ is a non-zero graded primary submodule of M, then $Y \subseteq \mathcal{PS}_G^p(M)$ for some graded maximal ideal p of R.*

Proof. Clearly, $\eta(Y) = Gr_M(\eta(Y))$. Since $\eta(Y)$ is a graded primary submodule of the graded multiplication module M, then $\eta(Y) \in Spec_G(M)$ by [15, Theorem 13] and hence $(\eta(Y) :_R M) \in Spec_G(R)$. If $(\eta(Y) :_R M) = \{0\}$, then $\eta(Y) = (\eta(Y) :_R M)M = \{0\}$, a contradiction. So $(\eta(Y) :_R M)$ is a non-zero graded prime ideal in the graded principle ideal domain R and thus $(\eta(Y) :_R M)$ is a graded maximal ideal of R. It follows that $\eta(Y)$ is a graded maximal submodule of M as M is a graded multiplication module. Now for any $Q \in Y \subseteq \mathcal{PS}_G(M)$, we have $\eta(Y) \subseteq Gr_M(Q) \neq M$ and thus $\eta(Y) = Gr_M(Q)$. This implies that $(Gr_M(Q) :_R M) = (\eta(Y) :_R M)$ for any $Q \in Y$. Take $p = (\eta(Y) :_R M)$. Therefore $Y \subseteq \mathcal{PS}^p_G(M)$. □

A topological space X is called a T_1-space if every singleton subset of X is closed. A G-graded R-module M is called graded finitely generated R-module if there are $m_1, m_2, ..., m_k \in h(M)$ such that $M = \sum_{i=1}^{k} Rm_i$.

Proposition 4.9. *Let M be a G-graded finitely generated R-module. If $\mathcal{PS}_G(M)$ is a T_1-space, then $\mathcal{PS}_G(M) = Max_G(M) = Spec_G(M)$, where $Max_G(M)$ is the set of all graded maximal submodule of M.*

Proof. It is clear that $Max_G(M) \subseteq \mathcal{PS}_G(M)$. Now, let $Q \in \mathcal{PS}_G(M)$. Since $\mathcal{PS}_G(M)$ is a T_1-space, then $Cl(\{Q\}) = \{Q\}$ and so $\nu_G(Q) = \{Q\}$ by Proposition 4.1. As $M \neq Q$ is a graded finitely generated module, we obtain M/Q is a non-zero graded finitely generated module and hence $\exists N \leq_G M$ with $Q \subseteq N$ such that $N/Q \in Max_G(M/Q)$ by [4, Lemma 2.7 (ii)]. Now, it is easy to see that $N \in Max_G(M) \subseteq \mathcal{PS}_G(M)$. Since $(Q :_R M) \subseteq (N :_R M) = (Gr_M(N) :_R M)$ and $N \in \mathcal{PS}_G(M)$, then $N \in \nu_G(Q) = \{Q\}$ and thus $N = Q \in Max_G(M)$. Therefore $Max_G(M) = \mathcal{PS}_G(M)$. Now, $Spec_G(M) \subseteq \mathcal{PS}_G(M) = Max_G(M) \subseteq Spec_G(M)$. Hence $Spec_G(M) = \mathcal{PS}_G(M) = Max_G(M)$. □

A topological space is called a T_0-space if the closure of any two distinct points are distinct. A topological space is called spectral space if it is homeomorphic to the prime spectrum of a ring equipped with the Zariski topology. Spectral spaces have been characterized by Hochster[9, Proposition 4] as the topological spaces X which satisfy the following conditions:
(a) X is a T_0-space.
(b) X is compact.
(c) The compact open subsets of X are closed under finite intersection and form an open base.
(d) each irreducible closed subset of X has a generic point.

