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Classical solutions for the Euler equations of compressible fluid dynamics: A new topological approach

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Abstract

In this article we study a class of Euler equations of compressible fluid dynamics. We give conditions under which the considered equations have at least one and at least two classical solutions. To prove our main results we propose a new approach based upon recent theoretical results.

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1. Introduction

In this paper, we investigate an initial value problem for Euler equations of compressible fluid dynamics, see [6], [10], [21]. Namely, we are concerned with

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the following system:

$$
(1.1) \qquad \begin{aligned}
\partial_t \rho + \partial_x(\rho u) &= 0, \\
\partial_t(\rho u) + \partial_x\left(\rho u^2 + p(\rho)\right) &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
\rho(0,x) &= \rho_0(x), \quad x \in \mathbb{R}, \\
u(0,x) &= u_0(x), \quad x \in \mathbb{R},
\end{aligned}
$$

where

(H1): $\rho_0, u_0 \in \mathcal{C}^1(\mathbb{R})$, $0 \leq \rho_0(x), u_0(x) \leq B$, $x \in \mathbb{R}$, with B is a given positive constant.

Here the unknowns $\rho = \rho(t,x) \geq 0$ and $u = u(t,x)$ denote respectively, the density and the velocity of the gas, while the pressure $p = p(\rho)$ is a given function so that

(H2): $p \in \mathcal{C}(\mathbb{R})$ is a nonnegative function for which $p(z) \leq Cz^q$, $z \geq 0$, C is a positive constant, $q \geq 0$.

Note that if

$$p(\rho) = C\rho^q, \ \rho \geq 0, \ C > 0, \ q \geq 1,$$

then, the fluid is called isentropic and isothermal when $q > 1$ and $q = 1$, respectively. For other possibilities of the pressure function, readers may refer to [5] and the references therein. Cauchy problem with bounded measurable initial data for (1.1):

$$(\rho_0, u_0) \in L^\infty \times L^\infty$$

where $u_0(x)$ and $\rho_0(x) \geq 0$ ($\not\equiv 0$) is studied in [4]. The authors established the convergence of a second-order shock-capturing scheme. In [7], a convergence result for the method of artificial viscosity applied to the isentropic equations of gas dynamics is established. In [20], some properties for solutions of (1.1) containing a portion of the $t - x$ plane in which $\rho = 0$ called vacuum state, were investigated. Conservation laws of the one-dimensional isentropic gas dynamics equations in Lagrangian coordinates are obtained in [16]. In [19], a 2×2 hyperbolic system of isentropic gas dynamics, in both Eulerian or Lagrangian variables is considered.

Whereas local existence results for problems of type (1.1) were obtained, see, for example, [2], [3], [13],[17], [18], [22], [23], the literature concerning global existence of solutions for such kind of problems does not seem to be very rich. The problem of the global in time existence of solutions of the equations of fluid mechanics in one space dimension was treated by Glimm in 1965 [12]. The equations (1.1) was investigated in [11] for existence of global periodic solutions. For Euler equations with damping, the global existence of solutions can be found in [24], [27], [30] and the references therein. In [5], a class of conditions for non-existence of global classical solutions is established for the initial-boundary value problem of a three-dimensional compressible Euler equations with (or without) time-dependent damping. We mention also the works [15], [26] and [28].

The aim of this paper is to investigate the IVP (1.1) for existence of global classical solutions. We call a solution a classical solution if it, along with its derivatives that appear in the equations, is of class $\mathcal{C}([0,\infty)\times\mathbb{R})$.

Our main result for existence of classical solutions of the IVP (1.1) is as follows.

Theorem 1.1. *Suppose* $(H1)$-$(H2)$. *Then, the IVP* (1.1) *has at least one nonnegative solution* $(\rho,u)\in\mathcal{C}^1([0,\infty)\times\mathbb{R})\times\mathcal{C}^1([0,\infty)\times\mathbb{R})$.

Theorem 1.2. *Suppose* $(H1)$-$(H2)$. *Then, the IVP* (1.1) *has at least two nonnegative solutions* $(\rho_1,u_1),(\rho_2,u_2)\in\mathcal{C}^1([0,\infty)\times\mathbb{R})\times\mathcal{C}^1([0,\infty)\times\mathbb{R})$.

The strategy for the proof of Theorem 1.1 and Theorem 1.2 which we develop in Section 2 uses the abstract theory of the sum of two operators. This basic and new idea yields global existence theorems for many of the interesting equations of mathematical physics.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we give an example to illustrate our main results.

