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Concrete functors that respect initiality and finality

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Abstract

We study concrete endofunctors of the category of convergence spaces and continuous maps that send initial maps to initial maps or final maps to final maps. The former phenomenon turns out to be fairly common while the latter is rare. In particular, it is shown that the pretopological modification is the coarsest hereditary modifier finer than the topological modifier and this is applied to give a structural interpretation of the role of Fréchet-Urysohn spaces with respect to sequential spaces and of k'-spaces with respect to k-spaces.

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1. Preliminaries

Except for details of presentation and some examples, everything in this section on preliminaries is well-known and most of it can be found under one form or another in [8], though in too scattered a form to easily refer the reader to everything needed.

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1.1. **Convergence spaces.** The context of this paper is that of the category **Conv** of convergence spaces and continuous maps. We use the terminology and notations of [8]. In particular, a *convergence* ξ on a set X is a relation between points of X and filters on X, denoted

$$x \in \lim_\xi \mathcal{F}$$

whenever x and $\mathcal{F}$ are ξ-related, subjected to two simple axioms: $x \in \lim_\xi \{x\}^\uparrow$ for every $x \in X$, and $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$ whenever $\mathcal{G}$ is a filter finer than the filter $\mathcal{F}$. If (X,ξ) and (Y,τ) are two convergence spaces, a map $f : X \to Y$ is *continuous* (from ξ to τ), in symbols $f \in C(\xi,\tau)$, if

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f[\mathcal{F}],$$

where $f[\mathcal{F}] = \{B \subset Y : f^-(B) \in \mathcal{F}\}$ is the image filter. A convergence ξ is called a *Kent convergence* if moreover

$$x \in \lim_\xi \mathcal{F} \Longrightarrow x \in \lim_\xi \mathcal{F} \wedge \{x\}^\uparrow.$$

Of course, every topology τ can be seen as a (Kent) convergence given by $x \in \lim_\tau \mathcal{F}$ if and only if $\mathcal{F} \geq \mathcal{N}_\tau(x)$, where $\mathcal{N}_\tau(x)$ denotes the neighborhood filter of x in the topology τ. This turns the category **Top** of topological spaces and continuous maps into a full subcategory of **Conv**.

We denote by $|\cdot| :$ **Conv** $\to$ **Set** the forgetful functor, so that $|\xi|$ denotes the underlying set of a convergence ξ and $|f|$ is the underlying function of a morphism. If $|\xi| = |\tau|$, we say that ξ is *finer than* τ or that τ is *coarser than* ξ, in symbols, $\xi \geq \tau$, if the identity map of their underlying set belongs to $C(\xi,\tau)$. This order turns the set of convergences on a given set into a complete lattice whose greatest element is the discrete topology, least element is the antidiscrete topology, and whose suprema and infima are given by

$$\lim_{\bigvee_{\xi\in\Xi}\xi} \mathcal{F} = \bigcap_{\xi\in\Xi} \lim_\xi \mathcal{F} \text{ and } \lim_{\bigwedge_{\xi\in\Xi}\xi} \mathcal{F} = \bigcup_{\xi\in\Xi} \lim_\xi \mathcal{F}. \tag{1.1}$$

Conv is a concrete topological category; in particular, for every $f : X \to |\tau|$, there is the coarsest convergence on X, called *initial convergence* for (f,τ) and denoted $f^-\tau$, making f continuous (to τ), and for every $f : |\xi| \to Y$, there is the finest convergence on Y, called *final convergence for* (f,ξ) and denoted $f\xi$, making f continuous (from ξ). Note that with these notations

$$f \in C(\xi,\tau) \iff \xi \geq f^-\tau \iff f\xi \geq \tau. \tag{1.2}$$

Moreover, the initial lift on X of a structured source $(f_i : X \to |\tau_i|)_{i\in I}$ turns out to be $\bigvee_{i\in I} f_i^-\tau_i$ and the final lift on Y of a structured sink $(f_i : |\xi_i| \to Y)_{i\in I}$ turns out to be $\bigwedge_{i\in I} f_i\xi_i$.

If $f : |\xi| \to Y$ is surjective, $f\xi$ is also called the *quotient convergence*. If $A \subset |\xi|$, the *subspace convergence* $\xi_{|A}$ or *induced convergence on* A is $i_A^-\xi$ where $i_A : A \to |\xi|$ is the inclusion map. If Ξ is a set of convergences, then the *product convergence* is

$$\Pi_{\xi\in\Xi}\xi = \bigvee_{\xi\in\Xi} p_\xi^-\xi,$$

where $p_\xi : \Pi_{\xi\in\Xi}|\xi| \to |\xi|$ is the projection, that is, $\Pi_{\xi\in\Xi}\xi$ is the initial convergence for the family of projections.

1.2. **Concrete functors.** Most functors considered will be *concrete endofunctors* of **Conv**, that is, functors $F : \mathbf{Conv} \to \mathbf{Conv}$ ([1]) such that $|Ff| = |f|$ for every continuous map f. As a result, $|F\xi| = |\xi|$ for every convergence space ξ and thus F can be seen as simply modifying the convergence structure on $|\xi|$. Similarly, we will not distinguish notationally between morphisms f and Ff, and the underlying map $|f|$, usually writing f for all of them. Unless specified otherwise, *functor* will mean concrete endofunctor. With this convention, $C(\xi,\tau) \subset C(F\xi, F\tau)$ for every convergences ξ and τ means that F preserves continuity. Functors do preserve continuity. In particular, if $\xi \geq \tau$ (that is, $\mathrm{id} : (X,\xi) \to (X,\tau)$ is continuous) then $F\xi \geq F\tau$ (because $\mathrm{id} : (X, F\xi) \to (X, F\tau)$ is continuous), that is, a functor is monotone on each fiber of $|\cdot|$. In other words, a concrete functor is a *modifier* in the sense of [8] ([2]) which additionally satisfies $C(\xi,\tau) \subset C(F\xi, F\tau)$ for every pair of **Conv**-objects ξ, τ ([3]).

It is easily seen that the latter condition can be rephrased in terms of initial and final convergences. Namely

$$\begin{aligned} \underset{\xi,\tau\in\mathbf{Ob}(\mathbf{Conv})}{\forall} C(\xi,\tau) \subset C(F\xi,F\tau) &\iff \underset{\xi,\tau\in\mathbf{Ob}(\mathbf{Conv})}{\forall}\ \underset{f\in|\tau|^{|\xi|}}{\forall} F(f^-\tau) \geq f^-(F\tau) \\ &\iff \underset{\xi,\tau\in\mathbf{Ob}(\mathbf{Conv})}{\forall}\ \underset{f\in|\tau|^{|\xi|}}{\forall} f(F\xi) \geq F(f\xi). \end{aligned} \tag{1.3}$$

We order functors "pointwise", that is, $F \leq G$ means that $F\xi \leq G\xi$ for every convergence ξ. Suprema and infima of functors are defined accordingly. We will denote by I the identity functor.

1.3. **Reflectors and coreflectors.**

1.3.1. *Concretely (co)reflective subcategory and concrete (co)reflector.* A full subcategory $\mathbf{R}$ of **Conv** is *concretely reflective* (See e.g., [1]) if for every convergence ξ there is an $\mathbf{R}$-object $R\xi$ with $R\xi \leq \xi$ with the universal property that for any $f \in C(\xi,\sigma)$ where σ is a $\mathbf{R}$-object, there is a unique morphism $f' \in C(R\xi,\sigma)$ making the following diagram commute:

[1] with the abuse that the codomain of F may be a subcategory $\mathbf{C}$ of **Conv**, but we do not distinguish F from $F \circ N$ where $N : \mathbf{C} \to \mathbf{Conv}$ is the inclusion functor.

