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Best proximity point for q-ordered proximal contraction in noncommutative Banach spaces

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Abstract

We introduce the concept of q-ordered proximal nonunique contraction for the non self mappings and then obtain some proximity point results for these mappings. We also furnish examples to support our claims.

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KEYWORDS: *best proximity point; proximal comparable; noncommutative Banach space; nonunique contraction.*

1. Introduction

In 1922, a polish mathematician Stefan Banach [9] established the Banach contraction principle (BCP) which has been a cynosure in the field of fixed point theory. The principle states that every contraction self-mapping T on a complete metric space (X, d) has a unique fixed point. It states the contraction condition as $d(Tx, Ty) \leq cd(x, y)$, where $x, y \in X (0 \leq c < 1)$. Also, every Picard sequence in X converges to a fixed point of T. BCP has various generalizations, extensions and applications given by eminent mathematicians. Since, the solution of nonlinear systems that are frequently used to solve real-life problems may not be unique. Therefore, theorems that do not guarantee

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the uniqueness of the fixed point are also considered. In 1974, Ćirić [18] proved a nonunique fixed point theorem for self-mapping T on a complete metric space (X, d) which satisfies the following contraction:

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leq qd(x, y)$$

for all $x, y \in X$ and $0 < q < 1$. But if T is not a self mapping, then the solution of the equation $Tx = x$ may or may not exist. This was one of the major problem of research during the past few decades. In this case one tries to find a point x that is close to Tx in some way. Basha and Veeramani [12] introduced the following notion. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ is a non-self mapping. Then $a \in A$ is said to be a best proximity point if $d(a, Ta) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Later, Basha [10] presented sufficient conditions to get the existence and uniqueness of best proximity point of T by considering the proximal version of the BCP. Recently, several mathematicians proved some novel best proximity point results in different metric space settings (see, for instance [3, 4, 5, 11, 21, 28, 30, 31, 32]).

Kurepa [26] introduced novel abstract metric spaces by defining a metric which takes values on an ordered vector space. After that several mathematicians introduced various vector valued metric spaces (see, for instance [16, 17, 27]). In 2007, Huang and Zhang [23] replaced the set of real numbers with an ordered Banach space to define the notion of cone metric spaces. Several fixed point results have been obtained in the setting of cone metric spaces using various contraction conditions ([1, 2, 6, 7, 8, 15, 19, 20, 22, 24, 25]). In 2014, Xin and Jiang [33] introduced a generalization of Banach spaces namely, noncommutative Banach spaces, and proved some fixed point results. Recently, Beg et al. [14] proved some best proximity point results in noncommutative Banach spaces. Also, Rawat et al. [29] proved some fixed point results in noncommutative Banach spaces.

In this paper, combining the ideas of Ćirić, Basha, and Xin and Jiang we obtain some nonunique best proximity point results in noncommutative Banach spaces. We first present the notion of q-ordered proximal nonunique contractions on noncommutative Banach spaces and prove several best proximity point results for q-ordered proximal contractions. Examples are also provided to show the significance of our results.

2. Preliminaries

In this section, we give some basic preliminaries regarding noncommutative Banach spaces (see [33]).

Definition 2.1 ([33]). Let X be a group with a unit element e and (X, d) be a complete metric space. Space X is called a noncommutative Banach space if it satisfies the following conditions:

(1) $d(xz, yz) = d(x, y)$ for any x, y, z in X.

(2) There exists $S : \mathbb{R} \times X \to X$, defined as $S(\alpha, x) = x^\alpha$ such that $S(-1, x)$ is inverse of x , $S(0, x)$ is unit element e, in the group X, and

$$S(pq, x) = S(p, S(q, x)), \qquad S(p + q, x) = S(p, x).S(q, x)$$

for all $p, q \in \mathbb{R}$, x in X.

(3) For each x in X, there exists a constant $M_x > 0$ such that

$$d(x^\alpha, e) \leq M_x|\alpha|, \qquad \text{for all } \alpha \in \mathbb{R}.$$

In case, if there exists $M > 0$ such that $d(x^\alpha, e) \leq M|\alpha|$, for all x in X, α in $\mathbb{R}$, then X is called uniformly bounded.