2. Preliminaries and auxiliary results

2.1. Preliminaries.

To prove our existence results we will use Theorem 2.1 and Theorem 2.8, that we will present and demonstrate in the sequel.

Theorem 2.1. *Let* $\epsilon>0$, $R>0$, E *be a Banach space and*

$$X=\{x\in E:\|x\|\le R\}.$$

Let also, $Tx=-\epsilon x$, $x\in X$, $S:X\to E$ *is a continuous,* $(I-S)(X)$ *resides in a compact subset of* E *and*

$$\{x\in E: x=\lambda(I-S)x,\quad \|x\|=R\}=\varnothing \tag{2.1}$$

for any $\lambda\in\left(0,\frac{1}{\epsilon}\right)$. *Then, there exists* $x^*\in X$ *so that*

$$Tx^*+Sx^*=x^*.$$

Proof. Define

$$r\left(-\frac{1}{\epsilon}x\right)=\begin{cases}-\frac{1}{\epsilon}x & \text{if}\quad \|x\|\le R\epsilon\\[2ex] \frac{Rx}{\|x\|} & \text{if}\quad \|x\|>R\epsilon.\end{cases}$$

Then, $r\left(-\frac{1}{\epsilon}(I-S)\right):X\to X$ is continuous and compact. Hence and the Schauder fixed point theorem, it follows that there exists $x^*\in X$ so that

$$r\left(-\frac{1}{\epsilon}(I-S)x^*\right)=x^*.$$

Assume that $-\frac{1}{\epsilon}(I-S)x^*\notin X$. Then,

$$\Big\|(I-S)x^*\Big\|>R\epsilon,\quad \frac{R}{\|(I-S)x^*\|}<\frac{1}{\epsilon}$$

and

$$x^* = \frac{R}{\|(I-S)x^*\|}(I-S)x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right)$$

and hence, $\|x^*\| = R$. This contradicts with (2.1). Therefore, $-\frac{1}{\epsilon}(I-S)x^* \in X$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = -\frac{1}{\epsilon}(I-S)x^*$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

This completes the proof. □

Let E be a real Banach space.

Definition 2.2. A closed, convex set $\mathcal{P}$ in E is said to be cone if

(1) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,
(2) $x, -x \in \mathcal{P}$ implies $x = 0$.

Every cone $\mathcal{P}$ defines a partial ordering $\leq$ in E defined by :

$$x \leq y \text{ if and only if } y - x \in \mathcal{P}.$$

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$.

Definition 2.3. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

In what follows, we give some results about the fixed point index theory for perturbation of a completely continuous mapping by expansive one. First, we recall the definition of an expansive mapping.

Definition 2.4. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

In the following lemma, we present the key property of the expansive mappings which allows to extend the notion of the fixed point index in the case of a completely continuous mapping perturbed by an expansive one.

Lemma 2.5. [29, Lemma 2.1] *Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. Assume that the mapping $T : D \to X$ is expansive with constant $h > 1$. Then, the inverse of $I - T : D \to (I-T)(D)$ exists and*

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq \frac{1}{h-1}\|x - y\|, \ \ \forall\, x, y \in (I-T)(D).$$

In the sequel, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$.

Assume that $S : \overline{U} \to E$ is a completely continuous mapping and $T : \Omega \to E$ is a expansive one with constant $h > 1$. By Lemma 2.5, the operator $(I-T)^{-1}$ is $(h-1)^{-1}$-Lipschtzian on $(I-T)(\Omega)$. Suppose that

$$S(\overline{U}) \subset (I-T)(\Omega), \tag{2.2}$$

and

$$x \neq Tx + Sx, \ \text{ for all } \ x \in \partial U \cap \Omega. \tag{2.3}$$

Then, $x \neq (I-T)^{-1}Sx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}S : \overline{U} \to \mathcal{P}$ is a completely continuous. From [14, Theorem 2.3.1], the fixed point index $i\left((I-T)^{-1}S, U, \mathcal{P}\right)$ is well defined. Thus we put

$$i_*(T+S, U\cap\Omega, \mathcal{P}) = \begin{cases} i\,(I-T)^{-1}S, U, \mathcal{P}), & \text{if } U\cap\Omega \neq \varnothing \\ 0, & \text{if } U\cap\Omega = \varnothing. \end{cases} \tag{2.4}$$

This integer is called the generalized fixed point index of the sum $T+S$ on $U\cap\Omega$ with respect to the cone $\mathcal{P}$.

