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Countable networks on Malykhin's maximal topological group

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Abstract

We present a solution to the following problem: Does every countable and non-discrete topological (Abelian) group have a countable network with infinite elements? In fact, we show that no maximal topological space allows for a countable network with infinite elements. As a result, we answer the question in the negative. The article also focuses on Malykhin's maximal topological group constructed in 1975 and establishes some unusual properties of countable networks on this special group G. We show, in particular, that for every countable network $\mathcal{N}$ for G, the family of finite elements of $\mathcal{N}$ is also a network for G.

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1. Introduction

One of our goals in this work is to present a solution to the following problem:

Problem 1.1. *Is it true that each countable non-discrete topological (Abelian) group has a countable network with infinite elements?*

This problem arises as a complement to [4, Lemma 2.27], which states that if a topological Abelian group G has a countable network and satisfies $|G| = \kappa$ and $cf(\kappa) > \omega$, then G has a countable network $\mathcal{N}$ such that $|N| = \kappa$, for each $N \in \mathcal{N}$.

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We will use [1, Theorem 4.5.22] and non-resolvability of maximal topological spaces to show that under the assumption $\mathfrak{p} = 2^\omega$, there exists a countable non-discrete topological group G that does not admit a countable network $\mathcal{N}$ with infinite elements.

For this group G, we prove in Proposition 2.8 that if $\mathcal{N}$ is a countable network for G and $\mathcal{N}_f$ is the subfamily of $\mathcal{N}$ consisting of finite sets, then $\mathcal{N}_f$ is also a network of G. This result gives rise to the following question:

Problem 1.2. *Let $\mathcal{N}$ be a countable network for the group G. For every $n \geq 1$, let $\mathcal{N}_n$ be the subfamily of $\mathcal{N}$ which consists of the sets $N \in \mathcal{N}$ with $|N| \leq n$. Is $\mathcal{N}_n$ a network of G, for some $n \geq 1$?*

In Examples 2.9 and 2.10 we solve Problem 1.2 in the negative.

1.1. **Notation and terminology.** The symbol $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^+$ stands for the set of positive integers. The set of all real numbers is $\mathbb{R}$.

The cardinality of a set A is denoted by $|A|$. The symbols ω and $\mathfrak{c}$ stand for the cardinality of $\mathbb{N}$ and $\mathbb{R}$, respectively.

A space X is called *resolvable* if it contains dense disjoint subsets A and B; otherwise X is said to be *irresolvable*. Let X be a space without isolated points and τ be the topology of X. The space X is called *maximal* if every topology τ' on X strictly finer than τ has isolated points.

Given a group G, the identity element of G is e_G or simply e. An Abelian group G is called *bounded* if there exists a positive integer m such that $mg = e$, for each $g \in G$. The least integer $m \geq 1$ with this property is called the *period* of G. In particular, a group G is called *Boolean* if G is a group of period 2.

Let us call a topological group topology τ on a group G *linear* if the topological group (G, τ) has a local base at the identity element e consisting of open subgroups.

The weight, character and π-character of a space X are denoted by $w(X)$, $\chi(X)$ and $\pi\chi(X)$, respectively. Also, $\chi(x, X)$ and $\pi\chi(x, X)$ are the character and π-character of X at the point $x \in X$.

In this paper, all spaces and topological groups are assumed to be Hausdorff.

2. Main results

In this section, we will start with some basic results. The following result shows that maximal spaces cannot be resolvable (see also [1, Proposition 4.5.19]).

Proposition 2.1. *If X is a maximal topological space, then X is irresolvable.*

Proof. Let A and B be dense subsets of X. Then A and B are open in X, by [1, Lemma 4.5.18]. Therefore, $A \cap B \neq \varnothing$. It follows that the space X is irresolvable. □

The following theorem presents an interesting property of irresolvable spaces.

Theorem 2.2. *If X is an irresolvable space, then X does not admit a countable network with infinite elements.*

Proof. Assume that the space X has a countable network $\mathcal{N} = \{N_k : k \in \omega\}$ such that $|N_k| \geq \omega$, for each $k \in \omega$. Take distinct elements $a_0, b_0 \in N_0$ and put $A_0 = \{a_0\}$ and $B_0 = \{b_0\}$. Clearly, A_0 and B_0 are disjoint.

