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$C(X)$ determines X - a unified theory

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Abstract

One of the fundamental problems in rings of continuous functions is to extract those spaces X for which $C(X)$ determines X, that is to investigate X and Y such that $C(X)$ isomorphic with $C(Y)$ implies X homeomorphic with Y. Later S. Banach and M. Stone proved independently with slight variance, that if X is a compact Hausdorff space, $C(X)$ also determine X. Their works were maximally extended by E. Hewitt who introduced realcompact spaces and later Melvin Henriksen and Biswajit Mitra solved the problem for locally compact and nearly realcompact spaces. In this paper we tried to develop a unified theory of this problem to cover up all the works in the literature introducing the notion $\mathcal{P}$-compact spaces.

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1. Introduction

In this paper we tried to develop a unified theory of $C(X)$ determining X in the sense that under what general condition(s), $C(X)$ is isomorphic with $C(Y)$ implies that X is homeomorphic with Y. It is a very age-old and a fundamental problem in the area of rings of continuous functions. M.H. Stone and E. Čech independently proved that for any space X, there exists a Tychonoff space Y such that $C(X)$ is isomorphic with $C(Y)$. Thus they directed us to investigate this problem within the class of Tychonoff spaces, we shall therefore always stick ourselves within the class of Tychonoff spaces unless otherwise mentioned. Banach and Stone independently proved with slight variance that if X is a compact and Hausdorff space, then $C(X)$ determines X. In the year 1948, E. Hewitt inroduced realcompact spaces and proved that within the class of realcompact spaces, $C(X)$ determines X [4, Theorem 8.3]. His theory is in some sense maximal as because he had shown that for any space X, $C(X)$ is isomorphic with $C(\upsilon X)$ [4, Theorem 8.8(a)], where υX is the Hewitt-realcompactification of X. This result particularly highlights that we can never extend "C(X) determining X"-type problem for those spaces which are the generalizations of realcompactness. More precisely, suppose P be a topological property such that every realcompact space satisfies that property and we call a space to be P-space if X satisfies the property P. Now Hewitt's above result suggests that it is never possible to investigate "$C(X)$ determines X"-type problem for P-spaces only, because if X is a P-space which is not realcompact then its υX, being always realcompact, is also a P-space and also $C(X)$ is isomorphic with $C(\upsilon X)$ but X is not homeomorphic to υX. So we need to impose additional restriction(s) on the P-space to study above type of problem. That is why, in the year 2005, Henriksen and Mitra proved that if X is locally compact and nearly realcompact, then $C(X)$ determines X, where nearly realcompact space is the generalization of realcompactness, introduced by E. K. van Douwen and R. L. Blair in [1]. In the year 1987 [10], Redlin and Watson, brought the isomorphism theorem of Banach and Stone and that of Hewitt into a common platform by introducing A-compact space for an intermediate subring $A(X)$ of $C(X)$ and proved that if X is A-compact and Y is B-compact and $A(X)$ is isomorphic with $B(Y)$, then X is homeomorphic with Y. If A and B are C^*, then Banach-Stone theorem follows and in case, A and B are C, then Hewitt's theorem follows.

In this paper, we develop a common theory which includes all the developments so far. By an intermediate subring of $C(X)$, we always mean that it lies in between $C^*(X)$ and $C(X)$. We usually denote these rings by $A(X)$, $B(X)$ etc. We here mainly concern about the subsets of the family of all maximal ideals $\mathcal{M}_A(X)$ (or simply $\mathcal{M}_A$) of intermediate subring $A(X)$ of $C(X)$. If $\mathcal{P}_A \subseteq \mathcal{M}_A$, accordingly we have defined $\mathcal{P}_A$-compact and locally-$\mathcal{P}_A$ [Definition 3.1]. In general, for a subset $\mathcal{P}$ of $\mathcal{M}_A$, $\mathcal{P}$-maximal ideal is not algebraic [Remark 3.4] and $\mathcal{P}$-compact, locally-$\mathcal{P}$ are not topological in the sense given in remark 3.14. So we have introduced concept of algebraic set [Definition

3.3] among the subsets of $\mathcal{M}_A$. As for example, $\mathcal{M}_A$ is itself an algebraic set. However, there are also many examples of non-trivial algebraic sets. We defined a relation, called conjugate, between subsets of $\mathcal{M}_A(X)$ and $\mathcal{M}_B(Y)$ when $A(X)$ is isomorphic with $B(Y)$. In general, a subset of $\mathcal{M}_A(X)$ may not have a conjugate in $\mathcal{M}_B(Y)$ [Remark 3.2], however, if the subset of $\mathcal{M}_A(X)$ is algebraic, then it always have unique conjugate in $\mathcal{M}_B(Y)$, which is also algebraic [Theorem 3.7]. Therefore we denote all the conjugates of an algebraic set $\mathcal{P}$ by the same symbol $\mathcal{P}$ itself and proved that $\mathcal{P}$-compact and locally-$\mathcal{P}$ are topological property [Theorem 3.15, 3.17]. We finally proved that if $\mathcal{P}$ be an algebraic set in $A(X)$ and $A(X)$ is isomorphic with $B(Y)$ and further X and Y both are locally-$\mathcal{P}$ and $\mathcal{P}$-compact, then X is homeomorphic with Y [Theorem 3.19]. This result unifies all the above mentioned isomorphism theorems on suitable choices of algebraic sets.