[2] Explicitly, a modifier M associates with each convergence ξ a convergence $M\xi$ with $|\xi| = |M\xi|$ and is monotone on fibers of $|\cdot|$.

[3] Note that preservation of composition is automatic because $|Ff| = |f|$ for every morphism.

$$\begin{tikzcd} \xi \arrow[r, "\mathrm{id}"] \arrow[dr, "f"'] & R\xi \arrow[d, "f'"] \\ & \sigma \end{tikzcd}$$

FIGURE 1. concretely reflective subcategory

Note that necessarily $|f'| = |f|$, so that $R\xi$ satisfies

$$\sigma \in \mathbf{Ob}(\mathbf{R}) \text{ and } f \in C(\xi, \sigma) \Longrightarrow f \in C(R\xi, \sigma).$$

The corresponding operator R is of course a modifier because if $\xi \geq \tau$ and $\sigma = R\tau$ with $f = \mathrm{id}$ in Figure 1, then $\mathrm{id} \in C(R\xi, R\tau)$, that is, $R\xi \geq R\tau$. Moreover, applying the scheme of Figure 1 to $R\xi$ in the role of ξ, $\sigma = R\xi$ and $f = \mathrm{id}$, yields $\mathrm{id} \in C(R(R\xi), R\xi)$,that is, $RR\xi \geq R\xi$ and $R\xi \geq RR\xi$ because R is contractive by definition, so that R is an idempotent modifier. The modifier R is also a functor. Indeed, for every f with domain ξ, $f \in C(\xi, R(f\xi))$ because $f\xi \geq R(f\xi)$, so that we can apply the definition of R with $\sigma = R(f\xi)$ to the effect that $f \in C(R\xi, R(f\xi))$, that is, $f(R\xi) \geq R(f\xi)$. In view of (1.3), R is a functor.

As a result R satisfies

$$\xi \geq \tau \Longrightarrow R\xi \geq R\tau \tag{monotone}$$

$$\xi \geq R\xi \tag{contractive}$$

$$R\xi = RR\xi \tag{idempotent}$$

for every ξ and τ, and

$$f(R\xi) \geq R(f\xi) \tag{1.4}$$

because R is a functor.

Conversely, if R is a contractive and idempotent concrete functor, then the concrete full subcategory $\mathbf{R} = \operatorname{fix} R = \{\xi \in \mathbf{Ob}(\mathbf{Conv}) : \xi = R\xi\}$ is concretely reflective and R acts as in the definition of concretely reflective subcategory ([4]). We say that R is a *reflector.* Note that for every convergence ξ, the convergence $R\xi$ is the finest convergence of $\mathbf{R}$ that is coarser than ξ (if $\sigma = R\sigma \leq \xi$ then $\sigma \leq R\xi$ because R is idempotent and monotone).

As a concretely reflective subcategory $\mathbf{R}$ of $\mathbf{Conv}$ is the class of fixed points for a reflector (a contractive and idempotent concrete functor) R, the category $\mathbf{R}$ is closed under suprema in $\mathbf{Conv}$. Indeed, if Ξ is a set of convergences of $\mathbf{R}$ on the same set, then $R\Big(\bigvee \Xi\Big) \geq R\xi \geq \xi$ for each $\xi \in \Xi$, and thus $R\Big(\bigvee \Xi\Big) \geq \bigvee \Xi$. Moreover, the antidiscrete topology on X is the smallest element of $|\cdot|^{-1}(X)$

[4]This is because if $\sigma = R\sigma$ and $f \in C(\xi, \sigma)$, that is, $f\xi \geq \sigma$, then $R(f\xi) \geq R\sigma = \sigma$ by (monotone) and thus $f(R\xi) \geq \sigma$ by (1.4), that is, $f \in C(R\xi, \sigma)$ as desired.

and thus belongs to $\mathbf{R} = \operatorname{fix} R$. So $\mathbf{R}$ is closed over *all* infima; even over the empty family, which yields the antidiscrete topology.

Moreover,

$$\tau = R\tau \Longrightarrow f^{-}\tau = R(f^{-}\tau) \tag{1.5}$$

for every τ and f with codomain $|\tau|$. Indeed, if $\tau \leq R\tau$ then $f^{-}\tau \leq f^{-}(R\tau) \leq R(f^{-}\tau)$, where the latter inequality follows from (1.3). In fact,

Proposition 1.1. *A concrete full subcategory* $\mathbf{A}$ *of* $\mathbf{Conv}$ *is concretely reflective in* $\mathbf{Conv}$ *if and only if it is closed under* all *suprema in* $\mathbf{Conv}$ *and satisfies* $f^{-}\tau \in \mathbf{Ob}(\mathbf{A})$ *whenever* $\tau \in \mathbf{Ob}(\mathbf{A})$ *and* f *has codomain* $|\tau|$.

Proof. We only need to show the converse part. Because $\mathbf{A}$ is closed under all infima, it contains all antidiscrete spaces. Hence, given a convergence ξ, $\{\tau \in \mathbf{Ob}(\mathbf{A}) : \tau \leq \xi\}$ is non-empty and its supremum, that we shall denote $R\xi$, belongs to $\mathbf{Ob}(\mathbf{A})$. It is clear that $R\xi$ is the finest convergence of $\mathbf{A}$ coarser than ξ and that R is a contractive idempotent modifier and that $\mathbf{Ob}(\mathbf{A}) = \operatorname{fix} R$. If now τ is a convergence $R\tau \in \mathbf{Ob}(\mathbf{A})$ so that

$$f^{-}(R\tau) \leq R(f^{-}(R\tau)) \leq R(f^{-}\tau),$$

where the first inequality follows from the fact that $f^{-}\tau \in \mathbf{Ob}(\mathbf{A})$ whenever $\tau \in \mathbf{Ob}(\mathbf{A})$ and the second follows from the fact that R is contractive. In view of (1.3), R is a functor, hence a reflector. $\square$

Dually, a concrete functor C that is idempotent and *expansive*, that is,

$$\xi \leq C\xi \tag{expansive}$$

for every convergence ξ is called a *coreflector*. The corresponding concrete full subcategory $\mathbf{C} = \operatorname{fix} C$ of $\mathbf{Conv}$ (whose objects satisfy $\xi = C\xi$, equivalently $\xi \geq C\xi$) is a concretely coreflective subcategory of $\mathbf{Conv}$ (in the usual sense of [1]). A concrete full subcategory $\mathbf{A}$ of $\mathbf{Conv}$ is coreflective if and only if its class of objects is the class of convergences fixed by such a functor, if and only if it is closed under all infima in $\mathbf{Conv}$ (in particular, it contains all discrete spaces) and $f\xi \in \mathbf{Ob}(\mathbf{A})$ whenever $\xi \in \mathbf{Ob}(\mathbf{A})$ and f has domain $|\xi|$.

1.3.2. *Reflective and coreflective parts of a (concrete) functor.* It is essentially an observation of Greco and Dolecki (in different terms) [6] and is proved in details in [8, Sections XIV.1 and XIV.2] that,

Theorem 1.2. *If* F *is a (concrete) functor then*

$$\mathbf{F}_{+} := \{\xi : F\xi \geq \xi\}$$

is a (concretely) reflective class and

$$\mathbf{F}_{-} := \{\xi : \xi \geq F\xi\}$$

is a (concretely) coreflective class.