Suppose that for some integer $m \geq 0$, we have defined finite disjoint subsets $A_m = \{a_0, \ldots, a_m\}$ and $B_m = \{b_0, \ldots, b_m\}$ of X such that $a_k, b_k \in N_k$, for each $k \leq m$. Then the set $N_{m+1} \setminus (A_m \cup B_m)$ is infinite, so we can choose two distinct points a_{m+1} and b_{m+1} in $N_{m+1} \setminus (A_m \cup B_m)$. Let $A_{m+1} = A_m \cup \{a_{m+1}\}$ and $B_{m+1} = B_m \cup \{b_{m+1}\}$. Clearly, the sets A_{m+1} and B_{m+1} are disjoint.

Continuing this process, we finally obtain the sets $A = \bigcup_{i=0}^{\infty} A_i$ and $B = \bigcup_{i=0}^{\infty} B_i$ and, by construction, $A \cap B = \varnothing$. Finally, if U is an open non-empty subset of X, there exists $N_k \in \mathcal{N}$ such that $N_k \subseteq U$. So $a_k \in A \cap U$ and $b_k \in B \cap U$. We conclude that A and B are dense disjoint subsets of X. Hence X is resolvable, which is a contradiction. This proves that the space X does not have a countable network with infinite elements. □

To continue, we need to present a brief overview of Malykhin's construction of a countable infinite topological Boolean group (G, τ) such that the topology τ is maximal, linear and Hausdorff. Our description follows the one given in [1, Theorem 4.5.22]. However, we require a few details that are not explicitly stated in the above-mentioned construction.

The following results are required for constructing the group (G, τ) and their proofs can be found in [1, Proposition 4.5.19] and [1, Lemma 4.5.21], respectively.

The first result is a characterization of maximal spaces.

Proposition 2.3. *Let X be a Hausdorff space without isolated points. Then X is maximal if and only if for every $x \in X$ and every disjoint subsets A and B of $X \setminus \{x\}$, the element x belongs to at most one of the sets $\overline{A}$, $\overline{B}$.*

The second required result is as follow.

Lemma 2.4. *Let K be a countable infinite Boolean group. Suppose that τ is a topological group topology on K such that τ is non-discrete, second countable and linear. If $K \setminus \{e_K\} = P_1 \cup P_2$ and $P_1 \cap P_2 = \varnothing$, then there exists a topological group topology τ' of K such that $\tau \subset \tau'$, τ' is non-discrete, second countable and linear and, in addition, at most one of the sets $cl_{\tau'} P_1$ or $cl_{\tau'} P_2$ contains the identity element e_K.*

Also, we will need a 'small' cardinal $\mathfrak{p}$ described below. A family γ of infinite subsets of ω is said to have a *strong intersection property* if the intersection of any finite subfamily of γ is infinite. An infinite subset A of ω is called a *pseudointersection* of γ if the complement $A \setminus B$ is finite, for each $B \in \gamma$. In other terms, the set A is almost contained in every element $B \in \gamma$. It is easy to verify that every countable family γ with the strong intersection property has a pseudointersection. Denote by $\mathfrak{p}$ the least cardinality of a family γ of subsets of

ω with the strong intersection property such that γ has no pseudointersection. It is known that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{c}$ (see [2]).

From now on we assume that the cardinal $\mathfrak{p}$ satisfies $\mathfrak{p} = \mathfrak{c}$. This assumption is equivalent to Martin's Axiom restricted to σ-centered partially ordered sets (see [3]) and, therefore, is compatible with the negation of the Continuum Hypothesis.