2. Preliminaries

We use most of the preliminary ideas, symbols and terminologies from the classic monograph of Leonard Gillman and Meyer Jerison, Rings of Continuous Functions [4]. For any $f \in C(X)$ or $C^*(X)$, $Z(f) = \{x \in X : f(x) = 0\}$, is called zero set of f and the complement of zero set is called cozero set or cozero part of f, denoted as $cozf$. For $f \in C(X)$, the set $cl_X(X \backslash Z(f))$ is known as the support of f. If h is a homomorphism from $C(X)$ or $C^*(X)$ into $C(Y)$, then the image of a bounded function on X is a bounded function on Y under h. A space is called pseudocompact if $C(X) = C^*(X)$. A maximal ideal M in $C(X)$ is called real maximal if $C(X)/M$ is isomorphic with $\mathbb{R}$, otherwise is called hyper-real. A maximal ideal M is called fixed if there exists $x \in X$ such that $f(x) = 0$, for all $f \in M$, usually denoted as M_x. Every fixed maximal ideal is real. However converse may not be true. A space is realcompact if every real maximal ideal in $C(X)$ is fixed. B. Mitra and S.K. Acharyya introduced the ring $\chi(X)$ in their paper [8]. $\chi(X)$ is the smallest subring of $C(X)$ containing $C^*(X)$ and $C_H(X)$, where $C_H(X) = \{f \in C(X) | cl_X(X \backslash Z(f))$ is hard in $X\}$. A subset H of X is hard in X if it is closed in $X \cup cl_{\beta X}(\upsilon X \backslash X)$.

They proved the following theorem

Theorem 2.1 ([8], Theorem 3.4)**.** *A space X is nearly pseudocompact if and only if $\chi(X) = C^*(X)$.*

In the same paper [8], Mitra and Acharyya defined hard pseudocompact spaces. A space is said to be hard pseudocompact if $C(X) = \chi(X)$. Then it is evident, hard pseudocompactness and nearly pseudocompactness together imply pseudocompactness of a space X and vice versa. Later Ghosh and Mitra in [3] worked in detail over hard pseudocompact spaces and their properties.

Henriksen and Mitra introduced Strongly real maximal ideal (in brief SRM ideal) in [5]. A maximal ideal M of $C(X)$ is called SRM ideal if there exists $g \notin M$ such that $fg \in C^*(X)$ for all $f \in C(X)$. There is, in general, no connection between real maximal ideal and SRM ideal. In fact not all fixed maximal ideal is SRM. If X is locally compact, all fixed maximal ideals are SRM. However SRM ideals help to characterize nearly realcompact spaces

Theorem 2.2 ([5], Theorem 2.9)**.** *A space X is nearly realcompact if and only if every SRM ideal is fixed.*

In $C(X)$, more generally in commutative ring with 1, if $\mathcal{M}(X)$ is the collection of all maximal ideal of $C(X)$, naturally topologized with hull-kernel topology, then $\mathcal{M}(X)$ with this topology is called structure space of $C(X)$. In this structure space $\{V_f : f \in C(X)\}$, where $V_f = \{M \in \mathcal{M}(X) : f \in M\}$, forms a base for closed sets and the structure space turns out to be compact T_1 space. In general structure space of a commutative ring with 1 may not be Hausdorff but the structure space of $C(X)$ or any intermediate subrings of $C(X)$, turns out to be Hausdorff.

Redlin and Watson in [10] discussed different properties of intermediate subrings of $C(X)$, almost similar to that of $C(X)$. As for instance, any intermediate subring $A(X)$ of $C(X)$ is a lattice ordered ring. Any maximal ideal M of $A(X)$ is absolutely convex. $A(X)/M$ is a totally ordered field containing $\mathbb{R}$ as a totally ordered subfield. For each $x \in X$, $M_x^A = \{f \in A(X) : f(x) = 0\}$ is precisely the collection of fixed maximal ideals of $A(X)$. A maximal ideal of $A(X)$ is real if $A(X)/M$ is isomorphic with $\mathbb{R}$. Every fixed maximal ideal of $A(X)$ is real. Redlin and Watson in [10] defined a space to be A-compact if every real maximal ideal of $A(X)$ is fixed and proved the following theorem

Theorem 2.3 ([10], Theorem 7)**.** *Let X be A-compact and Y be B-compact. If $A(X)$ is isomorphic with $B(Y)$, then X is homeomorphic with Y.*

3. Main results

We begin this section with the definition of $\mathcal{P}_A$-compactness. Let $A(X)$ be an intermediate subring of $C(X)$ and $\mathcal{M}_A$ be the family of all maximal ideals in $A(X)$. Let $\mathcal{P}$ be a subset of $\mathcal{M}_A$. A maximal ideal M is called $\mathcal{P}$-maximal ideal if $M \in \mathcal{P}$, otherwise M is called non-$\mathcal{P}_A$-maximal ideal.