Every uniformly bounded noncommutative Banach space X is bounded. Take $\alpha = 1$, then $d(x, e) \leq M$, now using the triangle inequality we obtain $d(x, y) \leq 2M$. It further implies that X is bounded.

Example 2.2. Consider the group $\mathbb{R}^n$ with respect to addition with usual unit element $\underbrace{(0, 0, \cdots, 0)}_{n-times}$ and define

$$d(x, y) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|},$$

for $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Obviously, $\mathbb{R}^n$ is a complete metric space. For $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$, we have

$$d(zx, zy) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|(x_i + z_i) - (y_i + z_i)|}{1 + |(x_i + z_i) - (y_i + z_i)|} = d(x, y).$$

Now define $S : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as $S(\alpha, x) = \alpha x$, we have $S(-1, x)$ is additive inverse of x, $S(0, x)$ is unit element $\underbrace{(0, 0, \cdots, 0)}_{n-times}$ in the group $\mathbb{R}^n$ and it is obvious that

$$S(pq, x) = S(p, S(q, x)), \qquad S(p + q, x) = S(p, x).S(q, x),$$

for all $p, q \in \mathbb{R}$. Now, we have to prove that for any $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$, there exists a constant $M_x > 0$ such that $d(kx, e) \leq M_x|k|$, for $k \in \mathbb{R}$, where e is unit element in $\mathbb{R}^n$. We know that $d(kx, e) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|k||x_i|}{1+|k||x_i|}$. Taking $M_x = \begin{cases} 1 \text{ if each } x_i = 0, \\ \sum_{i=1}^{n} \frac{1}{2^{i-1}} \frac{|x_i|}{1+|k||x_i|} \text{ otherwise} \end{cases}$, the inequality clearly holds. Thus, $\mathbb{R}^n$ is a noncommutative Banach space.

Definition 2.3 ([33])**.** Let E be a nonempty subset of a noncommutative Banach space X satisfying:

(1) E is closed and $E \neq \{e\}$.
(2) $x, y \in E$ and $\alpha, \beta \in \mathbb{R}^+ \implies x^\alpha y^\beta \in E$.
(3) $E \cap E^{-1} = \{e\}$, where $E^{-1} = \{x^{-1} : x \in E\}$.

Then E is called a cone in X.

Let E be a cone in a noncommutative Banach space X, then an order relation is introduced as follows:

$$x \lesssim y \iff y^{\beta}x^{-\beta} \in E \text{ for all } \beta \in [0,1]. \tag{2.1}$$

Order '$\lesssim$' is a partial ordering with respect to E. Also we have:

(i) For $x \in X$, $x^{\beta}x^{-\beta} = x^0 = e \in E$ for all $\beta \in [0,1]$. It further implies that $x \lesssim x$.

(ii) If $x \lesssim y$ and $y \lesssim x$, then $y^{\beta}x^{-\beta} \in E$ and $(y^{\beta}x^{-\beta})^{-1} = x^{\beta}y^{-\beta} \in E$ for all $\beta \in [0,1]$. By $E \cap E^{-1} = \{e\}$, we obtain $y^{\beta} = x^{\beta}$, which further implies that $y = x$.

(iii) If $x \lesssim y$ and $y \lesssim z$, then $y^{\beta}x^{-\beta} \in E$ and $z^{\beta}y^{-\beta} \in E$ for all $\beta \in [0,1]$, using condition 2 in definition 2.3 we have $z^{\beta}x^{-\beta} \in E$, i.e. $x \lesssim z$.

Definition 2.4 ([33])**.** A cone $E \subseteq X$ is said to be normal, if there is a number $N > 0$ such that

$$e \lesssim x \lesssim y \implies d(x,e) = Nd(y,e) \text{ for all } x, y \in X.$$

Normal constant of E is the least number N satisfying the above condition. Clearly $N \geq 1$.