The basic properties of the index i_* are collected in the following lemma

Lemma 2.6 ([8, Theorem 2.3])**.** *The fixed point index defined in (2.4) satisfies the following properties:*

(a) (Normalization). *If* $U = \mathcal{P}_r$, $0 \in \Omega$, *and* $Sx = z_0 \in \mathcal{B}(-T0, (h-1)r) \cap \mathcal{P}$ *for all* $x \in \overline{\mathcal{P}_r}$, *then,*

$$i_*\left(T+S, \mathcal{P}_r \cap \Omega, \mathcal{P}\right) = 1.$$

(b) (Additivity). *For any pair of disjoint open subsets* U_1, U_2 *in* U *such that* $T+S$ *has no fixed point on* $(\overline{U}\setminus(U_1\cup U_2))\cap\Omega$, *we have*

$$i_*\left(T+S, U\cap\Omega, \mathcal{P}\right) = i_*\left(T+S, U_1\cap\Omega, \mathcal{P}\right) + i_*\left(T+S, U_2\cap\Omega, \mathcal{P}\right),$$

where $i_*\left(T+S, U_j\cap\Omega, X\right) := i_*\left(T+S|_{\overline{U_j}}, U_j\cap\Omega, \mathcal{P}\right), \quad j = 1, 2.$

(c) (Homotopy Invariance). *The fixed point index* $i_*\left(T+H(t,.), U\cap\Omega, \mathcal{P}\right)$ *does not depend on the parameter* $t \in [0,1]$ *whenever*

(i) $H : [0,1]\times\overline{U} \to E$ *be a completely continuous mapping,*

(ii) $H([0,1]\times\overline{U}) \subset (I-T)(\Omega)$,

(iii) $Tx + H(t,x) \neq x$, *for all* $t \in [0,1]$ *and* $x \in \partial U \cap \Omega$.

(d) (Solvability). *If* $i_*\left(T+S, U\cap\Omega, \mathcal{P}\right) \neq 0$, *Then,* $T+S$ *has a fixed point in* $U\cap\Omega$.

Several considerations allowing computation of the index i_* are shown in [8]. The following result is an extension of [8, Proposition 2.11] in the case of a completely continuous mapping perturbed by an expansive one.

Proposition 2.7. *Let* U *be a bounded open subset of* $\mathcal{P}$ *with* $0 \in U$. *Assume that* $T : \Omega \subset \mathcal{P} \to E$ *is an expansive mapping,* $S : \overline{U} \to E$ *is a completely continuous one and* $S(\overline{U}) \subset (I-T)(\Omega)$.

If $T+S$ has no fixed point on $\partial U \cap \Omega$ and there exists $\varepsilon > 0$ small enough such that

$$Sx \neq (I-T)(\lambda x) \;\; \textit{for all} \;\; \lambda \geq 1+\varepsilon, \; x \in \partial U \;\; \textit{and} \;\; \lambda x \in \Omega,$$

then, the fixed point index $i_*(T+S, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I-T)^{-1}S : \overline{U} \to \mathcal{P}$ is a completely continuous without fixed point in the boundary ∂U and it is readily seen that the following condition is satisfied

$$(I-T)^{-1}Sx \neq \lambda x \quad \text{for all} \;\; x \in \partial U \;\; \text{and} \;\; \lambda \geq 1+\varepsilon.$$

Our claim then follows from the definition of i_* and [1, Lemma 2.3]. □

Now we are able to present a multiple fixed point theorem. The proof rely on Proposition 2.7 and [8, Proposition 2.16] producing the computation of the index i_*. This result will be used to prove Theorem 1.2.

Theorem 2.8. *Let U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous one and $S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \varnothing$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \varnothing$, and there exists $v_0 \in \mathcal{P}^*$ such that the following conditions hold:*

- **(i):** *$Sx \neq (I-T)(x - \lambda v_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda v_0)$,*
- **(ii):** *there exists $\varepsilon > 0$ small enough such that $Sx \neq (I-T)(\lambda x)$, for all $\lambda \geq 1+\varepsilon$, $x \in \partial U_2$, and $\lambda x \in \Omega$,*
- **(iii):** *$Sx \neq (I-T)(x - \lambda v_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda v_0)$.*

Then,$T+S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \;\; \textit{and} \;\; x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \;\; \textit{and} \;\; x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I-T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T+S$. Suppose that $Sx \neq (I-T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the conclusion has been proved. By Proposition 2.7 and [8, Proposition 2.16], we have

$$i_*(T+S, U_1 \cap \Omega, \mathcal{P}) = i_*(T+S, U_3 \cap \Omega, \mathcal{P}) = 0 \text{ and } i_*(T+S, U_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index i_* yields

$$i_*(T+S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}) = 1 \text{ and } i_*(T+S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}) = -1.$$