Definition 3.1. A space X is called $\mathcal{P}_A$-compact if every $\mathcal{P}_A$-maximal ideal is fixed. A space X is called locally-$\mathcal{P}_A$ if every fixed maximal ideal is $\mathcal{P}_A$-maximal ideal.

For $A(X) = C(X)$, we shall simply write $\mathcal{M}_A$ by $\mathcal{M}(X)$, $\mathcal{P}_C$-maximal ideal by $\mathcal{P}$-maximal, $\mathcal{P}_C$-compact by $\mathcal{P}$-compact and locally-$\mathcal{P}_A$ by locally-$\mathcal{P}$. It is clear that every compact space is $\mathcal{P}$-compact. However, in future, we shall often use $\mathcal{P}$-maximal instead of $\mathcal{P}_A$-maximal ideal to avoid notational complexity.

Remark 3.2. Suppose $A(X)$ and $B(Y)$ be two intermediate subrings of $C(X)$ and $C(Y)$ respectively. Let $\mathcal{P} \subseteq \mathcal{M}_A$. If ψ be an isomorphism between $A(X)$ and $B(Y)$, then ψ-image of any $\mathcal{P}$-maximal ideal is $\psi^*(\mathcal{P})$-maximal ideal, defined in the next paragraph. But choice of $\psi^*(P)$ strictly depends on ψ. For example, if we consider two homeomorphisms on $[0,1]$: $t \to t$ and $t \to 1-t$, then they induce two distinct isomorphisms ψ_1 and ψ_2 on $C([0,1])$, defined as

$\psi_1(f)(t) = f(f)$ and $\psi_2(f)(t) = f(1-t)$. Let $\mathcal{P}^{[a,b]} = \{M_x : a \leq x \leq b\}$. Then $\psi_1^*(\mathcal{P}^{[0,\frac{1}{2}]}) = \mathcal{P}^{[0,\frac{1}{2}]}$, while $\psi_2^*(\mathcal{P}^{[0,\frac{1}{2}]}) = \mathcal{P}^{[\frac{1}{2},1]}$. Thus $\mathcal{P}^{[0,\frac{1}{2}]}$-maximal ideals are preserved under the isomorphism ψ_1, but not preserved under ψ_2. So we can conclude that $\mathcal{P}$-maximal ideal, in general, may not be algebraic invariant. On the other hand, if we take $\mathcal{P}$ to be the collection of all real maximal ideals in $C(X)$ and $\mathcal{Q}$ to be the collection of all real maximal ideals in $C(Y)$, then for any isomorphism ϕ between $C(X)$ and $C(Y)$, $\phi^*(\mathcal{P}) = \mathcal{Q}$. In this case, we can identify $\mathcal{P}$ with $\mathcal{Q}$ and conclude that $\mathcal{P}$-maximal ideals are algebraic invariant. In next few paragraphs, we developed an analogical concept in more general set up to get hold of isomorphism theorems.

Let for any space X and Y, $A(X)$ and $B(Y)$ be intermediate subrings of $C(X)$ and $C(Y)$ respectively. Let $\sigma : A(X) \to B(Y)$ be an isomorphism. Then we can lift σ to $\sigma^* : \mathcal{P}(\mathcal{M}(A)) \to \mathcal{P}(\mathcal{M}(B))$ defined by $\sigma^*(\mathcal{E}) = \{\sigma(M) : M \in \mathcal{E}\}$. Let $\mathcal{P} \subseteq \mathcal{M}(A)$ of $A(X)$. $\mathcal{Q} \subseteq \mathcal{M}(B)$ is said to be σ-conjugate of P in $B(Y)$ if $\sigma^*(\mathcal{P}) = \mathcal{Q}$. $\mathcal{Q}$ is said to be conjugate of $\mathcal{P}$ in $B(Y)$ if $\mathcal{Q}$ is σ-conjugate of $\mathcal{P}$ in $B(Y)$ for any isomorphism σ between $A(X)$ and $B(Y)$. It is clear that if $\mathcal{Q}$ is conjugate of $\mathcal{P}$ if and only if $\mathcal{P}$ is conjugate of $\mathcal{Q}$. We shall, in future, simply write conjugate of $\mathcal{P}$ without mentioning what $B(Y)$ is, in order to mean that it is σ-conjugate of $\mathcal{P}$ in some $B(Y)$ for some space Y.

Definition 3.3. A subset $\mathcal{P}$ of $\mathcal{M}_A$ is called algebraic in $A(X)$ if for any automorphism ψ of $A(X)$, $\psi^*(P) = P$.