Remark 2.5. Let E be a cone in a noncommutative Banach space X and $x \in E, \alpha \in \mathbb{R}$, then the following condition holds:

$$\begin{cases} x \lesssim x^{\alpha}, & \alpha \geq 1, \\ x^{\alpha} \lesssim x, & \alpha < 1. \end{cases}$$

Also, for any $\beta \in [0,1]$, if $\alpha \geq 1$, then $(x^{\alpha})^{\beta}x^{-\beta} = x^{(\alpha-1)\beta} \in E$. Therefore we have $x \lesssim x^{\alpha}$; if $\alpha < 1$, then $x^{\beta}(x^{\alpha})^{-\beta} = x^{(1-\alpha)\beta} \in E$, which implies $x^{\alpha} \lesssim x$.

For $x, y \in X$, if either $x \lesssim y$ or $y \lesssim x$ holds, we say that x and y are comparable, denoted

$$\vee(x,y) = \begin{cases} y, & x \lesssim y \\ x, & y \lesssim x \end{cases} \quad \text{and} \quad \wedge(x,y) = \begin{cases} x, & x \lesssim y \\ y, & y \lesssim x \end{cases}$$

Lemma 2.6 ([33])**.** *Suppose that E is a cone in a noncommutative Banach space X. For $u, v \in X$, we have:*

(1) *Let $u \lesssim v$, then $u^{\alpha} \lesssim v^{\alpha}$, for any $0 \leq \alpha \leq 1$.*
(2) *If u and v are comparable, then $\vee(uv^{-1}, vu^{-1})$ exists and furthermore $e \lesssim \vee(uv^{-1}, vu^{-1})$.*
(3) *If u and v are comparable, then $d(\vee(uv^{-1}, vu^{-1}), e) = d(u,v)$ exists.*
(4) *Let $\{u_n\}, \{v_n\}$ be two sequences in X, u_n and v_n be comparable for all $n \in \mathbb{N}$. If $u_n \to u_0, v_n \to v_0$, then u_0 and v_0 are comparable.*

Henceforth, we give some notations for subsequent use. If F and G are nonempty subsets of X, then

$$\begin{aligned} d(\mu, G) &= \inf\{d(\mu, \varrho) : \varrho \in G\}, \text{ where } \mu \in F, \\ F_0 &= \{\mu \in F : d(\mu, \varrho) = d(F, G) \text{ for some } \varrho \in G\}, \\ G_0 &= \{\varrho \in G : d(\mu, \varrho) = d(F, G) \text{ for some } \mu \in F\}. \end{aligned}$$

Whenever U and V are closed subsets of a normed space X and $d(U, V) > 0$, then U_0 and V_0 are subset of boundaries of U and V respectively.

Definition 2.7 ([13])**.** A set V is said to be approximately compact with respect to U, if every sequence $\{\varrho_n\}$ of V with $d(\mu, \varrho_n) \to d(\mu, V)$ for some $\mu \in U$ has a convergent subsequence.

Definition 2.8 ([14])**.** Let $T : U \to V$ be a mapping. T is said to be proximal comparable if

$$\left.\begin{aligned} x_1 &\lesssim x_2 \\ d(y_1, Tx_1) &= d(U, V) \\ d(y_2, Tx_2) &= d(U, V) \end{aligned}\right\} \text{imply} \quad y_1 \lesssim y_2,$$

where x_1, x_2, y_1 and $y_2 \in U$.

3. Main results

Throughout in this section, we always suppose that (X, d) is a noncommutative Banach space with a partial ordering '$\lesssim$' induced by a normal cone E with the normal constant N.

Definition 3.1. Let (X, d) be a noncommutative Banach space, $U, V (\neq \phi)$ be two subsets of X, $T : U \to V$ be a mapping and $x \in U$. Then $Q_T(x)$, the set of iterative sequences such that

$$Q_T(x) = \{x_n \subseteq U : x_0 = x, x_n \lesssim x_{n+1} \text{ and } d(x_{n+1}, Tx_n) = d(U, V) \text{ for all } n \in \mathbb{N}\}$$

is called the comparable orbit of x.