Let R_F and C_F denote the corresponding reflector and coreflectors on $\mathbf{F}_+$ and $\mathbf{F}_-$ respectively.

Given a functor F, we define by transfinite induction $(F)^1 = (F)_1 = F$ and

$$(F)^\alpha = F\Big(\bigvee_{\beta<\alpha} F^\beta\Big) \text{ and } (F)_\alpha = F\Big(\bigwedge_{\beta<\alpha} F_\beta\Big).$$

Proposition 1.3. *Let F be a functor. For every convergence ξ, there are ordinals α and β such that*

$$\mathrm{R}_F\,\xi = (F\wedge \mathrm{I})_\alpha \xi \textit{ and } \mathrm{C}_F = (F\vee \mathrm{I})^\beta \xi.$$

In particular, R_F is the smallest among functors G with $\mathbf{G}_+ = \mathbf{F}_+$ and C_F is the greatest among functors G with $\mathbf{G}_- = \mathbf{F}_-$.

Proof. This follows immediately from [8, Propositions XIV.1.18 and XIV.1.19], after observing that

$$(\mathbf{F}\wedge \mathbf{I})_+ = \mathbf{F}_+ \text{ and } (\mathbf{F}\wedge \mathbf{I})_- = \mathbf{F}_-.$$

□

1.3.3. *Classes of filters.* Recall that the powerset $\mathbb{P}X = \{\varnothing\}^{\uparrow_X}$ is the *degenerate filter* on X and we denote by $\overline{\mathbb{F}X}$ the set of all (degenerate or proper) filters on X. Note that $\overline{\mathbb{F}} : \mathbf{Rel} \to \mathbf{Rel}$ is a functor that associates to a set $X \in \mathbf{Ob}(\mathbf{Rel})$ the set $\overline{\mathbb{F}X}$ and to a relation $R \subset X \times Y$ the relation $\overline{\mathbb{F}}R : \overline{\mathbb{F}X} \to \overline{\mathbb{F}Y}$ defined by

$$(\overline{\mathbb{F}}R)(\mathcal{F}) = R[\mathcal{F}] = \{R(F) : F \in \mathcal{F}\}^{\uparrow_Y}.$$

We will denote by $\mathbb{D} \subset \overline{\mathbb{F}}$ the fact that $\mathbb{D}$ is a subfunctor, that is, $\mathbb{D}X \subset \overline{\mathbb{F}X}$ for every set X and $\overline{\mathbb{F}}R(\mathcal{D}) \in \mathbb{D}Y$ for every $\mathcal{D} \in \mathbb{D}X$ and every relation $R \subset X\times Y$. We can alternatively say that $\mathbb{D}$ is an $\mathbb{F}_0$-*composable class of filters.* Such a class must contain all principal filters, in particular every principal ultrafilter (See e.g., [8, Lemma XIV.3.7] for this and other properties of $\mathbb{F}_0$-composable classes). Among such classes, we distinguish the class $\mathbb{F}_0$ of principal filters, $\mathbb{F}_1$ of countably based filters, $\mathbb{F}_{\wedge 1}$ of countably deep filters. In contrast, the class $\mathbb{U}$ of ultrafilters and the class $\mathbb{E}$ of filters generated by a sequence are not $\mathbb{F}_0$-composable.

Given $\mathcal{F} \in \mathbb{F}X$ and $\mathbb{D}$ a class of filters, we write

$$\mathbb{D}(\mathcal{F}) := \{\mathcal{D} \in \mathbb{D}X : \mathcal{D} \geq \mathcal{F}\}.$$

As we assume the Axiom of Choice, $\mathbb{U}(\mathcal{F}) \neq \varnothing$ for every filter $\mathcal{F}$ and $\mathcal{F} = \bigcap_{\mathcal{U}\in\mathbb{U}(\mathcal{F})} \mathcal{U}$ while $\mathcal{F}^\# = \bigcup_{\mathcal{U}\in\mathbb{U}(\mathcal{F})} \mathcal{U}$.

1.3.4. *Reflectors and coreflectors determined by a class of filters.* Several important examples of concrete endofunctors of **Conv** are defined modulo a class $\mathbb{D} \subset \overline{\mathbb{F}}$. For instance, the functor $\mathrm{A}_\mathbb{D}$ defined (on objects) by

$$\lim\nolimits_{\mathrm{A}_\mathbb{D}\,\xi} \mathcal{F} = \bigcap_{\mathbb{D}\ni\mathcal{D}\#\mathcal{F}} \mathrm{adh}_\xi\, \mathcal{D},$$

where the *adherence of* $\mathcal{D}$ is defined by

$$\operatorname{adh}_\xi \mathcal{D} = \bigcup_{\mathbb{F}X \ni \mathcal{H} \# \mathcal{D}} \lim_\xi \mathcal{H} = \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{D})} \lim_\xi \mathcal{U},$$

is a reflector, while $\mathrm{B}_{\mathbb{D}}$ defined (on objects) by

$$\lim_{\mathrm{B}_{\mathbb{D}}\, \xi} \mathcal{F} = \bigcup_{\mathbb{D} \ni \mathcal{D} \leq \mathcal{F}} \lim_\xi \mathcal{D}$$

is a coreflector. The table below gathers notations and names for these functors for specific classes of filters.

$\mathbb{D}$	$\mathrm{A}_{\mathbb{D}}$	reflector's name	$\mathrm{B}_{\mathbb{D}}$	coreflector's name
$\mathbb{F}$	S	pseudotopologizer	I	identity
$\mathbb{F}_1$	S_1	paratopologizer	I_1	modifier of countable character
$\mathbb{F}_{\wedge 1}$	$\mathrm{S}_{\wedge 1}$	hypotopologizer	$\mathrm{I}_{\wedge 1}$	P-modifier
$\mathbb{F}_0$	S_0	pretopologizer	I_0	modifier on finitely generated spaces

Table 1. reflectors and coreflectors

Note that $\mathbb{U}$ and $\mathbb{U} \cap \mathbb{F}_0$ are not $\mathbb{F}_0$-composable, so that we cannot directly conclude from the above considerations that the modifiers $\mathrm{A}_{\mathbb{D}}$ and $\mathrm{B}_{\mathbb{D}}$ are reflectors and coreflectors respectively for these classes. However, [8, XIV.3.1 and XIV.4] provides sufficient conditions on $\mathbb{D}$ (weaker than $\mathbb{F}_0$-composability) to ensure that $\mathrm{A}_{\mathbb{D}}$ is a reflector and that $\mathrm{B}_{\mathbb{D}}$ is a coreflector. In particular, $\mathrm{B}_{\mathbb{U}}$ and $\mathrm{B}_{\mathbb{U} \cap \mathbb{F}_0}$ are coreflectors.

We call $\mathrm{B}_{\mathbb{U}}$ the *ultrafilter convergence modifier* and $\mathrm{G} := \mathrm{B}_{\mathbb{U} \cap \mathbb{F}_0}$ the *graph modifier* as it is characterized by the underlying digraph of the convergence in which $x \to y$ if $y \in \lim\{x\}^{\uparrow}$ (See, e.g., [8, Example XIV.4.6])([5]).

Similarly, though the class $\mathbb{E}$ of filters generated by sequences is not $\mathbb{F}_0$-composable, $\mathrm{Seq} := \mathrm{B}_{\mathbb{E}}$ is a coreflector.