Remark 3.4. It is immediate that $\mathcal{P}$ is algebraic in $A(X)$ if and only if for any two isomorphism π and μ from $A(X)$ onto $B(Y)$, $\pi^*(\mathcal{P}) = \mu^*(\mathcal{P})$. In general, a subset of $\mathcal{M}_A$ may not have a conjugate as for instance, we may refer $\mathcal{P}^{[0,\frac{1}{2}]}$ in the above remark 3.2 . But for algebraic sets, existence of conjugate is assured, once you have an isomorphism from $A(X)$ onto some $B(Y)$. So, in particular, any algebraic set is conjugate to itself.

Theorem 3.5. *$\mathcal{P}$ is algebraic in $A(X)$ if and only if any σ-conjugate of $\mathcal{P}$ is conjugate to $\mathcal{P}$.*

Proof. Let $\mathcal{Q} = \sigma^*(\mathcal{P})$. Let h be any isomorphism between $A(X)$ and $B(Y)$. Then $\sigma^{-1} \circ h$ is an automorphism on $A(X)$. So $(\sigma^{-1} \circ h)^*(\mathcal{P}) = \mathcal{P}$. We shall show that $h^*(\mathcal{P}) = \mathcal{Q}$. Let $T \in h^*(\mathcal{P})$. There exists $S \in \mathcal{P}$, such that $T = h(S)$. So $\sigma^{-1}(T) \in \mathcal{P}$. So $T \in \sigma^*(\mathcal{P})$. Hence $h^*(\mathcal{P}) \subseteq \sigma^*(\mathcal{P})$. Interchanging h and σ we get the opposite inequality. Thus $h^*(\mathcal{P}) = \mathcal{Q}$.

The converse part trivially follows by taking σ to be the identity map. $\square$

Corollary 3.6. *Suppose $A(X)$ is isomorphic with $B(Y)$ and $\mathcal{P}$ is an algebraic set in $A(X)$. Then a maximal ideal is $\mathcal{P}$-maximal ideal if and only if its image is $\mathcal{Q}$-maximal ideal, where $\mathcal{Q}$ is conjugate of $\mathcal{P}$.*

Theorem 3.7. *If $\mathcal{P}$ be an algebraic set in $A(X)$ and whenever $A(X)$ is isomorphic with $B(Y)$, for any space Y, then there exists a unique subset $\mathcal{Q}$ of $\mathcal{M}(B)$, which is conjugate of $\mathcal{P}$ and is also algebraic in $B(Y)$.*

Proof. Uniqueness directly follows from the definition. Suppose π be an isomorphism between $A(X)$ and $B(Y)$. From the above theorem 3.5, it follows that $\mathcal{Q} = \pi^*(\mathcal{P})$ is the conjugate of $\mathcal{P}$. It is enough to show that $\mathcal{Q}$ is algebraic in $B(Y)$. Then for any automorphism σ of $B(Y)$, $\pi^{-1} \circ \sigma \circ \pi$ is an automorphism of $A(X)$. Let $M \in \mathcal{Q}$. There exists N in $\mathcal{P}$, such that $\pi(N) = M$. Now $\pi^{-1} \circ \sigma \circ \pi(N) \in \mathcal{P}$. That is $\pi^{-1} \circ \sigma(M) \in \mathcal{P}$. So, $\sigma(M) \in \pi^*(P) = Q$. That is why, $\sigma^*(\mathcal{Q}) \subseteq \mathcal{Q}$. For the converse, let $T \in \mathcal{Q}$. Then, there exists $S \in \mathcal{P}$ such that $T = \pi(S)$. As $\mathcal{P}$ is algebraic, there exists $M \in \mathcal{P}$ such that $S = \pi^{-1} \circ \sigma \circ \pi(M)$. Then $T = \pi \circ \pi^{-1} \circ \sigma \circ \pi(M) = \sigma(\pi(M))$. Now $\pi(M) \in \mathcal{Q}$. Hence $T \in \sigma^*(\mathcal{Q})$. Thus $\mathcal{Q} \subseteq \sigma^*(\mathcal{Q})$. So $\mathcal{Q} = \sigma^*(\mathcal{Q})$, for any automorphism σ on $A(X)$ □

Note 3.8. *If $\mathcal{P}$ is algebraic in $A(X)$, then we will keep the same symbol $\mathcal{P}$ for its conjugate because of its uniqueness.*

We hereby construct few examples of algebraic sets of maximal ideals in $C(X)$.

Example 3.9.

(1) If $\mathcal{P}$ is the collection of all maximal ideals of $C(X)$. Then $\mathcal{P}$ is trivially an algebraic set. Then every maximal ideal is $\mathcal{P}$-maximal ideal and hence $\mathcal{P}$-compact spaces are precisely the compact spaces.

(2) If $\mathcal{P}$ is the collection of all maximal ideal of $C(X)$ such that $C(X)/M$ is isomorphic with the field of reals. Then again $\mathcal{P}$ is an algebraic set. Then we know that a maximal ideal is $\mathcal{P}$-maximal ideal if and only if it is real maximal ideal introduced by Hewitt in [7] and $\mathcal{P}$-compact spaces are precisely realcompact spaces [7].