Definition 3.2. Let (X, d) be a noncommutative Banach space, $U, V (\neq \phi)$ be two subsets of X. A mapping $T : U \to V$ is said to be best comparable orbitally continuous at a point $x^* \in U$ if for every $x \in U$ and $\{x_n\} \in Q_T(x)$ the following holds

$$x_{n_i} \to x^* \text{ implies } Tx_{n_i} \to Tx^*, \text{ as } i \to \infty,$$

for any subsequence x_{n_i} of x_n. If at every point of U, the mapping T is best comparable orbitally continuous, then T is said to be best comparable orbitally continuous on U.

Definition 3.3. Let (X, d) be a noncommutative Banach space, $U, V (\neq \phi)$ be two subsets of X, $T : U \to V$ be a mapping. The set U is said to be T-best comparable complete, if for all $x \in U$ and $\{x_n\} \in Q_T(x)$, every Cauchy subsequence x_{n_i} of x_n converges to a point in U_0.

Definition 3.4. Let (X, d) be a noncommutative Banach space, $U, V(\neq \phi)$ be two subsets of X, $T : U \to V$ and $g : U \to \mathbb{R}$ are two mappings, if for each $x \in U$ and $\{x_n\} \in Q_T(x)$ the following holds

$$x_{n_i} \to x^* \text{ implies } g(x^*) \leq \lim_{i\to\infty} \inf g(x_{n_i}), \text{ as } i \to \infty,$$

for any subsequence x_{n_i} of x_n, then g is said to be best comparable orbitally lower semicontinuous at x^* in U. If the mapping g is best comparable orbitally lower semicontinuous at every point in U, then it is said to be best orbitally lower semicontinuous on U.

Lemma 3.5. *Let (X, d) be metric space, $T : U \to V$ be a mapping, where $U, V(\neq \phi) \subseteq X$. If mapping T is best comparable orbitally continuous on U, then $g : U \to \mathbb{R}$ defined as $g(x) = d(x, Tx)$ is best comparable orbitally lower semicontinuous on U*

Proof. By taking sequence $\{x_n\}$ in $Q_T(x)$, following the lines of the proof of Lemma 1 in [31]. □

Remark 3.6. The converse of above Lemma 3.5 may not be true. For this fact we are presenting the following example.

Example 3.7. Let $X = \mathbb{R}^2$($\mathbb{R}$ be the set of real numbers). Define

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

for every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Define comparability of (x_1, x_2) and (y_1, y_2) as

$$(x_1, x_2) \lesssim (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Suppose

$$U = \left\{ \left(-\frac{1}{n}, 0\right) : n \in \mathbb{N} \right\} \cup \{(0, 0)\}$$

$$V = \left\{ \left(-\frac{1}{n}, 1\right), \left(-\frac{1}{n}, -1\right) : n \in \mathbb{N} \right\} \cup \{(0, 1), (0, -1)\}.$$

We have $d(U, V) = 1$. Now, define $T : U \to V$ as $T(0, 0) = (0, 1)$ and

$$T\left(-\frac{1}{n}, 1\right) = \begin{cases} \left(-\dfrac{1}{n+1}, 1\right), & \text{if } n \text{ is odd} \\ \left(-\dfrac{1}{n+1}, -1\right), & \text{if } n \text{ is even.} \end{cases}$$

Let $x = (-1, 0)$, then we have

$$\begin{aligned} Q_T(x) &= \{x_n \subseteq U : x_0 = x, x_n \lesssim x_{n+1} \text{ and } d(x_{n+1}, Tx_n) = d(U, V) \text{ for all } n \in \mathbb{N}\} \\ &= \left\{ \left\{ -\frac{1}{n}, 0 \right\} : n \in \mathbb{N} \right\}. \end{aligned}$$

Now, suppose $\{x_n\} = \left\{\left(-\frac{1}{n}, 0\right)\right\} \in Q_T(x)$, as $n \to \infty$, $x_n \to (0,0)$. But $\lim_{n\to\infty} Tx_n$ does not exist, i.e. mapping T is not best comparable orbitally continuous at $(0,0)$. Notice that, $g(x) = d(x, Tx)$ is best comparable orbitally lower semicontinuous at each point of V.