1.3.5. *Other important reflectors and coreflectors.* A subset A of a convergence space (X, ξ) is *closed* if

$$A \in \mathcal{F} \Longrightarrow \lim_\xi \mathcal{F} \subset A$$

and *open* if

$$\lim_\xi \mathcal{F} \cap A \neq \varnothing \Longrightarrow A \in \mathcal{F}.$$

The collection of all open subsets of a convergence space (X, ξ) is a topology on X called *topological modification of* ξ and denoted by $\mathrm{T}\,\xi$ (of course, we identify this topology with the convergence it defines). The modifier T is a reflector. Note that T can alternatively be seen as a type of *regularization.*

[5]Note also that $\mathrm{A}_{\mathbb{U}} = \mathrm{S}$.

Indeed, [8, VII.2] introduces a notion of regularity with respect to a family of sets. Namely, if $\mathcal{Z}$ is a family of subsets of X, then a convergence ξ is *$\mathcal{Z}$-regular* if

$$\lim_\xi \mathcal{F} = \lim_\xi (\mathcal{F} \cap \mathcal{Z})^\uparrow$$

for every $\mathcal{F} \in \mathbb{F}X$, with the convention that $\lim_\xi \varnothing^\uparrow = \varnothing$ ([6]). An *admissible assignment* $\mathcal{Z}(\cdot)$ associates with each convergence ξ a family $\mathcal{Z}(\xi)$ of subsets of $|\xi|$ in such a way that $Z(\xi)$ is closed under intersections of finitely many members,

$$\xi \leq \theta \Longrightarrow \mathcal{Z}(\xi) \subset \mathcal{Z}(\theta),$$

and

$$f^-[\mathcal{Z}(\tau)] \subset \mathcal{Z}(f^-\tau)$$

for every function f and convergence τ on its codomain.

It is easily checked that if $\mathcal{O}(\xi)$ denotes the set of ξ-open sets then $\mathcal{O}(\cdot)$ defines an admissible assignment. Similarly, the collection $\mathcal{C}(\xi)$ of ξ-closed sets, the collection $\mathcal{Z}(\xi)$ of ξ-zero sets (that is, sets of the form $f^-(0)$ for some continuous function $f : (X, \xi) \to \mathbb{R}$), and the collection $\mathcal{CZ}(\xi)$ of complements of zero-sets (cozero sets) define admissible assignments.

[8, Prop. VII.2.5 and VII.2.6] state that:

Proposition 1.4. *If $\mathcal{Z}(\cdot)$ is an admissible assignment then for every convergence ξ there is the finest $\mathcal{Z}$-regular convergence $\mathrm{Reg}_\mathcal{Z}\,\xi$ that is coarser than ξ and $\mathrm{Reg}_\mathcal{Z}$ is a reflector.*

Remark 1.5. Note that if $\mathcal{Z}(\cdot)$ and $\mathcal{B}(\cdot)$ are admissible assignments and $\mathcal{Z}(\xi) \subset \mathcal{B}(\xi)$ for every convergence ξ then $\mathrm{Reg}_\mathcal{Z} \leq \mathrm{Reg}_\mathcal{B}$.

Example 1.6. In particular, the reflector $\mathrm{Reg}_\mathcal{O} = \mathrm{T}$ is the reflector on topologies ([7]), while the reflector $\mathrm{Reg}_{\mathcal{CZ}} = \Omega$ is the reflector on completely regular topologies ([8]). The reflector $\mathrm{Reg}_\mathcal{C} = \mathrm{R_T}$ is the reflector on topologically regular convergences in the sense of [8, VI.4] and the reflector $\mathrm{Reg}_\mathcal{Z} = \mathrm{R}_\Omega$ is the reflector on Ω-regular convergences ([9]).

[6] Of course, $(\mathcal{F} \cap \mathcal{Z})^\uparrow$ is either the degenerate filter $\varnothing^\uparrow$ or a filter coarser than $\mathcal{F}$, so that $\lim_\xi(\mathcal{F} \cap \mathcal{Z})^\uparrow \subset \lim_\xi \mathcal{F}$ is always true, that is, ξ is $\mathcal{Z}$-regular if $\lim_\xi \mathcal{F} \subset \lim_\xi(\mathcal{F} \cap \mathcal{Z})^\uparrow$ for every filter $\mathcal{F}$.

[7] Indeed, it is clear that a topological space is $\mathcal{O}$-regular, and conversely, if ξ is $\mathcal{O}$-regular, then

$$x \in \lim\{x\}^\uparrow = \lim\left(\{x\}^\uparrow \cap \mathcal{O}_\xi\right)^\uparrow = \lim \mathcal{N}_\xi(x),$$

so that ξ is topological.

[8] In view of Remark 1.5, $\mathrm{Reg}_{\mathcal{CZ}}$ is a reflector on a subcategory of that of topological spaces. Moreover, if $\xi = \mathrm{Reg}_{\mathcal{CZ}}\,\xi$ then $\mathcal{O}_\xi(x)^\uparrow = (\mathcal{O}_\xi(x) \cap \mathcal{CZ})^\uparrow$ that is, $\mathcal{N}_\xi(x)$ has a filter-base composed of cozero sets. Hence, if F is closed and $x \notin F$ then $F^c \in \mathcal{O}_\xi(x)$ and there is a cozero set C with $x \in C \subset F^c$. As a result, there is $f \in C(\xi, \nu)$ with $f(x) = 1$ and $f(F) = \{0\}$ and thus ξ is completely regular. Conversely, if ξ is a completely regular topology and $x \in U \in \mathcal{O}_\xi(x)$, then there is $f \in C(\xi, \nu)$ with $f(x) = 1$ and $f(U^c) = \{0\}$ so that $x \in \{f = 0\}^c \subset U$, that is, ξ is $\mathcal{CZ}$-regular.

[9] Also called *ω-regular convergences* in [2].

It is easily checked that the modifier χ associating with a convergence ξ its *characteristic convergence* $\chi(\xi)$ defined by

$$\lim_{\chi(\xi)}\mathcal{F}=\begin{cases}|\xi| & \text{if } \lim_{\xi}\mathcal{F}\neq\varnothing\\ \varnothing & \text{if } \lim_{\xi}\mathcal{F}=\varnothing\end{cases}$$

is a reflector. The class $\operatorname{fix}\chi$ is the (concrete full) subcategory of *constant convergences*, that is, convergences with the same set of convergent filters at each point. This reflector is instrumental in connecting convergence structures with nearness structures [9] and in the study of compactness [5, 4].

The modifier Dis sending every convergence to the discrete topology on the same underlying set is a coreflector called *discretization*.

A subset K of a convergence space (X,ξ) is *compact* if $\operatorname{adh}_{\xi}\mathcal{F}\cap K\neq\varnothing$ whenever $K\in\mathcal{F}$. The *locally compact modifier* K defined by $x\in\lim_{\mathrm{K}\,\xi}\mathcal{F}$ if $x\in\lim_{\xi}\mathcal{F}$ and there is a compact set $K\in\mathcal{F}$, is a coreflector.