(3) If $\mathcal{P}$ is the collection of all maximal ideal M satisfying the following condition: there exists f outside M such that $fg \in C^*(X)$ for all $g \in C(X)$. This kind of maximal ideal is called SRM ideal [5, Theorem 2.12]. Then $\mathcal{P}$ is also algebraic. The reason is as follows: suppose ψ be an automorphism of $C(X)$. Let M be an SRM ideal. There exists f outside M such that $fg \in C^*(X)$ for all $g \in C(X)$. Then $\psi(f)$ is not in $\psi(M)$. Let $h \in C(X)$, there exists unique $g \in C(X)$ such that $\psi(g) = h$. Now $fg \in C^*(X)$. By [4, Theorem 1.7], $\psi(fg) \in C^*(X)$. That is, $\psi(f).\psi(g) = \psi(f)h \in C^*(X)$, for $h \in C(X)$. Thus $\psi(M)$ is also an SRM ideal. Thus $\psi^*(\mathcal{P}) \subseteq \mathcal{P}$. As ψ^{-1} is also an automorphism of $C(X)$, the other side of the inequality trivially follows. So $\psi^*(\mathcal{P}) = \mathcal{P}$, for any automorphism ψ of $A(X)$. Thus $\mathcal{P}$-compact spaces are precisely the nearly realcompact spaces [5, Theorem 2.9].

We now topologize $\mathcal{M}(A)$ with the hull-kernel topology with $\{\mathcal{A}_f | f \in A(X)\}$ as its base for closed sets in $\mathcal{M}(A)$, where $\mathcal{A}_f = \{M \in \mathcal{M}(A) | \ f \in M\}$ and consider the subspace $\mathcal{P}$ of $\mathcal{M}(A)$. The base for closed sets in $\mathcal{P}$ is of the form $\mathcal{A}_f \cap \mathcal{P} : f \in A(X)$.

Lemma 3.10. *If ψ is an isomorphism between $A(X)$ and $B(Y)$ and $\mathcal{P} \subseteq \mathcal{M}(A)$ then $\mathcal{P}$ is homeomorphic with $\psi^*\mathcal{P}$.*

Proof. Let $\psi : A(X) \to B(Y)$ be the isomorphism. Define $\phi : \mathcal{P} \to \psi^*\mathcal{P}$ by $\phi(M) = \psi(M)$. Therefore ϕ is clearly well-defined. ϕ is evidently one-one and onto. Now we have to show that ϕ is closed and continuous. Here $\mathcal{B}_f \cap \psi^*\mathcal{P}$ is a basic closed set in $\psi^*\mathcal{P}$ and $\phi^{-1}(\mathcal{B}_f \cap \psi^*\mathcal{P}) = \phi^{-1}(\mathcal{B}_f) \cap \phi^{-1}(\psi^*\mathcal{P}) = \mathcal{A}_{\psi^{-1}(f)} \cap \mathcal{P}$, which is a basic closed set in $\mathcal{P}$. Again $\phi(\mathcal{A}_f \cap \mathcal{P}) = \mathcal{B}_{\psi(f)} \cap \mathcal{P}$. Hence ϕ is closed and continuous. Therefore $\mathcal{P}$ and $\psi^*\mathcal{P}$ are homeomorphic. □

Corollary 3.11. *Let $\mathcal{P}$ be an algebraic set in $A(X)$. If $A(X)$ is isomorphic with $B(Y)$, then $\mathcal{P}$ and its conjugate are homeomorphic.*

For $\mathcal{P} \subseteq \mathcal{M}(A)$, let $N_{\mathcal{P}_A}(X)$ be the set of all those points x in X for which M_x^A is not $\mathcal{P}_A$-maximal ideal. Then X is locally-$\mathcal{P}_A$ if and only if $N_{\mathcal{P}_A}(X) = \varnothing$. For $A(X) = C(X)$, we denote $N_{\mathcal{P}_A}(X) = N_{\mathcal{P}}(X)$. For example if $\mathcal{P}$ is the class of all real maximal ideals of $C(X)$, then X is locally-$\mathcal{P}$. If we take $\mathcal{P}$ to be family of all SRM ideal in $C(X)$, then X may not be locally-$\mathcal{P}$ as the fixed maximal ideals corresponding to the points in $J \cap X$, if non-empty, where $J = cl_{\beta X}(\beta X \backslash \upsilon X)$ are not SRM ideal [5, Theorem 2.6].