Consequently, by the existence property of the index i_*, $T+S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

2.2. **Auxiliary results.** In this subsection, we give some properties of solutions of IVP (1.1). Let $X^1 = \mathcal{C}^1([0,\infty)\times\mathbb{R})$ be endowed with the norm

$$\|u\|_{X^1} = \max\Bigg\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_t(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_x(t,x)| \Bigg\},$$

provided it exists. Let $X^2 = X^1 \times X^1$ be endowed with the norm

$$\|(\rho,u)\|_{X^2} = \max\{\|\rho\|_{X^1}, \quad \|u\|_{X^1}\}, \quad (\rho,u)\in X^2,$$

provided it exists. For $(\rho,u)\in X^2$, we will write $(\rho,u)\geq 0$ if $\rho(t,x)\geq 0$, $u(t,x)\geq 0$ for any $(t,x)\in[0,\infty)\times\mathbb{R}$. For $(\rho,u)\in X^2$, define the operators

$$\begin{aligned}
S_1^1(\rho,u)(t,x) &= \int_0^x (\rho(t,x_1)-\rho_0(x_1))\,dx_1 + \int_0^t \rho(t_1,x)u(t_1,x)dt_1,\\
S_1^2(\rho,u)(t,x) &= \int_0^x (\rho(t,x_1)u(t,x_1)-\rho_0(x_1)u_0(x_1))\,dx_1\\
&\quad + \int_0^t \big(\rho(t_1,x)(u(t_1,x))^2 + p(\rho(t_1,x))\big)\,dt_1,\\
S_1(\rho,u)(t,x) &= \left(S_1^1(\rho,u)(t,x), S_1^2(\rho,u)(t,x)\right), \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}$$

Lemma 2.9. *Suppose $(H1)$ and $p\in\mathcal{C}(\mathbb{R})$. If $(\rho,u)\in X^2$ satisfies the equation*

$$S_1(\rho,u)(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}, \tag{2.5}$$

then it is a solution of the IVP (1.1).

Proof. Let $(\rho,u)\in X^2$ is a solution to the equation (2.5). Then

$$S_1^1(\rho,u)(t,x)=0, \quad S_1^2(\rho,u)(t,x)=0, \quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{2.6}$$

We differentiate the first equation of (2.6) with respect to t and x and we find

$$\rho_t(t,x) + (\rho u)_x(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

We put $t=0$ in the first equation of (2.6) and we arrive at

$$\int_0^x (\rho(0,x_1)-\rho_0(x_1))\,dx_1 = 0, \quad x\in\mathbb{R},$$

which we differentiate with respect to x and we find

$$\rho(0,x) = \rho_0(x), \quad x\in\mathbb{R}.$$

Now, we differentiate the second equation of (2.6) with respect to t and x and we find

$$(\rho u)_t(t,x) + (\rho u^2 + p(\rho))_x(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

We put $t = 0$ in the second equation of (2.6) and we get

$$\int_0^x (\rho(0,x_1)u(0,x_1) - \rho_0(x_1)u_0(x_1))\, dx_1 = 0, \quad x \in \mathbb{R},$$

which we differentiate with respect to x and we obtain

$$\rho(0,x)u(0,x) - \rho_0(x)u_0(x) = 0, \quad x \in \mathbb{R},$$

whereupon

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}.$$

Thus, (ρ, u) is a solution to the IVP (1.1). This completes the proof. $\square$

Lemma 2.10. *Suppose $(H1)$ and let $h \in \mathcal{C}([0,\infty) \times \mathbb{R})$ be a positive function almost everywhere on $[0,\infty) \times \mathbb{R}$. If $(\rho, u) \in X^2$ satisfies the following integral equations:*

$$\int_0^t \int_0^x (t-t_1)^2(x-x_1)^2 h(t_1,x_1) S_1^1(\rho,u)(t_1,x_1)dx_1dt_1 \quad = \quad 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}$$

and

$$\int_0^t \int_0^x (t-t_1)^2(x-x_1)^2 h(t_1,x_1) S_1^2(\rho,u)(t_1,x_1)dx_1dt_1 \quad = \quad 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

then, (ρ, u) is a solution to the IVP (1.1).