Definition 3.8. Let (X, d) be a noncommutative Banach space, $U, V (\neq \phi)$ be two subsets of X, $T : U \to V$ be a mapping. Space (X, d) is said to satisfy orbitally $q-$property, if for each $x \in U$ and a sequence $\{x_n\}$ in $Q_T(x)$ with $\lim x_n \to x^*$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $t \in U_0$, with $d(t, Tx^*) = d(U, V)$ such that

$$\vee(x_{n_k} t^{-1}, t x_{n_k}^{-1}) \lesssim \vee(x_{n_k} x^{*-1}, x^{*-1} x_{n_k}^{-1})^q$$

Definition 3.9. Let (X, d) be a noncommutative Banach Space and $\phi \neq U, V \subseteq X$. A mapping $T : U \to V$ is said to be q-ordered proximal nonunique contraction if there exists $q \in (0, 1)$ such that for all y_1, y_2, x_1, x_2 in U, if x_1 and x_2 are comparable, then

$$\left.\begin{array}{l} d(y_1, Tx_1) = d(U, V) \\ d(y_2, Tx_2) = d(U, V) \end{array}\right\} \text{imply } M(x_1, x_2, y_1, y_2) N^{-1}(x_1, x_2, y_1, y_2) \lesssim \vee(x_1 x_2^{-1}, x_2 x_1^{-1})^q, \tag{3.1}$$

where

$M(x_1, x_2, y_1, y_2) = \wedge\{\vee(y_1 y_2^{-1}, y_2 y_1^{-1}), \vee(x_1 y_1^{-1}, y_1 x_1^{-1}), \vee(x_2 y_2^{-1}, y_2 x_2^{-1})\}$

and

$N(x_1, x_2, y_1, y_2) = \wedge\{\vee(x_2 y_1^{-1}, y_1 x_2^{-1}), \vee(x_1 y_2^{-1}, x_1 y_2^{-1})\}$.

Theorem 3.10. *Let $U, V (\neq \phi)$ be two subsets of a noncommutative Banach space (X, d) and V be approximately compact with respect to U. Suppose $T : U \to V$ is a proximal comparable mapping satisfying the conditions:*

1. *$T(U_0) \subseteq V_0$.*
2. *There exist $x_0, x_1 \in U_0$ such that $d(x_0, Tx_1) = d(U, V)$ and $x_0 \lesssim x_1$.*
3. *Mapping T satisfies q-ordered proximal nonunique contraction condition.*
4. *U is T-best comparable orbitally complete and $g(x) = d(x, Tx)$ is best comparable orbitally lower semicontinuous on U.*

Then, T admits a best proximity point in U.

Proof. From condition (2), there exist $x_0, x_1 \in U_0$ such that

$$d(x_1, Tx_0) = d(U, V) \text{ and } x_0 \lesssim x_1. \tag{3.2}$$

Since $T(U_0) \subseteq V_0$, there exists $x_2 \in U_0$ such that

$$d(x_2, Tx_1) = d(U, V). \tag{3.3}$$

As T is proximal comparable mapping, therefore, from (3.2) and (3.3), we have

$$x_1 \lesssim x_2.$$

Again, $T(U_0) \subseteq V_0$, so there exists $x_3 \in U_0$ such that

$$d(x_3, Tx_2) = d(U, V). \tag{3.4}$$

Now, T is proximal comparable mapping and $x_1 \lesssim x_2$, therefore, from (3.3) and (3.4), we have

$$x_2 \lesssim x_3.$$

On repeating the above process, we get $\{x_n\} \subseteq U_0$ such that

$$d(x_{n+1}, Tx_n) = d(U, V) \text{ and } x_n \lesssim x_{n+1} \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