1.4. **Functorial inequalities and quotient maps.** In [3], various topological properties were characterized in terms of functorial inequalities of the form

$$\xi\geq RC\xi$$

where R is a reflector and E is a coreflector (of **Conv**), and various classes of quotient maps naturally appearing in topology were characterized in similar terms: given a reflector R, an onto map $f:|\xi|\to|\tau|$ is *R-quotient* if $\tau\geq R(f\xi)$. Here is a partial list of such maps and topological properties:

reflector R	R -quotient $f:\lvert\xi\rvert\to\lvert\tau\rvert$	I_1	Seq	K
I	almost open $\tau\geq f\xi$	first-countable $\xi\geq\mathrm{I}_1\,\xi$	sequentially based $\xi\geq\operatorname{Seq}\xi$	locally compact $\xi\geq\mathrm{K}\,\xi$
S	biquotient $\tau\geq\mathrm{S}(f\xi)$	bisequential $\xi\geq\mathrm{S}\,\mathrm{I}_1\,\xi$	sequentially based $\xi\geq\mathrm{S}\operatorname{Seq}\xi$	locally compact $\xi\geq\mathrm{S}\,\mathrm{K}\,\xi$
S_1	countably biquotient $\tau\geq\mathrm{S}_1(f\xi)$	countably bisequential $\xi\geq\mathrm{S}_1\,\mathrm{I}_1\,\xi$	countably bisequential $\xi\geq\mathrm{S}_1\operatorname{Seq}\xi$	strongly k' $\xi\geq\mathrm{S}_1\,\mathrm{K}\,\xi$
$\mathrm{S}_{\wedge 1}$	weakly biquotient $\tau\geq\mathrm{S}_{\wedge 1}(f\xi)$	weakly bisequential $\xi\geq\mathrm{S}_{\wedge 1}\,\mathrm{I}_1\,\xi$	weakly bisequential	
S_0	hereditarily quotient $\tau\geq\mathrm{S}_0(f\xi)$	Fréchet-Urysohn $\xi\geq\mathrm{S}_0\,\mathrm{I}_1\,\xi$	Fréchet-Urysohn $\xi\geq\mathrm{S}_0\operatorname{Seq}\xi$	k' $\xi\geq\mathrm{S}_0\,\mathrm{K}\,\xi$
T	quotient $\tau\geq\mathrm{T}(f\xi)$	sequential $\xi\geq\mathrm{T}\,\mathrm{I}_1\,\xi$	sequential $\xi\geq\mathrm{T}\operatorname{Seq}\xi$	k $\xi\geq\mathrm{T}\,\mathrm{K}\,\xi$

TABLE 2. quotient maps and topological properties

See [8, Sections XIV.6, XV.1, XV.2] for details.

2. Functors respecting initiality and finality

We have seen (1.3) that the functoriality condition on a modifier F amounts to

$$F(f^{-}\tau) \geq f^{-}(F\tau) \tag{2.1}$$

for every f with codomain $|\tau|$, equivalently,

$$f(F\xi) \geq F(f\xi) \tag{2.2}$$

for every f with domain $|\xi|$.

In contrast, the reverse inequalities do not hold in general, and moreover, they are not equivalent to each other (as we will see shortly).

We say that a functor F *respects initiality* if

$$f^{-}(F\tau) \geq F(f^{-}\tau) \tag{2.3}$$

for every f with codomain $|\tau|$, and *respect finality* if

$$F(f\xi) \geq f(F\xi) \tag{2.4}$$

for every f with domain $|\xi|$.

Lemma 2.1. *A functor F respects finality if and only if it sends every final map to a final map and respects initiality if and only if it sends every initial map to an initial map.*

Proof. If $f:(X,\xi)\to(Y,f\xi)$ then $Ff=f:(X,F\xi)\to(Y,F(f\xi))$. If F is a functor that respects finality then $F(f\xi)=f(F\xi)$ and the conclusion follows. Similarly, if $f:(X,f^{-}\tau)\to(Y,\tau)$ then $Ff=f:(X,F(f^{-}\tau))\to(Y,F\tau)$. If F is a functor that respects initiality then $F(f^{-}\tau)=f^{-}(F\tau)$ and the result follows. □

Example 2.2 (A reflector that respects initiality but not finality)**.** If $\mathbb{D}$ is an $\mathbb{F}_0$-composable class of filters then the concrete reflector $\mathrm{A}_{\mathbb{D}}$ respects initiality (e.g., [8, Corollary XIV.3.8]), but not finality. For instance, if ξ and τ are topologies and $f:|\xi|\to|\tau|$ is continuous, $\mathrm{A}_{\mathbb{D}}$-quotient but not almost-open, then

$$\tau=\mathrm{A}_{\mathbb{D}}(f\xi)\ngeq f\xi=f(\mathrm{A}_{\mathbb{D}}\,\xi).$$

Example 2.3 (A coreflector that respects finality but not initiality)**.** The discretization Dis respects finality: because Dis is a functor, $f(\mathrm{Dis}\,\xi)\geq\mathrm{Dis}(f\xi)$, that is, $f(\mathrm{Dis}\,\xi)$ is the discrete topology, which is also $\mathrm{Dis}(f\xi)$. But Dis does not respect initiality. Indeed, if X has more than 2 points and $f:X\to\{*\}$ then the codomain τ is discrete but the initial convergence $f^{-}\tau$ is antidiscrete, so that $f^{-}\tau=f^{-}(\mathrm{Dis}\,\tau)\ngeq\mathrm{Dis}(f^{-}\tau)$.

Note that the later part of this argument can be extended to the effect that:

Proposition 2.4. *Let F be a modifer. If F respects initiality then the image of every antidiscrete space under F is antidiscrete. If F respects finality then the image of every discrete space under F is discrete.*

Recall that we denote by o_X the antidiscrete topology on X and by ι_X the discrete topology on X.

Proof. There is only one convergence structure $\tau = \iota = o$ on $\{*\}$, so $F\tau = \tau$ and $f^-\tau$ is antidiscrete. Hence, if F respects initiality and $f : X \to \{*\}$ then $f^-\tau = f^-(F\tau) = o_X \geq F(f^-\tau) = F(o_X)$ and thus $o_X = F(o_X)$.

Note that for any $g : \{*\} \to X$, $g\tau$ is discrete. If F respects finality then $F(g\tau) = F(\iota_X) \geq g(F\tau) = g(\tau) = \iota_X$, and then $F(\iota_X) = \iota_X$. □

Example 2.5 (applications of Proposition 2.4)**.** The reflector χ on constant convergences does not preserve the discrete topology and thus χ does not respect finality. It is easily seen that χ preserves initiality.

Similarly, the coreflectors $\mathrm{B}_{\mathbb{U}}$, Seq and G do not preserve antidiscrete spaces and therefore they do not respect initiality. We will see (Corollary 2.19) that $\mathrm{B}_{\mathbb{U}}$ respects finality but Seq and G do not.

Note that every contractive modifier preserves antidiscrete spaces and every expansive modifier preserves discrete spaces. Preserving finality turns out to be a rare phenomenon. Indeed, the following immediately follows from the definitions:

Lemma 2.6. *If there are $\xi > F\xi$ and $\sigma = F\sigma$ with $\xi = f\sigma$, then F does not respect finality.*

As all Kent convergences are the final image of a coproduct of prime topologies ([10]), if fix F contains all prime topologies and their coproducts (for instance, all topologies) and there is a Kent convergence $\xi > F\xi$ then the lemma above applies. In particular, no reflector greater or equal to T that is "non-trivial" in the sense that it doesn't leave all Kent convergences fixed can preserve finality.

