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On setwise betweenness

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Abstract

In this article, we investigate the notion of setwise betweenness, a concept introduced by P. Bankston as a generalisation of pointwise betweenness. In the context of continua, we say that a subset $\mathcal{C}$ of a continuum X is between distinct points a and b of X if every subcontinuum K of X containing both a and b intersects $\mathcal{C}$. The notion of an interval $[a, b]$ then arises naturally. Further interesting questions are derived from considering such intervals within an associated hyperspace on X. We explore these ideas within the context of the Vietoris topology and n-fold symmetric product hyperspaces on all nonempty closed subsets of a topological space X, $CL(X)$. Moreover, an alternative pointwise interval, derived from setwise intervals, is introduced.

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1. Introduction

When does an element lie between two other elements? This question arises naturally in various situations according to the elements themselves and the relation which relates them together. However, betweenness, as a mathematical concept, first emerged from geometry. In 1882, the geometer, Pasch [15] introduced the mathematical foundation of betweenness. More than three decades later, a refining investigation on axioms of betweenness was carried by Huntington and Kline [11]. Then it was followed by another examination which was conducted by Huntington [10].

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After the essential foundation of betweenness was established, many investigations then followed to explore this concept via a wide range of mathematical structures. Betweenness relations that emerge from ordered set were intensively investigated. In [18], Sholander investigated the betweenness induced by partially ordered sets. In [8], Fishburn investigated betweenness relations induced by semiorder and weak order relations. Other studies in the same vein can be found, for example, in [21] and [7].

Betweenness plays a significant role in lattice theory, and many investigations have been carried out to examine betweenness in lattice theory. In [19], Smiley investigated betweenness on lattices and also made a comparison between betweenness relations that arise from lattices and other mathematical structures. Pitcher and Smiley [16] investigated betweenness relations induced by lattice, however, they needed to introduce axioms that involve five points to deal with lattices. Many other studies which investigated betweenness relations arising from lattices were initiated, among them [5] [18], [9] and [17].

The notion of betweenness also attracted the attention of many researchers who worked on investigating metric spaces. For instance, Wald in [22] characterised metric betweenness using certain five axioms. Moszyńska in [13] studied the first-order theory of betweenness and equidistance relations in metric spaces. In [12], it was shown that the class of all metrizable betweenness spaces can be axiomatized by a set of universal Horn sentences. In topology, betweenness was also investigated within a series of studies. For example, Bankston examined gap freeness and antisymmetric axiom of specific betweenness relations induced by the topology of Hausdorff continua in [2] and [3], respectively. In [6], betweenness relations were examined within a topological categorical point of view.

Bankston in [1] introduced the concept of road systems and examined betweenness relations induced by various road systems. This article adopts the notion of a specific road system to introduce our interested betweenness relations.

2. Preliminaries

Bankston in [1], developed the concept of *road system* and used such system to show various known mathematical structures which induced betweenness have a common behaviour in general. A road sysem is a pair $\langle X, \mathcal{R} \rangle$, where X is a nonempty set and $\mathcal{R}$ is a collection of nonempty subsets of X, called the roads, such that for each $a \in X$, the singleton set $\{a\}$ is a road and each two points $a, b \in X$ belong to at least one road. Therefore, if $\langle X, \mathcal{R} \rangle$ is a road system with $a, b, c \in X$, then $c \in [a, b]_{\mathcal{R}}$ if every road containing a and b also contains c. Then $c \in [a, b]_{\mathcal{R}}$ if $c \in \cap\{R \in \mathcal{R} : R \in \mathcal{R}(a, b)\}$, where $\mathcal{R}(a, b)$ denotes the set of roads that contain both a and b. This ternary relation tells us that a point lies between two other points a and b whenever each road containing a and b is also containing c.

Many road systems which are induced from various mathematical structures were investigated in [1]. In this article, we use a road system that arises from topology. Specifically, we use a connected system to define our two kinds of intervals, i.e. setwise and pointwise. We mean by $\langle X, \mathcal{CO}\rangle$ a road system, where $\mathcal{CO}$ is a collection of nonempty connected subsets of X.