Suppose $d(x_{n_0}, x_{n_0+1}) = 0$ for some n_0, then $x_{n_0} = x_{n_0+1}$ and we have $d(x_{n_0}, Tx_{n_0}) = d(U, V)$, i.e. x_{n_0} is a best proximity point of T. So, we assume that $d(x_{n+1}, x_n) > 0$ for $n \geq 0$. Now, as T is a q-ordered proximal nonunique contraction, therefore, from (3.5), we have

$$M(x_{n-1}, x_n, x_n, x_{n+1})N^{-1}(x_{n-1}, x_n, x_n, x_{n+1}) \lesssim \vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1})^q,$$

for all $n \in \mathbb{N} \cup \{0\}$ and so we have

$$\begin{aligned}(\wedge\{\vee(x_n x_{n+1}^{-1}, x_{n+1}x_n^{-1}), \vee(x_{n-1}x_n^{-1}, x_{n-1}x_n^{-1})\})(\wedge\{\vee(e, e), \vee(x_{n-1}x_{n+1}^{-1},\\ x_{n-1}x_{n+1}^{-1})\})^{-1} \lesssim \vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1})^q.\end{aligned}$$

Now, by the reflexivity of partial ordering $\lesssim$ in P, x_{n-1} and x_{n+1} are comparable and using Lemma 2.6, we have

$$\wedge\{\vee(x_n x_{n+1}^{-1}, x_{n+1}x_n^{-1}), \vee(x_{n-1}x_n^{-1}, x_{n-1}x_n^{-1})\} \lesssim \vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1})^q.$$

Since $q < 1$, $\vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1}) \lesssim \vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1})^q$ is impossible, so we have

$$\vee(x_n x_{n+1}^{-1}, x_{n+1}x_n^{-1}) \lesssim \vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1})^q.$$

Again using Lemma 2.6, we have

$$\begin{aligned} e \lesssim \vee(x_n x_{n+1}^{-1}, x_{n+1}x_n^{-1}) &\lesssim \vee(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1})^q \\ &. \\ &. \\ &. \\ &\lesssim \vee(x_0 x_1^{-1}, x_1 x_0^{-1})^{q^n}. \end{aligned}$$

Since P is a normal cone with N as a normal constant,

$$d(\vee(x_n x_{n+1}^{-1}, x_{n+1}x_n^{-1}), e) \leq N.d(\vee(x_0 x_1^{-1}, x_1 x_0^{-1})^{q^n}, e).$$

Using Definition 2.1 and Lemma 2.6, we have

$$d(x_n, x_{n+1}) \leq Nq^n.d(x_0, x_1), \quad n = 0, 1, 2, \cdots$$

Then for $n, p \in \mathbb{N}$ we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq N.q^n(q^{p-1} + q^{p-2} + \cdots + q + 1)d(x_1, x_0) \\ &= \frac{N.q^n}{1-q}(1 - q^p)d(x_1, x_0). \end{aligned}$$

Since $q \in (0, 1)$, therefore, we conclude that $\{x_n\}$ is a Cauchy sequence. Now, U is T- best comparable orbitally complete, then there exists $x^* \in U_0$ such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty.$$

In addition, from (3.5), we have

$$\begin{aligned} d(x^*, V) &\leq d(x^*, Tx_n) \\ &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x^*, x_{n+1}) + d(U, V) \\ &\leq d(x^*, x_{n+1}) + d(x^*, V), \end{aligned}$$

as $n \to \infty$, $d(x^*, Tx_n) \to d(x^*, V)$. Now, V is approximately compact with respect to U, there exists $\{Tx_{n_k}\}$ of Tx_n such that $Tx_{n_k} \to \gamma$ for some γ in V. Also, using (3.5), we have

$$d(x^*, \gamma) = d(U, V).$$

On other hand, since $g(x) = d(x, Tx)$ is best comparable orbitally lower semi-continuous on U, we have

$$\begin{aligned} d(U, V) &\leq d(x^*, Tx^*) \\ &= g(x^*) \\ &\leq \liminf g(x_{n_i}) \\ &= \liminf d(x_{n_i}, Tx_{n_i}) \\ &= d(x^*, \gamma) \\ &= d(U, V). \end{aligned}$$

Hence, we have $d(x^*, Tx^*) = d(U, V)$, i.e. x^* is a best proximity point of T. $\square$

Example 3.11. Let $X = \mathbb{R}^2$ ($\mathbb{R}$ be the set of real numbers). Define