Proof. We differentiate three times with respect to t and three times with respect to x the integral equations of Lemma 2.10 and we find

$$h(t,x)S_1(\rho,u)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1(\rho,u)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Hence and Lemma 2.9, we conclude that (ρ, u) is a solution to the IVP (1.1). This completes the proof. $\square$

Let

$$B_1 = \max\{B, B^2, B^3, CB^q\}.$$

Lemma 2.11. *Suppose $(H1)$ and $(H2)$. For $(\rho, u) \in X^2$ with $\|(\rho,u)\|_{X^2} \leq B$, we have*

$$|S_1^1(\rho,u)(t,x)| \quad \leq \quad 2B_1(1+t)(1+|x|),$$

$$|S_1^2(\rho,u)(t,x)| \quad \leq \quad 2B_1(1+t)(1+|x|), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. We have

$$
\begin{aligned}
|S_1^1(\rho,u)(t,x)| &= \Big|\int_0^x (\rho(t,x_1)-\rho_0(x_1))\,dx_1 + \int_0^t \rho(t_1,x)u(t_1,x)dt_1\Big| \\
&\le \Big|\int_0^x (\rho(t,x_1)-\rho_0(x_1))\,dx_1\Big| + \Big|\int_0^t \rho(t_1,x)u(t_1,x)dt_1\Big| \\
&\le \Big|\int_0^x (|\rho(t,x_1)|+\rho_0(x_1))\,dx_1\Big| + \int_0^t |\rho(t_1,x)||u(t_1,x)|dt_1 \\
&\le 2B|x| + B^2 t \\
&\le 2B_1|x| + B_1 t \\
&\le 2B_1(1+|x|)(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|S_1^2(\rho,u)(t,x)\Big| &= \Big|\int_0^x (\rho(t,x_1)u(t,x_1)-\rho_0(x_1)u_0(x_1))\,dx_1 \\
&\quad + \int_0^t \big(\rho(t_1,x)(u(t_1,x))^2 + p(\rho(t_1,x))\big)\,dt_1\Big| \\
&\le \Big|\int_0^x (\rho(t,x_1)u(t,x_1)-\rho_0(x_1)u_0(x_1))\,dx_1\Big| \\
&\quad + \Big|\int_0^t \big(\rho(t_1,x)(u(t_1,x))^2 + p(\rho(t_1,x))\big)\,dt_1\Big| \\
&\le \Big|\int_0^x (|\rho(t,x_1)||u(t,x_1)|+\rho_0(x_1)u_0(x_1))\,dx_1\Big| \\
&\quad + \int_0^t \big(|\rho(t_1,x)|(u(t_1,x))^2 + C(\rho(t_1,x))^q\big)\,dt_1 \\
&\le 2B^2|x| + B^3 t + CB^q t \\
&\le 2B_1|x| + 2B_1 t \\
&\le 2B_1(1+|x|)(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}
$$

This completes the proof. $\square$

3. Proof of Theorem 1.1

(A1): Let A be a positive constant such that $A \leq 1$ and $g \in \mathcal{C}([0,\infty)\times\mathbb{R})$ is a nonnegative function such that

$$16(1+t)\left(1+t+t^2\right)(1+|x|)\left(1+|x|+x^2\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \leq A,$$

$(t,x)\in[0,\infty)\times\mathbb{R}$.

In the last section, we will give an example for a function g that satisfies $(A1)$.

For $(\rho,u)\in X^2$, define the operators

$$\begin{aligned}
S_2^1(\rho,u)(t,x) &= \int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)S_1^1(\rho,u)(t_1,x_1)dx_1dt_1,\\
S_2^2(\rho,u)(t,x) &= \int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)S_1^2(\rho,u)(t_1,x_1)dx_1dt_1,\\
S_2(\rho,u)(t,x) &= \left(S_2^1(\rho,u)(t,x), S_2^2(\rho,u)(t,x)\right), \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}$$