Example 2.7 (A modifier that is not a functor but preserves initiality)**.** Consider the modifier $\mathrm{U}_{\mathbb{D}}$ defined by

$$\lim_{\mathrm{U}_{\mathbb{D}}\,\xi} \mathcal{F} = \bigcap_{\mathcal{D}\in\mathbb{D}(\mathcal{F})} \lim_{\xi} \mathcal{D},$$

introduced and studied in [7, Section 4]. In view of [7, Prop.20, Lemma 22], $\mathrm{U}_{\mathbb{D}}$ is generally not a functor ([11]). In particular, it is not a functor if $\mathbb{D} = \mathbb{F}_1$. However, $\mathrm{U}_{\mathbb{D}}$ preserves initiality, as long as $\mathbb{D}$ is $\mathbb{F}_0$-composable. Indeed, if $x \in \lim_{f^-(\mathrm{U}_{\mathbb{D}}\,\tau)} \mathcal{F}$ then $f(x) \in \lim_{\mathrm{U}_{\mathbb{D}}\,\tau} f[\mathcal{F}]$. Suppose $\mathcal{D} \in \mathbb{D}(\mathcal{F})$. Then $f[\mathcal{D}] \geq f[\mathcal{F}]$ and $f[\mathcal{D}] \in \mathbb{D}$ because $\mathbb{D}$ is $\mathbb{F}_0$-composable, so that $f[\mathcal{D}] \in \mathbb{D}(f[\mathcal{F}])$. Since $f(x) \in \lim_{\mathrm{U}_{\mathbb{D}}\,\tau} f[\mathcal{F}]$, we conclude that $f(x) \in \lim_{\tau} f[\mathcal{D}]$, that is, $x \in \lim_{f^-\tau} \mathcal{D}$. Hence $x \in \lim_{\mathrm{U}_{\mathbb{D}}\,f^-\tau} \mathcal{F}$.

[10]Given a Kent convergence ξ and $x \in \lim_{\xi} \mathcal{F}$, let $\pi[x, \mathcal{F}]$ denote the prime topology in which the only non-isolated point is x and $\mathcal{N}_{\pi[\mathcal{F},x]}(x) = \mathcal{F} \wedge \{x\}^{\uparrow}$. Then $\xi = f\sigma$ where $\sigma = \coprod_{(\mathcal{F},x)\in\xi} \pi[x, \mathcal{F}]$ is the topological sum and f is the obvious quotient map identifying all copies of the same point.

[11]but it can be: note that $\mathrm{U}_{\mathbb{D}}$ is the identity functor when $\mathbb{D} = \mathbb{F}$ and $\mathrm{U}_{\mathbb{D}} = \mathrm{S}$ when $\mathbb{D} = \mathbb{U}$. See [7] for a characterization of classes $\mathbb{D}$ for which $\mathrm{U}_{\mathbb{D}}$ is a functor.

Definition 2.8. Let $\mathcal{C}$ be a class of convergences. We say that $\tau \in \mathcal{C}$ is $\mathcal{C}$-*initially stable* if $f^-\tau \in \mathcal{C}$ for every map f with codomain $|\tau|$. The *initial kernel* $\ker_i \mathcal{C}$ of $\mathcal{C}$ is the subclass of $\mathcal{C}$ formed by $\mathcal{C}$-initially stable convergences. The class $\mathcal{C}$ is *initially closed* if $\ker_i \mathcal{C} = \mathcal{C}$.

Notions of $\mathcal{C}$-*finally stable* convergence, *final kernel* $\ker_f \mathcal{C}$, and *finally closed* class are defined dually.

Remark 2.9. Since

$$g^-(f^-\tau) = (f \circ g)^-\tau \text{ and } g(f\xi) = (g \circ f)\xi,$$

$\ker_i \mathcal{C}$ is initially closed and $\ker_f \mathcal{C}$ is finally closed, for any class $\mathcal{C}$ of convergences.

Remark 2.10. Note that a class $\mathcal{C}$ is reflective if and only if it is initially closed and closed under suprema, and coreflective if and only if it is finally closed and closed under infima ([12]). In particular, given a functor F, $\mathbf{F}_+$ is initially closed and $\mathbf{F}_-$ is finally closed.

Theorem 2.11. *Let F be a functor.*

(1) *If F respects finality then $\mathbf{F}_+$ is finally closed. Moreover, if $\mathbf{F}_+$ is finally closed, then R_F respects finality. In particular a reflective class is finally closed if and only if the corresponding reflector respects finality.*
(2) *If F respects initiality then $\mathbf{F}_-$ is initially closed. Moreover, if $\mathbf{F}_-$ is initially closed, then C_F respects initiality. In particular, a coreflective class is initially closed if and only if the corresponding coreflector respects initiality.*

Proof. By duality we only need to prove (1). Assume that F respects finality and let $\xi \in \mathbf{F}_+$, that is, $\xi \leq F\xi$. Then for every f with $|\xi|$ as domain

$$f\xi \leq f(F\xi) \overset{(2.4)}{\leq} F(f\xi),$$

so that $f\xi \in \mathbf{F}_+$.

Assume now that $\mathbf{F}_+ = (\mathbf{R_F})_+$ is finally closed, and let $f : |\xi| \to Y$. Since $\mathrm{R}_F\, \xi \in \mathbf{F}_+$, $f(\mathrm{R}_F\, \xi) \in \mathbf{F}_+$, that is,

$$f(\mathrm{R}_F\, \xi) \leq \mathrm{R}_F(f(\mathrm{R}_F\, \xi)) \leq \mathrm{R}_F(f\xi),$$

where the second inequality follows from $\mathrm{R}_F\, \xi \leq \xi$. As a result, R_F respects finality. $\square$

Note that reflective classes are initially closed but reflectors need not respect initiality while coreflective classes are finally closed but coreflectors need not respect finality:

[12]Of course, suprema and infima can be seen as final and initial structures respectively, for a concrete sink or concrete source.

Example 2.12 (The reflector T on topologies does not respect initiality)**.** Let

$$X = \{x_{n,k} : (n,k) \in \mathbb{N}^2\} \cup \{x_n : n \in \mathbb{N}\} \cup \{\infty\}$$

with its usual bisequence pretopology π defined by $\mathcal{V}_\pi(x_{n,k}) = \{x_{n,k}\}^\uparrow$, $\mathcal{V}_\pi(x_n) = \{\{x_{n,k} : k \geq p\} \cup \{x_n\} : p \in \mathbb{N}\}^\uparrow$ and $\mathcal{V}_\pi(\infty) = \{\{x_n : n \geq p\} \cup \{\infty\} : p \in \mathbb{N}\}$. Let $A = \{x_{n,k} : (n,k) \in \mathbb{N}^2\} \cup \{\infty\}$ and $f : A \to X$ be the inclusion map. Then $f^-\pi = \pi_{|A}$ is the discrete topology, hence $\mathrm{T}(f^-\pi) = f^-\pi$ is discrete, while $f^-(\mathrm{T}\,\pi) = (\mathrm{T}\,\pi)_{|A}$ is not, as the neighborhood filter of ∞ for $\mathrm{T}\,\pi$ meshes with A.

Example 2.13 (The coreflector K on locally compact convergences does not respect finality)**.** Let ξ be the Arens topological space ([13]), so that its compact subsets are finite and thus $\mathrm{K}\,\xi$ is discrete. Let $f : |\xi| \to \omega$ be defined by $f(x_{n,k}) = k$ for all n and $f(\infty) = \omega$. Then $f\xi$ is the usual topology of ω because $f[\mathcal{N}(\infty)] = \{\omega\} \wedge (\omega)_0$ and thus is locally compact. But $f(\mathrm{K}\,\xi)$ is discrete because $\mathrm{K}\,\xi$ is. Hence $\mathrm{K}(f\xi) \ngeq f(\mathrm{K}\,\xi)$.