The ternary relation that we consider here concerns relating a set with two other points, and this can be accomplished using a road system as follows. Let $\langle X, \mathcal{R}\rangle$ be a road system with $a, b \in X$ and $\phi \neq C \subseteq X$. We say that C is between a and b if $C \cap R \neq \phi$ for all $R \in \mathcal{R}(a, b)$. However, such a relation brings points and sets together, which might cause inconsistency. To resolve such a problem, we use hyperspace theory, as it considers sets as points. Such an idea was first used by Bankston over continuum theory [4] as a road system and hyperspaces.

Theory of hyperspaces got much attention and intensive studies have been conducted in the literature, [14]. In this work, we consider only two hyperspaces, Vietoris and n-fold symmetric product hyperspaces. Consider, $CL(X)$, the collection of all non-empty closed subsets of X. Vietoris topology is the one that is generated by sets of the form $U^+ = \{A \in CL(X) : A \subset U\}$ and $U^- = \{A \in CL(X) : A \bigcap U \neq \varnothing\}$, where U is an open subset of a topological space X. A basis of the Vietoris topology consists of the collection of sets of the form

$$\langle U_1, U_2, ..., U_n\rangle = \{A \in CL(X) : A \subseteq \bigcup_{i=1}^{n} U_i \text{ and if } 1 \leq i \leq n, A \bigcap U_i \neq \varnothing\}$$

where $U_1, U_2, ..., U_n$ are non-empty open subsets of X.

We also consider n-fold symmetric product of X, denoted by $\mathcal{F}_n(X)$ and defined as $\mathcal{F}_n(X) = \{A \in X : |A| \leq n\}$. This hyperspace is a subspace of the Vietoris space 2^X. Notice that if $\langle U_1, U_2, ..., U_m\rangle$ is an open set in the Vietoris space 2^X, then $\langle U_1, U_2, ..., U_m\rangle_n = \langle U_1, U_2, ..., U_m\rangle \bigcap \mathcal{F}_n(X)$ is an open set in $\mathcal{F}_n(X)$.

3. Setwise betweeness

Matching a road system with a specific hyperspace would relate sets to points in a sense that a betweenness relationship can be defined. We generalise a setwise betweenness concept given in [4], where continuum system is considered as a road system. Instead of considering continuum system, we use throughout this article connected systems to act as road systems. In the following, we set a definition of setwise betweenness with respect to a road system and a hyperspace.

Let X be a topological space and $a, b \in X$, the collection of sets that satisfies a topological property $\mathcal{P}$ forms a road system. The collection of sets that contain a and b and satisfy $\mathcal{P}$ is denoted by $\mathcal{P}(a, b)$. Thus, we define the setwise interval with respect to property $\mathcal{P}$ and a hyperspace $\mathcal{H}$ as follows.

$$[a, b]^{S\mathcal{P}}_{\mathcal{H}} = \{C \in \mathcal{H} : C \bigcap K \neq \varnothing \text{ for every } K \in \mathcal{P}(a, b)\} \tag{3.1}$$

We elaborate on the interval notation in 3.1; A superscript is used to distinguish between two kinds of betweenness intervals. For the notion $S\mathcal{P}$, S is used to mention a setwise betweenness and $\mathcal{P}$ represents a topological property while $P\mathcal{P}$ is used to mention pointwise betweenness. Notice that the interval defined in 3.1 contains elements of a hyperspace that lie between two points in X. Throughout this article, we apply two widely used hyperspaces, namely Vietoris topology and n-fold symmetric product hyperspace. However, other kinds of hyperspaces can be used to define other types of setwise and pointwise intervals in the same sense as the intervals defined here. Now, we can describe in some details the two types of setwise intervals, and we start with one that is defined via Vietoris topology 2^X, i.e. $[a,b]^{S\mathcal{CO}}_{2^X}$, where $\mathcal{CO}$ refers to the collection of all connected sets that contains a and b, i.e. $\mathcal{CO}(a,b)$.

Example 3.1. Let $X = C \bigcup B$ be a subspace of $\mathbb{R}^2$ where $C = (\frac{1}{2}, 1]$ and $B = \bigcup_{i=1}^{\infty} C_i$ where $C_i = \{\text{closed line segment joining } (0,0) \text{ and } (1, \frac{1}{i})\}$ for $i = 1, 2, \dots$. Now, if $a \in C_i$ and $b \in C_j$ with $i \neq j$ then for a set $A \in 2^X$ to be lie in the interval $[a,b]^{S\mathcal{CO}}_{2^X}$ it is necessary and sufficient that $(0,0) \in A$, Figure 1.