Lemma 3.1. *Suppose $(H1)$-$(H2)$. For $(\rho,u)\in X^2$, $\|(\rho,u)\|_{X^2}\leq B$, we have*

$$\|S_2(\rho,u)\|_{X^2}\leq AB_1,$$

where

$$B_1=\max\{B,B^2,B^3,CB^q\}.$$

Proof. We have

$$\begin{aligned}
|S_2^1(\rho,u)(t,x)| &= \left|\int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)S_1^1(\rho,u)(t_1,x_1)dx_1dt_1\right|\\
&\leq \int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)|S_1^1(\rho,u)(t_1,x_1)|dx_1\right|dt_1\\
&\leq 2B_1\int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)(1+t_1)(1+|x_1|)dx_1\right|dt_1\\
&\leq 8B_1(1+t)t^2(1+|x|)x^2\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\leq 16B_1(1+t)(1+t+t^2)(1+|x|)(1+|x|+x^2)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\leq AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial t}S_2^1(\rho,u)(t,x)\right| &= 2\left|\int_0^t\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)S_1^1(\rho,u)(t_1,x_1)dx_1dt_1\right|\\
&\le 2\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)|S_1^1(\rho,u)(t_1,x_1)|dx_1\right|dt_1\\
&\le 4B_1\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)(1+t_1)(1+|x_1|)dx_1\right|dt_1\\
&\le 16B_1(1+t)t(1+|x|)x^2\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le 16B_1(1+t)(1+t+t^2)(1+|x|)(1+|x|+x^2)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial x}S_2^1(\rho,u)(t,x)\right| &= 2\left|\int_0^t\int_0^x (t-t_1)^2(x-x_1) g(t_1,x_1)S_1^1(\rho,u)(t_1,x_1)dx_1dt_1\right|\\
&\le 2\int_0^t\left|\int_0^x (t-t_1)^2|x-x_1| g(t_1,x_1)|S_1^1(\rho,u)(t_1,x_1)|dx_1\right|dt_1\\
&\le 4B_1\int_0^t\left|\int_0^x (t-t_1)^2|x-x_1| g(t_1,x_1)(1+t_1)(1+|x_1|)dx_1\right|dt_1\\
&\le 8B_1(1+t)t^2(1+|x|)|x|\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le 16B_1(1+t)(1+t+t^2)(1+|x|)(1+|x|+x^2)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}$$

As above,

$$\left|S_2^2(\rho,u)(t,x)\right|, \quad \left|\frac{\partial}{\partial t}S_2^2(\rho,u)(t,x)\right|, \quad \left|\frac{\partial}{\partial x}S_2^2(\rho,u)(t,x)\right| \le AB_1,$$

$(t,x)\in[0,\infty)\times\mathbb{R}$. Therefore,

$$\|S_2(\rho,u)\|_{X^2} \le AB_1.$$

This completes the proof. □

Below, let

(A2): $\epsilon\in(0,1)$, A, B, B_1 and q satisfy the inequalities $\epsilon B_1(1+A)<1$ and $AB_1<B$.

Let $\widetilde{Y}$ denotes the union of the set $\{(\rho_0, u_0)\}$ and the closure of the set of all equi-continuous families in X^2 with respect to the norm $\|\cdot\|_{X^2}$. Let also,

$$Y = \{(\rho, u) \in \widetilde{Y} : (\rho, u) \geq 0, \quad \|(\rho, u)\|_{X^2} \leq B\}.$$

Note that Y is a compact set in X^2. For $(\rho, u) \in X^2$, define the operators

$$\begin{aligned} T(\rho, u)(t, x) &= -\epsilon(\rho, u)(t, x), \\ S(\rho, u)(t, x) &= (\rho, u)(t, x) + \epsilon(\rho, u)(t, x) + \epsilon S_2(\rho, u)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}. \end{aligned}$$

For $(\rho, u) \in Y$, using Lemma 3.1, we have

$$\begin{aligned} \|(I - S)(\rho, u)\|_{X^2} &= \|\epsilon(\rho, u) + \epsilon S_2(\rho, u)\|_{X^2} \\ &\leq \epsilon\|(\rho, u)\|_{X^2} + \epsilon\|S_2(\rho, u)\|_{X^2} \\ &\leq \epsilon B_1 + \epsilon A B_1 \\ &= \epsilon B_1(1 + A) \\ &< B. \end{aligned}$$

Thus, $S : Y \to X^2$ is continuous and $(I - S)(Y)$ resides in a compact subset of X^2. Now, suppose that there is a $(\rho, u) \in X^2$ so that $\|(\rho, u)\|_{X^2} = B$ and

$$(\rho, u) = \lambda(I - S)(\rho, u)$$

or

$$\frac{1}{\lambda}(\rho, u) = (I - S)(\rho, u) = -\epsilon(\rho, u) - \epsilon S_2(\rho, u),$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)(\rho, u) = -\epsilon S_2(\rho, u)$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2(\rho, u)\|_{X^2} \leq AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right) \|(\rho, u)\|_{X^2} = \epsilon\|S_2(\rho, u)\|_{X^2} < \epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T + S$ has a fixed point $(\rho^*, u^*) \in Y$. Therefore,

$$\begin{aligned} (\rho^*, u^*)(t, x) &= T(\rho^*, u^*)(t, x) + S(\rho^*, u^*)(t, x) \\ &= -\epsilon(\rho^*, u^*)(t, x) + (\rho^*, u^*)(t, x) + \epsilon(\rho^*, u^*)(t, x) + \epsilon S_2(\rho^*, u^*)(t, x), \end{aligned}$$