On the other hand,

Proposition 2.14. *The coreflector* K *respects initiality.*

Proof. If $x \in \lim_{f^-(\mathrm{K}\,\tau)} \mathcal{F}$ then $f(x) \in \lim_{\mathrm{K}\,\tau} f[\mathcal{F}]$, that is, $f(x) \in \lim_\tau f[\mathcal{F}]$ and there is $K \in \mathcal{K}(\tau) \cap f[\mathcal{F}]$. Then there is $F \in \mathcal{F}$ with $f(F) \subset K$ so that $F \subset f^-(K) \in \mathcal{F}$. Remains to see that $f^-(K)$ is $f^-\tau$-compact to conclude that $x \in \lim_{\mathrm{K}(f^-\tau)} \mathcal{F}$. If an ultrafilter $\mathcal{U} \# f^-[K]$ then $f[\mathcal{U}] \# K$ so that $\lim_\tau f[\mathcal{U}] \cap K \neq \varnothing$ and thus $\lim_{f^-\tau} \mathcal{U} \cap f^-(K) \neq \varnothing$. □

Recall that I_κ is the coreflector on convergences based in filters with a filter-base of cardinality less than $\aleph_\kappa$.

Proposition 2.15. *Let κ be a cardinal. Then* I_κ *respects initiality.*

Proof. If $x \in \lim_{f^-(\mathrm{I}_\kappa\,\tau)} \mathcal{F}$ then $f(x) \in \lim_{\mathrm{I}_\kappa\,\tau} f[\mathcal{F}]$, that is, there is a filter $\mathcal{D} \in \mathbb{F}_\kappa$ with $\mathcal{D} \leq f[\mathcal{F}]$ and $f(x) \in \lim_\tau \mathcal{D}$. Let $\mathcal{B}$ denote a filter-base of $\mathcal{D}$ of cardinality less than $\aleph_\kappa$. For each $B \in \mathcal{B}$ there is $F_B \in \mathcal{F}$ with $f(F_B) \subset B$. Then $\{F_B : B \in \mathcal{B}\}^\uparrow \in \mathbb{F}_\kappa$ and this filter converges $\mathrm{I}_\kappa f^-\tau$ to x and the conclusion follows. □

Recall from [7] that a filter $\mathcal{F}$ is $\mathbb{D}$-*rich* if for every $f : X \to Y$ and every $\mathcal{G} \in \mathbb{D}(f[\mathcal{F}])$ there is $\mathcal{D} \in \mathbb{D}(\mathcal{F})$ with $\mathcal{G} \geq f[\mathcal{D}]$. Let $\mathcal{R}(\mathbb{D})$ denote the class of $\mathbb{D}$-rich filters. If $\mathbb{D} \subset \overline{\mathbb{F}}$ then $\mathbb{D} \subset \mathcal{R}(\mathbb{D})$ [7, Lemma 19], so that $\mathrm{B}_{\mathcal{R}(\mathbb{D})} \leq \mathrm{B}_\mathbb{D}$.

[13] that is, the prime topology on

$$\{x_{n,k} : n,k \in \omega\} \cup \{\infty\}$$

defined by $V \in \mathcal{N}(\infty)$ if there is $h : \mathbb{N} \to \mathbb{N}$ and $p \in \mathbb{N}$ with $\{\infty\} \cup \{x_{n,k} : k \geq h(n), n \geq p\} \subset V$.

Theorem 2.16. *If* $\xi \geq \mathrm{B}_{\mathcal{R}(\mathbb{D})}\,\xi$ *then*

$$\mathrm{B}_{\mathbb{D}}(f\xi) \geq f(\mathrm{B}_{\mathbb{D}}\,\xi),$$

for every function f *with domain* $|\xi|$.

On the other hand, if there is a filter $\mathcal{F}$ *that is not* $\mathbb{D}$*-rich then there is a convergence* ξ *with*

$$\mathrm{B}_{\mathbb{D}}(f\xi) \ngeq f(\mathrm{B}_{\mathbb{D}}\,\xi).$$

Proof. If $y \in \lim_{\mathrm{B}_{\mathbb{D}}(f\xi)} \mathcal{H} \subset \lim_{\mathrm{B}_{\mathbb{D}}\, f(\mathrm{B}_{\mathcal{R}(\mathbb{D})}\,\xi)} \mathcal{H}$ then there is $\mathcal{G} \in \mathbb{D}(|f\xi|)$ with $\mathcal{G} \leq \mathcal{H}$ and $x \in f^{-}y$ and $\mathcal{F} \in \mathcal{R}(\mathbb{D})|\xi|$ with $x \in \lim_{\xi} \mathcal{F}$ and $f[\mathcal{F}] \leq \mathcal{G}$. As $\mathcal{F}$ is $\mathbb{D}$-rich, there is $\mathcal{D} \in \mathbb{D}(\mathcal{F})$ with $\mathcal{G} \geq f[\mathcal{D}]$. Since $\mathcal{D} \geq \mathcal{F}$, $x \in \lim_{\mathrm{B}_{\mathbb{D}}\,\xi} \mathcal{D}$ so that $f(x) \in \lim_{f(\mathrm{B}_{\mathbb{D}}\,\xi)} \mathcal{G}$.

If there is $\mathcal{F} \in \mathbb{F}X$ that is not $\mathbb{D}$-rich, then there is $f : X \to Y$ and there is $\mathcal{G} \in \mathbb{D}(f[\mathcal{F}])$ such that $\mathcal{G} \ngeq f[\mathcal{D}]$ whenever $\mathcal{D} \in \mathbb{D}(\mathcal{F})$. Pick $x_0 \in X$. Let ξ be the pretopology on X where $\mathcal{V}_{\xi}(t) = \mathcal{F} \wedge \{t\}^{\uparrow}$ if $f(t) = f(x_0)$ and $\mathcal{V}_{\xi}(t) = \{t\}^{\uparrow}$ if $f(t) \neq f(x_0)$. Then

$$f(x_0) \in \lim_{\mathrm{B}_{\mathbb{D}}\, f\xi} \mathcal{G} \setminus \lim_{f(\mathrm{B}_{\mathbb{D}}\,\xi)} \mathcal{G}.$$

□

Corollary 2.17. *The coreflector* $\mathrm{B}_{\mathbb{D}}$ *respects finality if and only if all filters are* $\mathbb{D}$*-rich.*

It is clear that all filters are $\mathbb{F}$-rich and also $\mathbb{U}$-rich (see e.g., [8, Lemma II.6.6] for $\mathbb{U}$-rich).

Lemma 2.18. *If all filters are* $\mathbb{D}$*-rich then* $\mathbb{U} \subset \mathbb{D}$.

Proof. Recall from [7, Lemma 22] that if the class $\mathbb{D}$ satisfies $\mathbb{D}X \neq \varnothing$ for all set X and $\mathcal{F}$ is a filter with $\mathbb{D}(\mathcal{F}) = \varnothing$, then $\mathcal{F}$ is not $\mathbb{D}$-rich ([14]). In particular, if $\mathbb{D}$ does not contain all ultrafilters and $\mathcal{U}$ is an ultrafilter that is not a $\mathbb{D}$-filter, then $\mathbb{D}(\mathcal{U}) = \varnothing$ and thus $\mathcal{U}$ is not $\mathbb{D}$-rich. In other words if every filter is $\mathbb{D}$-rich then $\mathbb{U} \subset \mathbb{D}$. □

In view of Corollary 2.17 and Lemma 2.18:

Corollary 2.19. *The coreflector* $\mathrm{B}_{\mathbb{U}}$ *respects finality but* G, $\mathrm{I}_0, \mathrm{I}_1$ *and* Seq *do not.*

To summarize the main examples we went over:

[14]consider $f : X \to \{*\}$.

modifier	respects initiality	respects finality
Dis	No	Yes
I_κ	Yes	No
$\mathrm{B}_{\mathbb{U}}$	No	Yes
G	No	No
Seq	No	No
K	Yes	No
χ	Yes	No
S	Yes	No
S_1	Yes	No
$\mathrm{S}_{\wedge 1}$	Yes	No
S_0	Yes	No
T	No	No

3. Heredity

As already noticed, in contrast to preserving initiality, preserving finality is a rare phenomenon. Thus we focus now on questions related to preserving initiality.