We also have to fix a particular Boolean group G such that $|G| = \omega$. For this, denote by $\mathbb{Z}(2)$ the discrete two-element group $\{0, 1\}$. Let $\sigma\mathbb{Z}(2)^\omega$ be the subgroup of the compact group $\mathbb{Z}(2)^\omega$ which consists of all elements $x = (x_n)_{n\in\omega} \in \mathbb{Z}(2)^\omega$ such that $x_n \neq 0$ for at most finitely many coordinates $n \in \omega$. Then $\sigma\mathbb{Z}(2)^\omega$ is a countable dense subgroup of $\mathbb{Z}(2)^\omega$. Let us take G as $\sigma\mathbb{Z}(2)^\omega$. Let τ_0 be the topology of G inherited from the compact group $\mathbb{Z}(2)^\omega$. Then τ_0 is a non-discrete, Hausdorff, linear, second-countable topological group topology on G.

Let $\mathcal{P} = \{(P_{\alpha,1}, P_{\alpha,2}) : \alpha < \mathfrak{c}\}$ be an enumeration of all pairs $P = (P_1, P_2)$ such that $P_1 \cap P_2 = \varnothing$, $P_1 \cup P_2 = G \setminus \{e\}$, and $(P_{0,1}, P_{0,2}) = (G \setminus \{e\}, \varnothing)$, where e is the identity element of G. Such an enumeration exists since the group G is countable. The required topology τ on G is constructed by a recursion of length $\mathfrak{c}$.

Our aim is to define a family $\{\tau_\alpha : \alpha < \mathfrak{c}\}$ of non-discrete, second countable and linear topological group topologies on G satisfying the following conditions for each $\alpha, \beta < \mathfrak{c}$:

(i) $\tau_\alpha \subset \tau_\beta$ if $\alpha < \beta$;
(ii) the identity element e of G belongs to the closure in (G, τ_α) of at most one of the sets $P_{\alpha,1}$, $P_{\alpha,2}$.

Suppose that for some $\alpha < \mathfrak{c}$ we have defined a sequence $\{\tau_\nu : \nu < \alpha\}$ of non-discrete, second countable and linear topological group topologies on G satisfying (i) and (ii). It follows from (i) that the topological group topology γ_α on G with base $\bigcup_{\nu<\alpha} \tau_\nu$ is linear and non-discrete. Since for each $\nu < \alpha$, the topology τ_ν has a countable base in e, (G, γ_α) has a base at e of cardinality less than $\mathfrak{c}$. Denote by $\mathcal{B}_\alpha$ a local base at the identity of the group (G, γ_α) consisting of open subgroups and satisfying $|\mathcal{B}_\alpha| < \mathfrak{c}$.

The group G is countable, so there exists a bijection $f : G \setminus \{e\} \longrightarrow \omega$. For every $U \in \mathcal{B}_\alpha$, let $U^* = U \setminus \{e\}$ and consider the family $\mathcal{F} = \{f(U^*) : U \in \mathcal{B}_\alpha\}$ of infinite subsets of ω. Since the group (G, γ_α) is non-discrete and Hausdorff, the family $\mathcal{F}$ has the strong intersection property. Then, applying $|\mathcal{F}| < \mathfrak{p} = \mathfrak{c}$, we see that the family $\mathcal{F}$ has a pseudointersection, let's say, A. Let $X = \{x_n : n \in \omega\}$ be a faithful enumeration of the infinite set $f^{-1}(A)$. Clearly, the set $X \setminus U$ is finite for every $U \in \mathcal{B}_\alpha$. Therefore, if $U \in \mathcal{B}_\alpha$, then there exists $m \in \omega$ such that $\{x_k : m \leq k \in \omega\} \subset U$. Since U is a subgroup of G, the subgroup H_m of G generated by the set $X_m = \{x_k : m \leq k \in \omega\}$ is contained in the open subgroup U. If γ'_α is the topology of G with base $\mathcal{B}'$ which consists of the sets $g + H_n$, where $g \in G$ and $n \in \omega$, then $\mathcal{B}'$ is countable and γ'_α is a non-discrete, second countable linear topological group topology on G finer than γ_α.

It follows from the definition of γ'_α that $W = \langle X \rangle$ is an open subgroup of (G, γ'_α). Take an arbitrary element $U \in \mathcal{B}_\alpha$. There exists $n \in \omega$ such that the subgroup $H_n = \langle X_n \rangle$ of G is contained in U. Hence, $W \setminus U \subseteq W \setminus H_n$. Since $X \setminus X_n$ is finite and the group G is Boolean, we see that $W \setminus H_n$ is finite, for every $n \in \omega$. Therefore, $|W \setminus U| < \omega$. The aforementioned property will be utilized in the proof of Lemma 2.7.