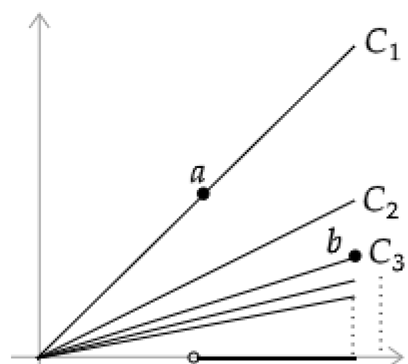

FIGURE 1. Any set $A \in 2^X$ that contains the origin point will be in the interval $[a,b]^{S\mathcal{CO}}_{2^X}$.

Now, we consider setwise interval with respect to the n-fold symmetric product hyperspace $\mathcal{F}_n(X)$, i.e. $[a,b]^{S\mathcal{CO}}_{n(X)}$.

Examples 3.2. We present the following three examples that concerns the interval $[a,b]^{S\mathcal{CO}}_{n(X)}$.

(1) Consider the real numbers with the standard topology and $a, b \in \mathbb{R}$, then $[a,b]^{S\mathcal{CO}}_{1(\mathbb{R})} = [a,b]$.

(2) Let X be the comb space and $K = \{[x,0] \bigcup [0.2, y] : 0.2 \leq x \leq 0.6 \text{ and } 0 \leq y \leq 0.4\}$. It is clear that $K \in \mathcal{CO}(a,b)$ where $a = (0.2, 0.4)$ and $b = (0.6, 0)$. Now, for a $C \in \mathcal{F}_n(X)$ to lie between a and b, it is enough that C intersects K, Figure 2.

Proposition 3.3. *Let X be a topological space with $a, b \in X$. Then*

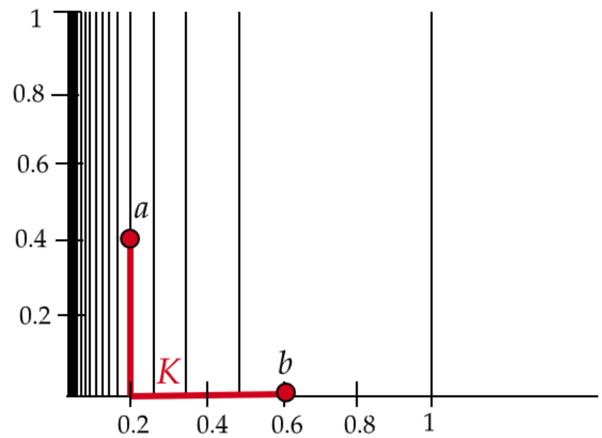

FIGURE 2. A set $C \in [a,b]^{\mathcal{SCO}}_{2(X)}$ should intersects K.

(1) $\{a\}, \{b\} \in [a,b]^{\mathcal{SCO}}_{2^X}$
(2) $\langle [a,b]^{\mathcal{SCO}}_{n(X)} \rangle^- \subset [a,b]^{\mathcal{SCO}}_{n(X)}$ *for* $n \geq 2$
(3) $\langle [a,b]^{\mathcal{SCO}}_{2(X)} \rangle^- = \langle \{a,b\} \rangle^-$
(4) $\langle \{a,b\} \rangle^- \subset [a,b]^{\mathcal{SCO}}_{2(X)}$
(5) $[a,a]^{\mathcal{SCO}}_{1(X)} = \{a\}$
(6) *For* $n \geq 3$, *we have* $[a,b]^{\mathcal{SCO}}_{n(X)} \bigcap [b,c]^{\mathcal{SCO}}_{n(X)} \neq \varnothing$
(7) $[a,b]^{\mathcal{SCO}}_{1(X)} \subseteq [a,b]^{\mathcal{SCO}}_{2(X)} \subseteq ... \subseteq [a,b]^{\mathcal{SCO}}_{n(X)}$