Lemma 3.12. *If X is $\mathcal{P}_A$-compact then $X \backslash N_{\mathcal{P}_A}(X)$ and $\mathcal{P}$ are homeomorphic.*

Proof. Let X be $\mathcal{P}_A$-compact, for $\mathcal{P} \subseteq \mathcal{M}(A)$. Then $\mathcal{P} = \{M_x^A | x \in X \backslash N_{\mathcal{P}_A}(X)\}$. Now define $\sigma : X \backslash \mathcal{N}_{\mathcal{P}_A}(X) \to \mathcal{P}$ by $\sigma(x) = M_x^A$. Then σ is bijective. Let $y \in \sigma^{-1}(\mathcal{A}_f \cap \mathcal{P}) \Leftrightarrow \sigma(y) \in \mathcal{A}_f \cap \mathcal{P} \Leftrightarrow M_y^A \in \mathcal{A}_f \cap \mathcal{P} \Leftrightarrow M_y^A \in \mathcal{A}_f$ and $M_y^A \in \mathcal{P} \Leftrightarrow f \in M_y^A$ and $y \notin N_{\mathcal{P}_A}(X) \Leftrightarrow f(y) = 0$ and $y \notin N_{\mathcal{P}_A}(X) \Leftrightarrow y \in Z(f)$ and $y \notin N_{\mathcal{P}_A}(X) \Leftrightarrow y \in Z(f) \cap (X \backslash N_{\mathcal{P}_A}(X))$. Therefore $\sigma^{-1}(\mathcal{A}_f \cap \mathcal{P}) = Z(f) \cap (X \backslash N_{\mathcal{P}_A}(X))$. Also $\sigma(Z(f) \cap (X \backslash N_{\mathcal{P}_A}(X))) = \mathcal{A}_f \cap \mathcal{P}$. Hence σ is a homeomorphism i.e $X \backslash N_{\mathcal{P}_A}(X)$ and $\mathcal{P}$ are homeomorphic. □

Hence the following theorem is immediate.

Theorem 3.13. *Let $\psi : A(X) \to B(Y)$ be an isomorphism and $\mathcal{P} \subseteq \mathcal{M}(A)$. If X is $\mathcal{P}_A$-compact and Y is $\psi^*\mathcal{P}_B$-compact, then $X \backslash \mathcal{N}_{\mathcal{P}_A}(X)$ is homeomorphic with $Y \backslash N_{\mathcal{P}_B}(Y)$.*

Proof. As $X \backslash N_{\mathcal{P}_A}(X)$ is homeomorphic with $\mathcal{P}$ and $\mathcal{P}$ is homeomorphic with $\psi^*(\mathcal{P})$. Further as, Y is $\psi^*\mathcal{P}_B$-compact, $\psi^*(P)$ is homeomorphic with $Y \backslash N_{\mathcal{P}_B}(Y)$, by transitivity, $X \backslash N_{\mathcal{P}_A}(X)$ and $Y \backslash N_{\mathcal{P}_B}(Y)$ □

Remark 3.14. One thing is to be noted from the previous theorem 3.13, that to assume Y to be $\psi^*(\mathcal{P})$-compact, we need explicit information about the isomorphism ψ. It might be possible that for some isomorphism ϕ, X being ϕ-compact and Y without being $\phi^*(\mathcal{P})$-compact, is homeomorphic with X. Let us consider an example in support of it. Let $\omega_1 = [0, \omega_1)$ denotes the space of 1^{st} uncountable ordinal. Each continuous function in $C(\omega_1)$ is eventually constant, in the sense that for each $f \in C(\omega_1)$, there exists an unique $\alpha < \omega_1$ such that $f(x) = f(\alpha), \forall x \geq \alpha$. Define $\psi : C(\omega_1) \to C(\omega_1)$ by

$$\psi(f)(x) = \begin{cases} f(\alpha), & \text{if } x = 0 \\ f(x), & \text{if } 0 < x < \alpha + 1 \\ f(0), & \text{if } \alpha < x < \omega_1 \end{cases} \tag{3.1}$$

Then ψ is an isomorphism. Let $\mathcal{P} = \{M_0\}$. Then $\psi^*(\mathcal{P}) = \{M^{\omega_1}\}$, where, $M^{\omega_1} = \{f \in C(\omega_1) : \text{there exists an unique } \gamma < \omega_1 \text{ such that } f(x) = 0, \forall x \geq \gamma\}$, which is not fixed in $C(\omega_1)$. According to our terminology, ω_1 is $\mathcal{P}$-compact but not $\psi^*(\mathcal{P})$-compact. Even though, ω_1 is trivially homeomorphic with ω_1. So it is clear that $\mathcal{P}$-compact and $\psi^*\mathcal{P}$-compact are not topologically same. If we take $\mathcal{P}$ to be the family of all fixed maximal ideal in $C(\omega_1)$, then ω_1 is locally-$\mathcal{P}$ but ω_1 is not locally-$\psi^*\mathcal{P}$ as $M_0 \notin \psi^*(\mathcal{P})$. So, locally-$\mathcal{P}$ and locally-$\psi^*(\mathcal{P})$ are also not topologically same. In remark 3.2, we saw that $\mathcal{P}$-maximal ideals may not be algebraic invariant. On the other side, the original isomorphism problem does not care about the nature of isomorphism. It says that if $C(X)$ is isomorphic with $C(Y)$, then under what restrictions on X and Y makes X to be homeomorphic with Y. No information about this isomorphism is mandatory. So all these logical conflicts can be easily managed if we consider $\mathcal{P}$ to be an algebraic set. The following two theorem depict the importance of algebraic sets.