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2},$$

for every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Clearly, $(\mathbb{R}^2, d)$ is a complete metric space and it is also a noncommutative Banach Space. Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. The partial ordering in $\mathbb{R}^2$ with respect to cone E is defined as

$$(x_1, x_2) \lesssim (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Now, suppose $U = \{(-\frac{1}{3^n}, 0) : n \in \mathbb{N}\} \cup \{(0, 0)\}$ and $V = \{(-\frac{1}{3^n}, 2), (-\frac{1}{3^n}, -2) : n \in \mathbb{N}\} \cup \{(0, 2), (0, -2)\}$.
Define $T : U \to V$ such that

$$\begin{aligned} T(0, 0) &= (0, 2), \\ T(-\frac{1}{3^n}, 0) &= \begin{cases} (-\frac{1}{3^{n+1}}, 1), & \text{n is odd}, \\ (-\frac{1}{3^{n+1}}, -1), & \text{n is even}. \end{cases} \end{aligned}$$

It is clear that $d(U, V) = 2$, $U_0 = U$ and $V_0 = V$.

We can easily observe that $T(U_0) \subseteq V_0$ and as V is compact, so it is also approximately compact.

The only cases in which $d(x, Ty) = d(U, V)$ for any $x, y \in X$ are $d((0, 0), T(0, 0)) = d(U, V)$ and $d(x_{n+1}, Tx_n) = d(U, V)$, where $n \in \mathbb{N}$. In all the cases, clearly the contraction condition is satisfied, for example take $x_1 = (-\frac{1}{3}, 0)$, $x_2 = (-\frac{1}{3^2}, 0)$, $y_1 = (-\frac{1}{3^2}, 0)$ and $y_2 = (-\frac{1}{3^3}, 0)$ then (3.1) becomes

$$M\left(\left(-\frac{1}{3},0\right),\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^3},0\right)\right)N^{-1}\left(\left(-\frac{1}{3},0\right),\left(-\frac{1}{3^2},0\right),\right. \tag{3.6}$$
$$\left.\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^3},0\right)\right) \precsim \vee\left(\left(-\frac{1}{3},0\right)\left(-\frac{1}{3^2},0\right)^{-1},\left(-\frac{1}{3^2},0\right)\left(-\frac{1}{3},0\right)^{-1}\right)^q.$$

Here,

$$M\left(\left(-\frac{1}{3},0\right),\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^3},0\right)\right) = \left(\frac{2}{27},0\right)$$
$$N\left(\left(-\frac{1}{3},0\right),\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^2},0\right),\left(-\frac{1}{3^3},0\right)\right) = (0,0).$$

Therefore, (3.6) reduces to $\left(\frac{2}{27}, 0\right) \precsim \left(\frac{2}{9}, 0\right)^q$, which is true for $q = \frac{1}{2}$ since $\frac{2}{27} \leq \frac{2}{9}^{\frac{1}{2}}$and $0 \leq 0$. Similarly, in all the cases the contraction condition will be satisfied. Also, all other conditions of Theorem 3.10 are satisfied. Hence, T admits a best proximity point which is $(0, 0)$.

Theorem 3.12. *Let $U, V(\neq \phi)$ be two subsets of a noncommutative Banach space (X, d) with orbitally $q-$property and V be approximately compact with respect to U. Suppose $T : U \to V$ is a proximal comparable mapping satisfying the conditions:*
1. *$T(U_0) \subseteq V_0$.*
2. *There exist $x_0, x_1 \in U_0$ such that $d(x_0, Tx_1) = d(U, V)$ and $x_0 \precsim x_1$.*
3. *Mapping T satisfies q-ordered proximal nonunique contraction condition.*
4. *U is T-best comparable orbitally complete.*

Then, T admits a best proximity point in U.