Applying Lemma 2.4, we find a non-discrete, second countable and linear topological group topology τ_α on G such that $\gamma'_\alpha \subset \tau_\alpha$ and the identity element e of G belongs to the closure in (G, τ_α) of at most one of the sets $P_{\alpha,1}$, $P_{\alpha,2}$. This completes the construction of the family $\{\tau_\alpha : \alpha < \mathfrak{c}\}$.

Finally, if τ is the topology in G with base $\bigcup_{\alpha<\mathfrak{c}} \tau_\alpha$, then applying condition (ii) of our construction and Proposition 2.3, we conclude that (G, τ) is a non-discrete, maximal, linear and Hausdorff topological group. By Lemma 2.1, G is irresolvable.

The next result follows directly from Theorem 2.2.

Proposition 2.5. *The group G does not admit a countable network with infinite elements.*

A few properties of the maximal linear topology τ of G are listed in the subsequent lemmas.

Lemma 2.6. *Let U be an open set in (G, τ). Then there exists an ordinal $\alpha < \mathfrak{c}$ such that $U \in \tau_\alpha$.*

Proof. If $U = \varnothing$, there is nothing to prove. We assume therefore that $U \neq \varnothing$. Then $|U| = \omega$. Since $\gamma = \bigcup_{\alpha<\mathfrak{c}} \tau_\alpha$ is a base for τ, U can be covered by countably many open basic sets from γ. It follows from $cf(\mathfrak{c}) > \omega$ that there exists $\alpha < \mathfrak{c}$ such that $U \in \tau_\alpha$. □

Lemma 2.7. *Let x be an element of the group G and $\{U_n : n \in \omega\}$ be a countable family of open neighborhoods of x in (G, τ). Then there exists an open neighborhood W of x in (G, τ) such that $|W \setminus U_n| < \omega$, for every $n \in \omega$.*

Proof. By the homogeneity of the group G, it suffices to prove the lemma for the special case $x = e$, where e is the identity element of the group G.

Let $\mathcal{B} = \{U_n : n \in \omega\}$ be a countable family of open neighborhoods of e in (G, τ). The group (G, τ) is linear, so we can suppose that each U_n is an open subgroup of the group (G, τ). By Lemma 2.6, for every $n \in \omega$, there exists $\alpha_n < \mathfrak{c}$ such that $U_n \in \tau_{\alpha_n}$. Take an ordinal $\alpha < \mathfrak{c}$ such that $\alpha_n < \alpha$ for each $n \in \omega$.

At the step α of our construction of the topology τ, we have defined an infinite subset X of G such that the set $W = \langle X \rangle$ is in τ_α and $W \setminus U$ is finite, for each $U \in \bigcup_{\nu<\alpha} \tau_\nu$ with $e \in U$. Hence, W is an open neighborhood of e in (G, τ) and $|W \setminus U_n| < \omega$, for every $n \in \omega$. □

We now formulate the next result that complements the conclusion of Theorem 2.2 for the group G.

Proposition 2.8. *Let $\mathcal{N}$ be a countable network for (G, τ) and $\mathcal{N}_f = \{N \in \mathcal{N} : |N| < \omega\}$. Then $\mathcal{N}_f$ is a network for the group G.*

Proof. By Proposition 2.5, the family $\mathcal{N}_f$ is not empty. Suppose, seeking a contradiction, that $\mathcal{N}_f$ is not a network for the group G. Hence, we can find a point $x \in G$ and an open neighborhood U of x in (G, τ) such that $N \setminus U \neq \varnothing$, for each $N \in \mathcal{N}_f$ with $x \in N$. Let

$$\mathcal{N}_x = \{N \in \mathcal{N} : x \in N \subset U\}.$$

Then every $N \in \mathcal{N}_x$ is infinite. Since the group G is maximal, each $N \in \mathcal{N}_x$ has the form $O_N \cup D_N$, where O_N is an open subset of (G, τ) and D_N is closed and discrete in (G, τ) (see the proof of [1, Lemma 4.5.19]).