Theorem 3.15. *Let $\mathcal{P}$ be algebraic in $A(X)$. Suppose X is homeomorphic with Y. Then there exists an intermediate subring $B(Y)$ of $C(Y)$ such that $A(X)$ is isomorphic with $B(Y)$. Then X is $\mathcal{P}$-compact if and only if Y is $\mathcal{Q}$-compact, where $\mathcal{Q}$ is the conjugate of $\mathcal{P}$ in $B(Y)$*

Note 3.16. *The above theorem tells that $\mathcal{P}$-compact and $\mathcal{Q}$-compact are topologically same, where $\mathcal{Q}$ is the conjugate of $\mathcal{P}$. This fact again justifies our agreement in note 3.8, that is, we can use same symbol $\mathcal{P}$ for the conjugate of $\mathcal{P}$. Then it follows that $\mathcal{P}$-compact is a topological property.*

Proof. Let $\mathcal{P}$ be an algebraic set in $\mathcal{M}_A(X)$. Suppose X is homeomorphic with Y. Let σ be the homeomorphism. $\psi_\sigma : C(X) \to C(Y)$ defined by $\psi_\sigma(f) = f \circ \sigma^{-1}$ is an isomorphism. Then $\sigma(A(X))$ is an intermediate subring of $C(Y)$, denoted as $B(Y)$ as follows from [4, Theorem 1.18] and $A(X)$ is isomorphic with $B(Y)$. By theorem 3.7, let $\mathcal{Q}$ be the unique conjugate of $\mathcal{P}$ in $B(Y)$. Let M be a $\mathcal{Q}$-maximal ideal in $B(Y)$, then there exists a unique $\mathcal{P}$-maximal ideal $N \in A(X)$ such that $\psi_\sigma(N) = M$ as by definition of the algebraic set, $\psi_\sigma^*(\mathcal{P}) = \mathcal{Q}$. Since X is $\mathcal{P}$-compact, $N = N_x^A$ for some $x \in X$. Let $f \in M_{\sigma(x)}^B$. Then $f(\sigma(x)) = 0$. Now there exists a $g \in A(X)$, such that $\psi_\sigma(g) = f$, that is, $g \circ \sigma^{-1} = f$. This follows that $f \circ \sigma$ is in $A(X)$. Thus $f \circ \sigma \in N_x^A$. So, $\psi_\sigma(f \circ \sigma) \in M$. But $\psi_\sigma(f \circ \sigma) = f \circ \sigma \circ \sigma^{-1} = f$. So $f \in M$. Thus $M_{\sigma(x)} \subseteq M$ and due to maximality, $M = M_{\sigma(x)}^B$. Hence every $\mathcal{Q}$-maximal ideal in $B(Y)$ is fixed. So Y is $\mathcal{Q}$-compact.

Converse part is the replicate of the above proof just by interchanging $\mathcal{P}$ and $\mathcal{Q}$ as by theorem 3.7, conjugate of an algebraic set is also algebraic.

□

In the statement of the next theorem we now justifiably keep the same symbol for an algebraic set and its conjugate and the theorem says that locally-$\mathcal{P}$ is also a topological property. More explicitly, in the following statement when we say that X is locally- $\mathcal{P}$ if and only if Y is locally- $\mathcal{P}$, then the $\mathcal{P}$ attached with Y is actually the conjugate of $\mathcal{P}$ attached with X. But instead of using different symbols, we use the same symbol.

Theorem 3.17. *Let $\mathcal{P}$ be an algebraic set in $A(X)$. Suppose X is homeomorphic with Y. Then X is locally-$\mathcal{P}$ if and only if Y is locally-$\mathcal{P}$.*

Proof. As X is homeomorphic with Y, by the proof of the above theorem, this homeomorphism induces a canonical isomorphism between $C(X)$ and $C(Y)$ which on restriction over $A(X)$, makes $A(X)$ isomorphic with $B(Y)$. Let M_y^B be a fixed maximal ideal in $B(Y)$. Let $g \in A(X)$, such that $g \in \psi^{-1}(M_y^B)$. Then $g \circ \sigma^{-1} \in M_y^B$. That is, $g(\sigma^{-1}(y)) = 0$. Thus $g \in M^A_{\sigma^{-1}}$. Again due to maximality, it follows that $\psi^{-1}(M_y^B) = M^A_{\sigma^{-1}}$, that is, a fixed maximal ideal in $A(X)$. As X is locally-$\mathcal{P}$ and $\mathcal{P}$ is algebraic, $M_y^B \in \mathcal{P}$. Hence Y is locally-$\mathcal{P}$. □

We hereby restate the theorem 3.13, for algebraic set.

Corollary 3.18. *Let $A(X)$ and $B(Y)$ are isomorphic and $\mathcal{P}$ is an algebraic set in $A(X)$. If X and Y are $\mathcal{P}$-compact then $X \backslash \mathcal{N}^A_{\mathcal{P}}(X)$ is homeomorphic with $Y \backslash \mathcal{N}^B_{\mathcal{P}}(Y)$.*

The following corollary is immediate.