Theorem 4.10. *Let M be a G-graded R-module and ρ be surjective. Then the following statements are equivalent:*

(1) $\mathcal{PS}_G(M)$ is a T_0-space.
(2) If whenever $\nu_G(Q) = \nu_G(Q')$ with $Q, Q' \in \mathcal{PS}_G(M)$, then $Q = Q'$.
(3) ρ is injective.
(4) $|\mathcal{PS}^p_G(M)| \leq 1$ for every $p \in Spec_G(R)$.
(5) $\mathcal{PS}_G(M)$ is a spectral space.

Proof. The equivalence of (2), (3) and (4) is proved in Proposition 2.6. Also (1), (5) are equivalent by Theorem 3.5, Theorem 3.6 and Theorem 4.5. For (1)⇒(2), assume that $\nu_G(Q) = \nu_G(Q')$ for $Q, Q' \in \mathcal{PS}_G(M)$, then $Cl(\{Q\}) = \nu_G(Q) = \nu_G(Q') = Cl(\{Q'\})$ and hence $Q = Q'$ as $\mathcal{PS}_G(M)$ is T_0 space. For

(2)⇒(1), let $Q, Q' \in \mathcal{PS}_G(M)$ with $Q \neq Q'$, then by the assumption (2) we have $\nu_G(Q) \neq \nu_G(Q')$. Hence $Cl(\{Q\}) \neq Cl(\{Q'\})$. Therefore $\mathcal{PS}_G(M)$ is a T_0-space. □

Acknowledgements. *The authors wish to thank sincerely the referees for their valuable comments and suggestions.*

References

[1] K. Al-Zoubi, The graded primary radical of a graded submodules, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 1 (2016), 395–402.
[2] K. Al-Zoubi, I. Jaradat and M. Al-Dolat, On graded P-compactly packed modules, Open Mathematics 13, no. 1 (2015), 487–492.
[3] S. E. Atani, On graded prime submodules, Chiang Mai J. Sci. 33, no. 1 (2006), 3–7.
[4] S. E. Atani and F. Farzalipour, Notes on the graded prime submodules, Int. Math. Forum. 1, no. 38 (2006), 1871–1880.
[5] S. E. Atani and F. Farzalipour, On graded secondary modules, Turk. J. Math. 31 (2007), 371–378.
[6] N. Bourbaki, Commutative Algebra. Chapter 1–7. Springer-Verlag, Berlin, 1989.
[7] A. Y. Darani, Topology on $Spec_g(M)$, Buletinul Academiei De Stiinte 3 (67) (2011), 45–53.
[8] J. Escoriza and B. Torrecillas, Multiplication objects in commutative Grothendieck categories, Comm. in Algebra 26, no. 6 (1998), 1867–1883.
[9] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43–60.
[10] M. Jaradat and K. Al-Zoubi, The Quasi-Zariski topology on the graded quasi-primary spectrum of a graded module over a graded commutative ring, preprint (arXiv:2106.09519).
[11] C. Nastasescu and V. F. Oystaeyen, Graded and filtered rings and modules, Lecture Notes in Mathematics Vol. 758, Springer-Verlag, Berlin, 2004.
[12] C. Nastasescu and F. Van Oystaeyen, Graded ring theory, Mathematical Library 28, North Holand, Amsterdam, 1982.
[13] C. Nastasescu and V. F. Oystaeyen, Methods of Graded Rings, Lecture Notes in Math., Vol. 1836, Berlin-Heidelberg: Springer-Verlag, 2004.
[14] N. A. Ozkirisci, K. H. Oral and U. Tekir, Graded prime spectrum of a graded module, Iran. J. Sci. Technol. 37A3 (2013), 411–420.
[15] K. H. Oral, U. Tekir and A. G. Agargun, On graded prime and primary submodules, Turk. J. Math. 35 (2011), 159–167.
[16] M. Refai, On properties of G-spec(R), Sci. Math. Jpn. 53, no. 3 (2001), 411–415.
[17] M. Refai and K. Al-Zoubi, On graded primary ideals, Turk. J. Math. 28 (2004), 217–229.
[18] M. Refai, M. Hailat and S. Obiedat, Graded radicals on graded prime spectra, Far East J. of Math. Sci., part I (2000), 59–73.