The inequality

$$(F\xi)_{|A} \leq F(\xi_{|A}) \tag{3.1}$$

is true for every functor F, as a particular case of (2.1).

We call a concrete functor F *hereditary* if (2.3) is restricted to inclusion maps, that is, if

$$(F\xi)_{|A} \geq F(\xi_{|A}) \tag{3.2}$$

for every convergence ξ and every $A \subset |\xi|$. Of course, every functor that respects initiality is hereditary, but not conversely:

Example 3.1 (An hereditary functor that does not respect initiality)**.** Dis is hereditary: $(\mathrm{Dis}\,\xi)_{|A} = \mathrm{Dis}(\xi_{|A})$ is the discrete topology on A, but as we have seen in Example 2.3, Dis does not respect initiality.

We know that $\mathrm{S}_0 \geq \mathrm{T}$ and S_0 respects initiality, hence is hereditary. On the other hand, this is the coarsest such modifier:

Theorem 3.2. *Let J be a modifier with $J \geq \mathrm{T}$. If J is hereditary then $J \geq \mathrm{S}_0$.*

Proof. Assume to the contrary that $J\xi \ngeq \mathrm{S}_0\,\xi$ for some convergence ξ. Then there is a filter $\mathcal{F}$ on $|\xi|$ and $x \in |\xi|$ with $x \in \lim_{J\xi} \mathcal{F}$ but $x \notin \lim_{\mathrm{S}_0\,\xi} \mathcal{F}$, that is, there is $A \in \mathcal{F}^{\#}$ with $x \notin \mathrm{adh}_\xi A$. Let $B = A \cup \{x\}$. Then, $x \in \lim_{(J\xi)_{|B}} \mathcal{G}$ where $\mathcal{G} = \{H \subset A : \exists F \in \mathcal{F}, F \cap A \subset H\}$ because $x \in \lim_{J\xi} \mathcal{F} \vee A$, but $x \notin \lim_{\mathrm{T}(\xi_{|B})} \mathcal{G}$ hence $x \notin \lim_{J(\xi_{|B})} \mathcal{G}$. Indeed, $x \notin \mathrm{adh}_{\xi_{|B}} A \subset \mathrm{adh}_\xi A$, so A is $\xi_{|B}$-closed. □

Remark 3.3. More specifically we showed that if $J \geq \mathrm{T}$ and for a certain convergence ξ we have

$$(J\xi)_{|A} \geq J(\xi_{|A})$$

for all $A \subset |\xi|$, then $J\xi \geq \mathrm{S}_0\,\xi$.

Definition 3.4. If $\mathbf{C}$ denotes a class of convergences, $\mathbf{C}X$ denotes the set of convergences on X that belong to $\mathbf{C}$. A convergence $\xi \in \mathbf{C}$ is $\mathbf{C}$-hereditary if for every $A \subset |\xi|$, the subspace convergence $\xi_{|A} \in \mathbf{C}$. The *hereditary kernel* $\ker_H \mathbf{C}$ *of* $\mathbf{C}$ is the subclass of $\mathbf{C}$ formed by $\mathbf{C}$-hereditary convergences. The class $\mathbf{C}$ is *hereditary* if it is equal to its hereditary kernel, that is, if every $\xi \in \mathbf{C}$ is $\mathbf{C}$-hereditary.

Note that reflective classes are initially closed, hence hereditary. In particular, given a functor F, the associated reflective class $\mathbf{F}_+$ is hereditary. On the other hand, we prove ([15]) in a way similar to Theorem 2.11 that:

Lemma 3.5. *Let F be a (concrete) functor. If F is hereditary then so is $\mathbf{F}_-$. On the other hand, if $\mathbf{F}_-$ is hereditary then the corresponding coreflector C_F is hereditary.*

Theorem 3.6. *Let E be an expansive functor that respects initiality. Then*

$$\mathbf{Top} \cap \ker_i \mathbf{TE}_- = \mathbf{Top} \cap \ker_H \mathbf{TE}_- = \mathbf{Top} \cap \mathbf{S}_0\mathbf{E}_-.$$

Proof. Because S_0 and E respects initiality, so does $\mathrm{S}_0\, E$, hence $\mathbf{S}_0\mathbf{E}_-$ is initially closed by Theorem 2.11 (2). Since $\mathrm{S}_0 \geq \mathrm{T}$, $\mathbf{S}_0\mathbf{E}_- \subset \mathbf{TE}_-$ hence

$$\mathbf{S}_0\mathbf{E}_- \subset \ker_i \mathbf{TE}_- \subset \ker_H \mathbf{TE}_-.$$

Moreover, if $\xi \in \mathbf{Top} \cap \ker_H \mathbf{TE}_-$ then $\xi = \mathrm{T}\, E\xi$ and for every $A \subset |\xi|$ we have

$$\xi_{|A} = (\mathrm{T}\, E\xi)_{|A} \geq \mathrm{T}\, E(\xi_{|A}) \geq \mathrm{T}(E\xi)_{|A}.$$

In view of Remark 3.3, for $J = \mathrm{T}$ and $E\xi$ playing the role of ξ, we conclude that $\mathrm{T}\, E\xi = \mathrm{S}_0\, E\xi$ and thus $\xi \in \mathbf{Top} \cap \mathbf{S}_0\mathbf{E}_-$. □

In view of Propositions 2.14 and 2.15, we can apply Theorem 3.6 with $E = \mathrm{K}$ and with $E = \mathrm{I}_1$ to the effect that:

Corollary 3.7. *The hereditary kernel and initial kernel of*

(1) *the class of sequential topologies is the class of Fréchet topologies.*
(2) *the class of k-topologies is the class of k'-topologies.*

In other words, as is well-known, hereditarily sequential topologies are Fréchet topologies and hereditarily k-topologies are k'-topologies.

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Proof. Suppose F is hereditary and let ξ in $\mathbf{F}_-$, that is, $\xi \geq F\xi$. Then for every $A \subset |\xi|$, $\xi_{|A} \geq (F\xi)_{|A} \geq F(\xi_{|A})$ because F is hereditary, that is, $\xi_{|A}$ is in $\mathbf{F}_-$.

Assume and $\mathbf{F}_- = (\mathbf{C_F})_-$ is hereditary. Then for every ξ and $A \subset |\xi|$, $\mathrm{C}_F\, \xi$ is in $\mathbf{F}_-$, which is hereditary, so that

$$(\mathrm{C}_F\, \xi)_{|A} \geq \mathrm{C}_F((\mathrm{C}_F\, \xi)_{|A}) \geq \mathrm{C}_F(\xi_{|A})$$

because $\mathrm{C}_F\, \xi \geq \xi$ and thus $(\mathrm{C}_F\, \xi)_{|A} \geq \xi_{|A}$. □

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