Proof. (1) Obvious.
(2) Let $A \in \langle [a,b]^{\mathcal{SCO}}_{n(X)} \rangle^-$. So $A \bigcap [a,b]^{\mathcal{SCO}}_{n(X)} \neq \varnothing$ and therefore $A \in [a,b]^{\mathcal{SCO}}_{n(X)}$.
(3) Let $C \in \langle [a,b]^{\mathcal{SCO}}_{2(X)} \rangle^-$. So, $C \bigcap [a,b]^{\mathcal{SCO}}_{2(X)} \neq \varnothing$ which means for every $K \in \mathcal{CO}(a,b)$ we have $C \bigcap K \neq \varnothing$. But $C \in \mathcal{F}_2(X)$, therefore, $C \bigcap \{a,b\} \neq \varnothing$. Consequently, $C \in \langle \{a,b\} \rangle^-$. Notice that the other direction can easily follows.
(4) Let $C \in \langle \{a,b\} \rangle^-$. So $C \bigcap \{a,b\} \neq \phi$. It is clear that any $K \in \mathcal{C}(a,b)$ we have $C \bigcap K \neq \phi$. Thus, $C \in [a,b]^{\mathcal{SCO}}_{2(X)}$.
(5) Obvious.
(6) This follows from being $\{b\} \in [a,b]^{\mathcal{SCO}}_{n(X)} \bigcap [b,c]^{\mathcal{SCO}}_{n(X)}$
(7) Let $C \in [a,b]^{\mathcal{SCO}}_{i(X)}$, for $1 \leq i \leq n-1$. Thus, for each $K \in \mathcal{CO}(a,b)$, we have $C \bigcap K \neq \varnothing$. But $\mathcal{F}_i(X) \subset \mathcal{F}_{i+1}(X)$ which leads to $C \in [a,b]^{\mathcal{SCO}}_{i+1(X)}$.

□

Proposition 3.4. *Let* X *be a topological space with* $a,b \in X$ *and* $C_i \in \mathcal{F}_n(X)$ *for* $i = 1,2,\ldots$ *such that* $C_1 \subset C_2 \subset \ldots$. *If* $C_1 \in [a,b]^{\mathcal{SCO}}_{n(X)}$ *then* $C_i \in [a,b]^{\mathcal{SCO}}_{n(X)}$ *for each* $i = 2,3,\ldots$.

Proof. Since $C_1 \in [a,b]^{\mathcal{SCO}}_{n(X)}$, so any connected subset K of X that contains a and b yields $C_1 \bigcap K \neq \phi$. But $C_1 \subset C_2 \subset ...$, so $C_i \bigcap K \neq \phi$ for each $i = 2, 3, \ldots$. Therefore, $C_i \in [a,b]^{\mathcal{SCO}}_{n(X)}$. □

Proposition 3.5. *Let X and Y be two homeomorphic spaces with $a, b \in X$. Let $f : X \longrightarrow Y$ be a homeomorphism, then*

(1) $f([a,b]^{\mathcal{SCO}}_{2^X}) = [f(a), f(b)]^{\mathcal{SCO}}_{2^Y}$
(2) $f([a,b]^{\mathcal{SCO}}_{n(X)}) = [f(a), f(b)]^{\mathcal{SCO}}_{n(Y)}$

Proof. We only proceed the proof of (1).
$\Rightarrow$ Let $A \in f([a,b]^{\mathcal{SCO}}_{2^X})$. Let $K \in \mathcal{CO}(f(a), f(b))$ in Y. To show that $A \bigcap K \neq \varnothing$. Let $A = f(S)$ for some $S \in [a,b]^{\mathcal{SCO}}_{2^X}$.
Since f^{-1} is continuous then $f^{-1}(K) \in \mathcal{CO}(a,b)$ in X.
Thus, $S \bigcap f^{-1}(K) \neq \varnothing$ which means that $f(S) \bigcap K \neq \varnothing$ i.e. $A \bigcap K \neq \varnothing$. Therefore $A \in [f(a), f(b)]^{\mathcal{SCO}}_{2^Y}$
$\Leftarrow$ Let $B \in [f(a), f(b)]^{\mathcal{SCO}}_{2^Y}$ and $D \in \mathcal{CO}(a,b)$ in X.
Hence $f(D) \in \mathcal{CO}(f(a), f(b))$ in Y, so $f(D) \bigcap B \neq \varnothing$. So $D \bigcap f^{-1}(B) \neq \varnothing$. Thus $f^{-1}(B) \in [a,b]^{\mathcal{SCO}}_{2^X}$. Therefore $B \in f([a,b]^{\mathcal{SCO}}_{2^X})$
□