Proof. Following the lines of proof of Theorem 3.10, we have a Cauchy sequence $\{x_n\}$ in $Q_T(x_0)$. Since U is T- best comparable orbitally complete, then there exists $x^* \in U_0$ such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty.$$

In addition, we also have as $n \to \infty$, $d(x^*, Tx_n) \to d(x^*, V)$.
Now, V is approximately compact with respect to U, so there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\{Tx_{n_k}\} \to \gamma$ for some γ in V. Also, using (3.5), we have

$$d(x^*, \gamma) = d(x_{n_k+1}, Tx_{n_k}) = d(U, V) \text{ for each } k,$$

which implies $x^* \in U_0$. Since $T(U_0) \subseteq V_0$ we have

$$d(t, Tx^*) = d(U, V) \text{ for some element } t \in U_0. \tag{3.7}$$

Now, from orbitally $q-$property, there exists a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\vee(x_{n_k}t^{-1}, tx_{n_k}^{-1}) \lesssim \vee(x_{n_k}x^{*-1}, x^*x_{n_k}^{-1})^q.$$

From Lemma 2.6 we have

$$e \lesssim \vee(x_{n_k}t^{-1}, tx_{n_k}^{-1}) \lesssim \vee(x_{n_k}x^{*-1}, x^*x_{n_k}^{-1})^q.$$

Hence

$$d(t, x_{n_k}) = d(\vee(x_{n_k}t^{-1}, tx_{n_k}^{-1}), e) \leq N.d(\vee(x_{n_k}x^{*-1}, x^*x_{n_k}^{-1})^q, e) \leq N.qd(x_{n_k}, x^*).$$

So $d(t, x^*) = 0$, as $k \to \infty$, i.e. $t = x^*$. From (3.7), we have

$$d(x^*, Tx^*) = d(t, Tx^*) = d(U, V),$$

i.e. T has a best proximity point x^* in U.

□

Theorem 3.13. *Let $U, V (\neq \phi)$ be two subsets of a noncommutative Banach space (X, d). Suppose $T : U \to V$ is a proximal comparable mapping satisfying the conditions:*

1. $T(U_0) \subseteq V_0$.

2. There exist $x_0, x_1 \in U_0$ such that $d(x_0, Tx_1) = d(U, V)$ and $x_0 \lesssim x_1$.

3. Mapping T satisfies q-ordered proximal nonunique contraction condition.

4. U is T-best comparable orbitally complete and T is best comparable orbitally continuous on U.

Then, T admits a best proximity point in U.

Proof. Following the lines of proof of Theorem 3.10, we have a Cauchy sequence $\{x_n\}$ in $Q_T(x_0)$. Since U is T- best comparable orbitally complete, then there exists $x^* \in U_0$ such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty.$$

Now as T is best comparable orbitally continuous on U. Therefore, we have

$$Tx_n \to Tx^* \quad \text{as} \quad n \to \infty.$$

So, we obtain

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(U, V),$$

i.e. T has a best proximity point x^* in U. □

Example 3.14. Let $X = \mathbb{R}^2$ ($\mathbb{R}$ be the set of real numbers). Define

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

for every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Clearly, $(\mathbb{R}^2, d)$ is a complete metric space and it is also a noncommutative Banach Space. Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. The partial ordering in $\mathbb{R}^2$ with respect to cone E is defined as

$$(x_1, x_2) \lesssim (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Now, suppose

$U = \{(1,0),(2,0)\}$ and $V = \{(1,1),(2,1)\} \cup \left\{ \left(1+\frac{1}{n}, -1\right) : n \in \mathbb{N}, n \geq 3 \right\}$.

Define $T : U \to V$ such that $T(1,0) = (1,1)$ and $T(2,0) = (2,1)$. It is clear that $d(U,V) = 1$, $U_0 = U$ and

$$V_0 = V \setminus \left\{ \left(1+\frac{1}{n}, -1\right) : n \in \mathbb{N}, n \geq 3 \right\}.$$

We can easily observe that $T(U_0) \subseteq V_0$ and other conditions of Theorem 3.13 can be easily verified. Hence, T admits a best proximity point. Indeed, every point of U is a best proximity point.

It is important to notice that Theorem 3.10 and 3.12 can not be applied here, because considering $x = (1,0) \in U$ and the sequence $\{x_n\} = \left\{ \left(1+\frac{1}{n}, -1\right) \right\} \subseteq V$, it is clear that $\lim_{n\to\infty} d((1,0),(1+\frac{1}{n},-1)) \to 1 = d(U,V)$, but the sequence $\{x_n\}$ does not have any convergent subsequence in V, i.e. V is not approximately compact w.r.t. U.

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