Corollary 3.19. *Let $A(X)$ and $B(Y)$ are isomorphic and $\mathcal{P}$ is an algebraic set in $A(X)$. If X and Y are $\mathcal{P}$-compact and locally-$\mathcal{P}$ then X and Y are homeomorphic.*

The next theorem is important to us and is direct consequence of theorem 3.18, as special case for $A(X) = C(X)$ and $B(Y) = C(Y)$.

Theorem 3.20. *If X, Y are $\mathcal{P}$-compact spaces for an algebraic set $\mathcal{P}$ and $C(X)$ is isomorphic with $C(Y)$ then $X \backslash \mathcal{N}_{\mathcal{P}}(X)$* and *$Y \backslash \mathcal{N}_{\mathcal{P}}(Y)$* are homeomorphic.

The following theorem establishes that if $\mathcal{P}$ is an algebraic set containing all fixed maximal ideals, then if X is $\mathcal{P}$-compact, $C(X)$ determines X too.

Theorem 3.21. *Let $\mathcal{P}$ be an algebraic set and $C(X)$ is isomorphic with $C(Y)$. If X and Y both are locally-$\mathcal{P}$ and $\mathcal{P}$-compact, then X is homeomorphic with Y.*

Proof directly follows from the previous theorem 3.19.

Now we can easily deduce all the three results as mentioned in the beginning of this paper as special case.

Banach-Stone theorem: As we have already observed that if we choose $\mathcal{P}$ to be the collection of all maximal ideals of $C(X)$, then $\mathcal{P}$ is an algebraic set. Clearly X is locally-$\mathcal{P}$. $\mathcal{P}$-compact spaces are precisely the compact spaces. So, if X is compact space, then $C(X)$ determines X.

E. Hewitt: If we take $\mathcal{P}$ to be the collection of all real maximal ideals. Then $\mathcal{P}$ is an algebraic set and X is also locally -$\mathcal{P}$. Then it follows that if X is realcompact, then $C(X)$ determines X too.

M. Henriksen and B. Mitra: If we choose $\mathcal{P}$ to be the collection of SRM ideals, then $\mathcal{P}$ is an algebraic set. Then X is locally-$\mathcal{P}$ if and only X is locally pseudocompact [[5], Theorem 2.8]. But if X nearly realcompact, then X is locally-$\mathcal{P}$ if and only if X is locally compact [[5], Lemma 2.7] . Thus we conclude that if X locally compact and nearly realcompact, then $C(X)$ also determines X.

We call, in general, a maximal ideal M of a commutative ring A with unity is B-real maximal ideal, where B is a subring of A containing the unity of A if $M \cap B$ is a maximal ideal of B. If $A(X)$ and $B(X)$ are intermediate subrings of $C(X)$ with $A(X)$ being subring of $B(X)$, then we call X to be $B - A$-compact if every A-real maximal ideal of B is fixed. It is clear from [[4], theorem 7.9(c)], that real maximal ideals are precisely C^*-real maximal ideals of $C(X)$ and $C - C^*$-compact is precisely realcompact space. For any intermediate subring $A(X)$ of $C(X)$, real maximal ideal of $A(X)$ is precisely $C^*(X)$-real maximal ideal of $A(X)$ by [[2], corollary 3.8]. So $C^* - A$-compact is precisely the A-compact spaces.

Redlin and Watson: If we take $\mathcal{P}$ to be the family of real maximal ideals of $A(X)$. Then $\mathcal{P}$ is an algebraic set. Clearly X is locally-$\mathcal{P}$. If $A(X)$ is isomorphic with $B(Y)$, $\mathcal{P}$-compactness of X and Y are respectively A-compactness of X and B-compactness of Y. So, theorem 2.3 follows.

Now we shall try to build up another structurally similar example which is different from previous examples. In [8] Mitra and Acharyya introduced a subring of $C(X)$ containing $C^*(X)$, referred as $\chi(X)$ which is the smallest subring of $C(X)$ containing $C^*(X)$ and $C_H(X)$. It is clear that if we choose $\mathcal{P}$ to be the collection of χ-real maximal ideals of $C(X)$, then $\mathcal{P}$ is an algebraic set. X is trivially locally-$\mathcal{P}$. Then we have the following result.

Theorem 3.22. *If X and Y are χ-realcompact spaces, then $C(X)$ isomorphic with $C(Y)$ implies that X is homeomorphic with Y.*

Since, in hard pseudocompact space, $\chi(X) = C(X)$, χ-real maximal ideals are precisely all maximal ideals and therefore it is obvious that χ-realcompact and hard pseudocompact implies compactness and on the other hand, if X is nearly pseudocompact, $\chi(X) = C^*(X)$, χ-real maximal ideals are precisely real maximal ideals and hence χ-realcompact and nearly pseudocompact implies realcompactness.

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