Since G is not first countable and $\chi(x, G) = \pi\chi(x, G)$ (see [1, Proposition 5.2.6]), $\pi\chi(x, G)$ is uncountable. Hence, it follows from $|\mathcal{N}_x| \leq \omega$ that there exists an open neighborhood V of x satisfying $N \setminus V \neq \varnothing$, for every $N \in \mathcal{N}_x$ with $O_N \neq \varnothing$.

Further, denote by $\mathcal{N}_x^*$ the family of all elements $N \in \mathcal{N}_x$ that are closed and discrete in (G, τ). By considering the open neighborhood V of x mentioned in the previous paragraph, together with the fact that $\mathcal{N}$ is a network, we can conclude that $\mathcal{N}_x^*$ is non-empty. Let $\{N_m : m \in \omega\}$ be an enumeration of $\mathcal{N}_x^*$. Since N_0 is a discrete set, there exists an open neighborhood V_0 of x in (G, τ) contained in V such that $V_0 \cap N_0 = \{x\}$. Suppose that for some $k \geq 0$, we have defined open neighborhoods $V_0, \ldots, V_k$ of x such that $V_k \cap N_k = \{x\}$ and $V_0 \supseteq \cdots \supseteq V_k$. Since N_{k+1} is discrete, there exists an open neighborhood V_{k+1} of x in (G, τ) contained in V_k, such that $V_{k+1} \cap N_{k+1} = \{x\}$. Continuing this process, we obtain a decreasing sequence $\{V_n : n \in \omega\}$ of open neighborhoods of x in (G, τ).

By Lemma 2.7, there exists an open neighborhood W of x in (G, τ) such that $|W \setminus V_m| < \omega$, for every $m \in \omega$. Given an element $N \in \mathcal{N}_x^*$, there exists $k \in \omega$ such that $|V_k \cap N| = 1$. It follows from $|N| = \omega$ and $|W \setminus V_k| < \omega$ that $N \setminus W$ is infinite and, hence, $N \setminus W \neq \varnothing$. So we have proved that $N \not\subset W$ for every $N \in \mathcal{N}_x$. This contradicts our assumption that $\mathcal{N}$ is a network for G. ☐

Example 2.9 and Corollary 2.11 below provide a negative solution to Problem 1.2 because the group G has no isolated points.

Example 2.9. *Let X be a countable infinite regular space that contains infinitely many non-isolated points. Denote by F the set of all non-isolated points in X. Then F is closed in X. Since X is regular, there exists an infinite discrete set $A = \{x_n : n \in \mathbb{N}^+\}$ contained in F. We can assume that the complement $F \setminus A$ is infinite.*

For any positive integer n, we define $\mathcal{Y}_n$ to be the family of all subsets $\{y_1, \ldots, y_n\}$ of $X \setminus \{x_n, x_{n+1}, \ldots\}$, where $y_1, y_2, \ldots, y_n$ are pairwise distinct.

We are going to build a network for X using families of finite subsets of X defined as follows. Let $\mathcal{N}_0 = \big\{\{x\} : x \in X \setminus A\big\}$. In general, for every $n \geq 1$, let $\mathcal{N}_n = \big\{\{x_n\} \cup Y : Y \in \mathcal{Y}_n\big\}$.

We claim that $\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$ is a network for X. Take $x \in X$ and let U be an open neighborhood of x. If $x \notin A$, then $\{x\} \in \mathcal{N}_0 \subset \mathcal{N}$ and clearly $\{x\} \subset U$. If $x \in A$, then $x = x_n$ for some $n \in \mathbb{N}^+$. Then there exists an open neighborhood O of x in X such that $O \subseteq U$ and $O \cap A = \{x_n\}$. Therefore, if $y_1, y_2, \ldots, y_n$ are pairwise distinct elements of $O \setminus A$, then the set $N = \{x_n, y_1, y_2, \ldots, y_n\}$ is contained in O and $N \in \mathcal{N}_n$, by the definition of $\mathcal{N}_n$. Since $O \subseteq U$, this proves our claim.