Corollary 3.6. *Let X and Y be two topological spaces such that $X \cong Y$ and $a, b \in X$. Consider the interval $[a,b]^{\mathcal{SCO}}_{2^X}$ and $[a,b]^{\mathcal{SCO}}_{n(X)}$, then there exist $c, d \in Y$ such that $[a,b]^{\mathcal{SCO}}_{2^X} \cong [c,d]^{\mathcal{SCO}}_{2^Y}$ and $[a,b]^{\mathcal{SCO}}_{n(X)} \cong [c,d]^{\mathcal{SCO}}_{n(Y)}$, respectively.*

4. Pointwise betweenness via setwise intervals

In the following, we discuss a new type of betweenness relation that relates three points via the betweenness relation that we introduced in Section 3. Let x be a point in a topological space X. We define a *hyperstar* collection of x with respect to a hyperspace $\mathcal{H}$ to be $st(x, \mathcal{H}) = \{C \in \mathcal{H} : x \in C\}$. We also define the hyperstar collection of a set $C \subset X$ as $st(C, \mathcal{H}) = \bigcup_{c \in C} st(c, \mathcal{H})$. In the following, we investigate some properties of this hyperstar collection.

Proposition 4.1. *Let $A, B \subseteq X$ such that $A \subseteq B$, then $st(A, 2^X) \subseteq st(B, 2^X)$.*

Proof. Let $C \in st(A, 2^X)$. So $C \in \bigcup_{a \in A} st(a, 2^X)$. Thus, there is some $a_0 \in A$ such that $C \in st(a_0, 2^X)$. But $a_0 \in B$ which implies that $st(a_0, 2^X) \subseteq \bigcup_{b \in B} st(b, 2^X)$. Hence, $C \in st(B, 2^X)$. □

Now, we consider a hyperstar collection with respect to the n-fold symmetric product hyperspace, namely $st(x, \mathcal{F}_n(X))$.

Example 4.2. Let $X = \{1, 2, 3, 4\}$ and $\tau = \{\phi, \{1\}, \{2\}, \{1, 2\}, X\}$. Hence, $CL(X) = \{\{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, X\}$, $\mathcal{F}_3(X) = \{\{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}\}$. It is clear that $st(3, \mathcal{F}_3(X)) = \mathcal{F}_3(X)$ while $st(1, \mathcal{F}_3(X)) = \{\{1, 3, 4\}\}$.

Proposition 4.3. *Let X be a topological space, $x \in X$ and C, K be two subsets in X. Then*

(1) $st(x, \mathcal{F}_1(X)) = \{\{x\}\}$
(2) $st(C, \mathcal{F}_1(X)) = \{C\}$
(3) $st(x, \mathcal{F}_n(X)) \subset st(x, \mathcal{F}_m(X))$ *where* $m > n$
(4) *If* $K \subseteq C$, *then* $st(K, \mathcal{F}_n(X)) \subseteq st(C, \mathcal{F}_n(X))$

Proof. (1) Obvious.

(2) $st(C, \mathcal{F}_1(X)) = \bigcup_{c\in C} st(c, \mathcal{F}_1(X)) = \bigcup_{c\in C} \{c\} = \{C\}$.
(3) Since $\mathcal{F}_n(X) \subset \mathcal{F}_m(X)$.
(4) Let $A \in st(K, \mathcal{F}_n(X))$. So there is some $k_0 \in K$ such that $A \in st(k_0, \mathcal{F}_n(X))$. But $k_0 \in C$, so there is some $c_0 \in C$ with $c_0 = k_0$ such that $A \in st(c_0, \mathcal{F}_n(X))$. Hence $A \in \bigcup_{c\in C} st(c, \mathcal{F}_n(X))$. Consequently we get $A \in st(C, \mathcal{F}_n(X))$.

□

Now, we are ready to introduce the following betweenness relation. Suppose $a, b, c \in X$. We say that c lies between a and b with respect to a hyperspace $\mathcal{H}$ if $st(c, \mathcal{H}) \subseteq [a, b]^{\mathcal{SP}}_{\mathcal{H}}$. This betweenness relation induces the pointwise betweenness interval $[a, b]^{\mathcal{PP}}_{\mathcal{H}} = \{c \in X : st(c, \mathcal{H}) \subset [a, b]^{\mathcal{SS}}_{\mathcal{H}}\}$.