$(t, x) \in [0, \infty) \times \mathbb{R}$, whereupon

$$0 = S_2(\rho^*, u^*)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

From here and from Lemma 2.10, it follows that (ρ^*, u^*) is a solution to the IVP (1.1). This completes the proof.

4. Proof of Theorem 1.2

Let X^2 be the space used in the previous section. Let

(A3): $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 + \frac{A}{m} B_1 + \frac{L}{5m} > \left(\frac{2}{5m} + 1\right) L,$$

$$AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{(\rho, u) \in X^2 : (\rho, u) \geq 0 \quad \text{on} \quad [0, \infty) \times \mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(\rho, v) \in X^2$, define the operators

$$\begin{aligned} T_1(\rho, v)(t, x) &= (1 + m\epsilon)(\rho, v)(t, x) - \left(\epsilon \frac{L}{10}, \epsilon \frac{L}{10}\right), \\ S_3(\rho, v)(t, x) &= -\epsilon S_2(\rho, v)(t, x) - m\epsilon(\rho, v)(t, x) - \left(\epsilon \frac{L}{10}, \epsilon \frac{L}{10}\right), \end{aligned}$$

$(t, x) \in [0, \infty) \times \mathbb{R}$. Note that any fixed point $(\rho, v) \in X^2$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{(\rho, v) \in \mathcal{P} : \|(\rho, v)\|_{X^2} < r\}, \\ U_2 &= \mathcal{P}_L = \{(\rho, v) \in \mathcal{P} : \|(\rho, v)\|_{X^2} < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{(\rho, v) \in \mathcal{P} : \|(\rho, v)\|_{X^2} < R_1\}, \\ R_2 &= R_1 + \frac{A}{m} B_1 + \frac{L}{5m}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{(\rho, v) \in \mathcal{P} : \|(\rho, v)\|_{X^2} \leq R_2\}. \end{aligned}$$

(1) For $(\rho_1, v_1), (\rho_2, v_2) \in \Omega$, we have

$$\|T_1(\rho_1, v_1) - T_1(\rho_2, v_2)\|_{X^2} = (1 + m\epsilon)\|(\rho_1, v_1) - (\rho_2, v_2)\|_{X^2},$$

whereupon $T_1 : \Omega \to X^2$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(2) For $(\rho, v) \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned}\|S_3(\rho,v)\|_{X^2} &\leq \epsilon\|S_2(\rho,v)\|_{X^2} + m\epsilon\|(\rho,v)\|_{X^2} + \epsilon\frac{L}{10} \\ &\leq \epsilon\left(AB_1 + mR_1 + \frac{L}{10}\right).\end{aligned}$$

Therefore, $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X^2$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently, $S_3 : \overline{\mathcal{P}_{R_1}} \to X^2$ is completely continuous.

(3) Let $(\rho_1, v_1) \in \overline{\mathcal{P}_{R_1}}$. Set

$$(\rho_2, v_2) = (\rho_1, v_1) + \frac{1}{m} S_2(\rho_1, v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

Note that $S_2^1(\rho_1, v_1) + \frac{L}{5} \geq 0$, $S_2^2(\rho_1, v_1) + \frac{L}{5} \geq 0$ on $[0,\infty) \times \mathbb{R}$. We have $\rho_2, v_2 \geq 0$ on $[0,\infty) \times \mathbb{R}$ and

$$\begin{aligned}\|(\rho_2,v_2)\|_{X^2} &\leq \|(\rho_1,v_1)\|_{X^2} + \frac{1}{m}\|S_2(\rho_1,v_1)\|_{X^2} + \frac{L}{5m} \\ &\leq R_1 + \frac{A}{m}B_1 + \frac{L}{5m} \\ &= R_2.\end{aligned}$$

Therefore, $(\rho_2, v_2) \in \Omega$ and

$$-\varepsilon m(\rho_2, v_2) = -\varepsilon m(\rho_1, v_1) - \varepsilon S_2(\rho_1, v_1) - \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$\begin{aligned}(I - T_1)(\rho_2, v_2) &= -\varepsilon m(\rho_2, v_2) + \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right) \\ &= S_3(\rho_1, v_1).\end{aligned}$$