Let $n \geq 0$ be an arbitrary integer. Then $\mathcal{A}_n = \bigcup_{i=0}^n \mathcal{N}_i$ is not a network for X, because no element $N \in \mathcal{A}_n$ contains x_{n+1}. In particular, for no $n \in \omega$ can the family $\{N \in \mathcal{N} : |N| \leq n\}$ be a network for X. □

In a topological space X, an element $x \in X$ is called *P-point* if any countable intersection of open neighborhoods of x is again a (not necessarily open) neighborhood of x. We also say that X is a *P-space* if every element $x \in X$ is a P-point. Clearly, X is a P-space if and only if every G_δ-set in X is open.

In the following example, the space X is not necessarily countable. Assuming that X is not a P-space, we construct a network $\mathcal{N}$ of finite sets for X such that for every integer $n \geq 1$, the subfamily $\mathcal{N}_n = \{N \in \mathcal{N} : |N| \leq n\}$ of $\mathcal{N}$ is not a network for X.

Example 2.10. *Let X be an infinite Hausdorff space. Suppose that the element $y^* \in X$ is not a P-point in the space X. Consequently, there is a decreasing sequence $\{U_n : n \in \mathbb{N}^+\}$ of open neighborhoods of y^* in X such that $\bigcap_{n=1}^\infty U_n$ is not a neighborhood of y^*. Consider the following families of sets:*

- $\mathcal{N}_0 = \{\{x\} : x \neq y^*\}$;
- $\mathcal{N}_1 = \{\{y^*, x\} : x \notin U_1\}$;
- $\mathcal{N}_2 = \{\{y^*, x_1, x_2\} : \text{either } x_1 \notin U_2 \text{ or } x_2 \notin U_2\}$.

In general, let $\mathcal{N}_n$ be the family of sets of the form $\{y^*, x_1, x_2 \ldots, x_n\}$ such that $x_j \notin U_n$ for some $j \in \{1, 2, \ldots, n\}$. Let $\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$ and V be an open neighborhood of y^* in X. Then there exists $n \in \mathbb{N}^+$ such that $V \setminus U_n \neq \varnothing$. Take $x_n \in V \setminus U_n$ and elements $\{x_1, x_2, \ldots, x_{n-1}\} \subset V$, hence $\{y^*, x_1, x_2, \ldots, x_{n-1}, x_n\} \in \mathcal{N}_n \subset \mathcal{N}$ and $\{y^*, x_1, x_2, \ldots, x_{n-1}, x_n\} \subset V$. If $x \in X$ and $x \neq y^*$, then $\{x\} \in \mathcal{N}_0$ and, clearly, $x \in \{x\} \subset W$, for every open neighborhood W of x. Therefore, $\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$ is a network for the space X.

For every $n \in \omega$, let $\mathcal{M}_n = \bigcup_{i\leq n} \mathcal{N}_i$. Then $\mathcal{M}_n$ is not a network for the space X because for every $A \in \mathcal{M}_n$ with $y^* \in A$ and the open neighborhood U_n of y^*, the inclusion $A \subset U_n$ does not hold. □

Since a non-discrete countable T_1-space cannot be a P-space, Example 2.10 implies the following fact that improves upon Example 2.9.

Corollary 2.11. *Every countably infinite non-discrete Hausdorff space X admits a countable network $\mathcal{N}$ of finite sets such that for every integer $n \geq 1$, the subfamily $\mathcal{N}_n = \{N \in \mathcal{N} : |N| \leq n\}$ is not a network for X.*

This section concludes with some unresolved problems. In the first, we weaken the requirement in Propositions 2.5 and 2.8 for a network $\mathcal{N}$ to be countable.

Problem 2.12. *Are the conclusions of Propositions 2.5 and 2.8 valid for networks with cardinalities less than* $\mathfrak{p}$*?*

In Proposition 2.8 we provided a negative solution to Problem 1.1 assuming that the pseudointersection number $\mathfrak{p}$ is equal to $\mathfrak{c}$. This naturally leads to the following question:

Problem 2.13. *Is it possible to construct in ZFC a countable, non-discrete topological (Abelian) group* G *that does not admit a countable network* $\mathcal{N}$ *with infinite elements?*

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