Proposition 4.4. *Let a and b are two points in a topological space X. Then*

(1) $\{a, b\} \subset [a, b]^{\mathcal{PCO}}_{n(X)}$
(2) $[a, b]^{\mathcal{PCO}}_{n(X)} \subset [a, b]^{\mathcal{PCO}}_{m(X)}$ *for* $m > n$

Note that the interval $[a, b]^{\mathcal{PCO}}_{\mathcal{H}}$ is a set of points which lie between a and b. Thus, it is more convenient to rename such a set and try to investigate some of its properties. We fix a topological property, $\mathcal{P}$ and a hyperspace $\mathcal{H}$. Let a and b be two points in X, define $C^{\mathcal{H}}_{a,b} = \{c \in X : c \in [a, b]^{\mathcal{PCO}}_{\mathcal{H}}\}$.

Example 4.5.
Consider the unit circle S^1. Let A and B are two arcs of S^1 such that their union is S^1 and $a, b \in S^1$ and $A \bigcap B = \{a, b\}$. A set $C \in \mathcal{F}_2(S^1)$ lies in the interval $[a, b]^{\mathcal{SCO}}_{2(S^1)}$ if one point of C lies in A and the other point of C lies in B, Figure 3. It is clear that the interval $[a, b]^{\mathcal{SCO}}_{1(S^1)}$ contains only $\{a\}$ and $\{b\}$. Hence $C^{n(1)}_{a,b} = \{a, b\}$. On the other hand, $C^{n(1)}_{a,a} = \{a\}$.

Proposition 4.6. *Let X be a topological space with $a, b \in X$. Then*

(1) $a, b \in C^{n(X)}_{a,b}$
(2) *If X is T_1 space and* $|C^{n(X)}_{a,b}| \leq n$, *then* $C^{n(X)}_{a,b} \in [a, b]^{\mathcal{SCO}}_{n(X)}$
(3) $C^{n(X)}_{a,b} \subseteq C^{n+1(X)}_{a,b}$

Proof. (1) We need to show that $st(a, \mathcal{F}_n(X)) \subseteq [a, b]^{\mathcal{SCO}}_{n(X)}$. Let $C \in \mathcal{F}_n(X)$ such that $a \in C$. It is clear that for every connected subset K of X

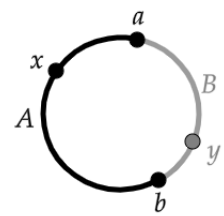

FIGURE 3. The set which contains $x \in A$ and $y \in B$ lies between a and b, i.e. $C = \{x, y\} \in [a, b]^{\mathcal{SCO}}_{2(X)}$.

containing both a and b, we have $\phi \neq \{a\} \subset C \bigcap K$. Hence $C \in [a, b]^{\mathcal{SCO}}_{n(X)}$. The same previous argument can be applied to show that $b \in C^{n(X)}_{a,b}$.

(2) Clearly, $C^{n(X)}_{a,b} \in \mathcal{F}_n(X)$. Hence, for every $K \in \mathcal{CO}(a, b)$, $\{a, b\} \subset C^{n(X)}_{a,b} \bigcap K \neq \varnothing$. Therefore, $C^{n(X)}_{a,b} \in [a, b]^{\mathcal{SCO}}_{n(X)}$.

(3) It follows from being $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$.

□

Proposition 4.7. *If $f : X \longrightarrow Y$ be a homeomorphism between two topological spaces X and Y with $a, b \in X$, then*

(1) $f(C^{2^X}_{a,b}) = C^{2^Y}_{f(a),f(b)}$

(2) $f(C^{n(X)}_{a,b}) = C^{n(Y)}_{f(a),f(b)}$

Proof. We only present the proof of (2). Clearly, $f(a) \neq f(b)$ (since f is one to one map). Let $y \in f(C^{n(X)}_{a,b})$, so $f^{-1}(y) \in C^{n(X)}_{a,b}$, i.e. $st(f^{-1}(y), \mathcal{F}_n(X)) \subseteq [a, b]^{\mathcal{SCO}}_{n(X)}$. Let $C \in st(f^{-1}(y), \mathcal{F}_n(X))$. Thus, $C \bigcap K \neq \varnothing$ for every $K \in \mathcal{CO}(a, b)$. But $f(C) \bigcap f(K) \neq \varnothing$ for every $f(K) \in \mathcal{CO}(f(a), f(b))$. This means, $f(C) \in [f(a), f(b)]^{\mathcal{SCO}}_{n(Y)}$. But $f(C) \in st(y, \mathcal{F}_n(Y))$, therefore, $y \in C^{n(Y)}_{f(a),f(b)}$.

□

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