Consequently, $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

(4) Assume that for any $(\rho_1, u_1) \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $(\rho, v) \in \partial\mathcal{P}_r \cap (\Omega + \lambda(\rho_1, u_1))$ or $(\rho, v) \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda(\rho_1, u_1))$ such that

$$S_3(\rho, v) = (I - T_1)((\rho, v) - \lambda(\rho_1, u_1)).$$

Then

$$-\epsilon S_2(\rho, v) - m\epsilon(\rho, v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((\rho, v) - \lambda(\rho_1, u_1)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$-S_2(\rho, v) = \lambda m(\rho_1, u_1) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2(\rho, v)\|_{X^2} = \left\| \lambda m(\rho_1, u_1) + \left(\frac{L}{5}, \frac{L}{5}\right) \right\|_{X^2} > \frac{L}{5}.$$

This is a contradiction.

(5) Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist a $(\rho_1, v_1) \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \varepsilon_1$ such that $\lambda_1(\rho_1, v_1) \in \overline{\mathcal{P}_{R_2}}$ and

$$S_3(\rho_1, v_1) = (I - T_1)(\lambda_1(\rho_1, v_1)). \tag{4.1}$$

Since $(\rho_1, v_1) \in \partial \mathcal{P}_L$ and $\lambda_1(\rho_1, v_1) \in \overline{\mathcal{P}_{R_2}}$, it follows that

$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|(\rho_1, v_1)\|_{X^2} \leq R_1 + \frac{A}{m} B_1 + \frac{L}{5m}.$$

Moreover,

$$-\epsilon S_2(\rho_1, v_1) - m\epsilon(\rho_1, v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -\lambda_1 m\epsilon(\rho_1, v_1) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right),$$

or

$$S_2(\rho_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) = (\lambda_1 - 1) m(\rho_1, v_1).$$

From here,

$$2\frac{L}{5} > \left\| S_2(\rho_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) \right\|_{X^2} = (\lambda_1 - 1) m \|(\rho_1, v_1)\|_{X^2} = (\lambda_1 - 1) mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore, all conditions of Theorem 2.8 hold. Hence, the IVP (1.1) has at least two solutions (ρ_1, u_1) and (ρ_2, u_2) so that

$$\|(\rho_1, u_1)\|_{X^2} = L < \|(\rho_2, u_2)\|_{X^2} \leq R_1$$

or

$$r \leq \|(\rho_1, u_1)\|_{X^2} < L < \|(\rho_2, u_2)\|_{X^2} \leq R_1.$$

5. An Example

Below, we will illustrate our main results. Let $q = 2$, $C = 1$ and

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.$$

Then

$$B_1 = \max\left\{10, 10^3\right\} = 10^3$$

and

$$AB_1 = \frac{1}{10^4} \cdot 10^3 < B, \quad \epsilon B_1(1 + A) = \frac{1}{10^4} \cdot 10^3 \left(1 + \frac{1}{10^4}\right) < B,$$

i.e., $(A2)$ holds. Next,

$$r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m}+1\right)L, \quad AB_1 < \frac{L}{5}.$$

i.e., $(A3)$ holds. Take

$$h(s) = \log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore,

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty}(1+s+s^2+s^3+s^4+s^5+s^6)h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty}(1+s+s^2+s^3+s^4+s^5+s^6)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2+s^3+s^4+s^5+s^6)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \quad \leq \quad C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [25] (pp. 707, Integral 79), we have

$$\int\frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then, there exists a constant $C_2 > 0$ such that

$$16(1+t)\left(1+t+t^2\right)(1+|x|)\left(1+|x|+|x|^2\right)$$

$$\int_0^t\left|\int_0^x g_1(t_1,x_1)dx_1\right|dt_1 \leq C_2, \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Let

$$g(t,x) = \frac{A}{C_2}g_1(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Then

$$16(1+t)\left(1+t+t^2\right)(1+|x|)\left(1+|x|+|x|^2\right)$$

$$\int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \leq A, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

i.e., $(A1)$ holds. Therefore, for the IVP

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + \rho^2\right) &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\ \rho(0,x) &= u(0,x) = \tfrac{1}{1+x^8}, \quad x \in \mathbb{R}, \end{aligned}$$

are fulfilled all conditions of Theorem 1.1 and Theorem 1.2.

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