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Ćirić-generalized contraction via $wt-$distance

Hosein Lakzian [a], Darko Kocev [b] and Vladimir Rakočević [c]

[a] Department of Mathematics, Payame Noor University, 19395-4697 Tehran, I.R. of Iran (lakzian@pnu.ac.ir)
[b] Department of Mathematics, NLA University College, Olav Bjordals veg 41, 5111 Breistein, Bergen, Norway (Darko.Kocev@nla.no)
[c] University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia. (vrakoc@sbb.rs)

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Abstract

In this present paper, besides other things, we introduce the concept of Ćirić-generalized contractions via $wt-$distance and then we will prove some new fixed point results for these mappings, which generalize and improve fixed point theorems by L. B. Ćirić in [9, 8, 10] and also, B. E. Rhoades in [23]. Some examples illustrate usefulness of the new results. At the end, we will give some applications to nonlinear fractional differential equations.

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1. Introduction and Preliminaries

Banach contraction principle is the starting point for Fixed point theory. This theory has been developed in many directions and we will point out some of the fixed point theorems that we want to generalize in the context of wt-distance.

In 1971, Ćirić [8] extended this idea in the following way:

Suppose that there exist nonnegative functions q_1, q_2, q_3, q_4 satisfying

$$\sup\{q_1(x,y) + q_2(x,y) + q_3(x,y) + 2q_4(x,y) : x, y \in X\} = \alpha < 1, \tag{1.1}$$

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such that, for each $x, y \in X$,

$$M(x,y) = q_1(x,y)d(x,y) + q_2(x,y)d(x,fx) + q_3(x,y)d(y,fy) + q_4(x,y)[d(x,fy) + d(fx,y)]. \quad (1.2)$$

The mapping $f : X \to X$ is said to be a α-generalized contraction if and only if

$$d(fx,fy) \leq \alpha M(x,y), \quad (1.3)$$

for all $x, y \in X$. Ćirić proved the following statement.

Theorem 1.1 ([8]). *Let $f : X \to X$ be a α-generalized contraction of a f-orbitally complete metric space. Then f has a unique fixed point.*

Inequality (1.3) can be written in the form

$$d(fx,fy) \leq M(x,y) - (1-\alpha)M(x,y). \quad (1.4)$$

Now, among other things, our further work is motivated by the papers of Alber and Guerre - Delabrieriere [1] and Rhoades [23] on weakly contractive maps. See also [20, 21].

Since we replace the usual metric with the notions of wt-distance in our statements, we should talk about the history of these concepts and some relations between them and the usual metric.

The history of wt-distance goes as follows:

Bakhtin in [7] and Czerwik in [11] and [12] introduced b-metric spaces (as a generalization of metric spaces) and established the contraction principle in this framework.

Definition 1.2. Let X be a set and let $d : X \times X \longrightarrow [0, \infty)$ be a map that satisfies the following:

(1) $d(x,y) = 0 \Longleftrightarrow x = y \ \forall x, y \in X$;
(2) $d(x,y) = d(y,x) \ \forall x, y \in X$;
(3) $d(x,y) \leq l[d(x,z) + d(z,y)] \ \forall x, y, z \in X$ for some constant $l \geq 1$.

The function d is called a b-metric with coefficient l and a triplet (X, d, l) is called a b-metric space.

Bakhtin and Czerwik gave examples of b-metric spaces that do not satisfy the triangle inequality. Note that, as in the classical metric case, every b-metric induces a topology. In this topology, in [6], [19] and [18] was shown by adequate examples that b-metric is not always continuous and that an open ball is not always an open set with respect to b-metric. We add the following example from [18] for the convenience of reader:

Example 1.3 ([18]). Let $X = \{0, 1, 1/2, ..., 1/n, ...\}$ and

$d(x,y) = 0$ if $x = y$, $d(x,y) = 1$ if $x \neq y \in \{0,1\}$, $d(x,y) = \mid x - y \mid$ if $x \neq y \in \{0, \frac{1}{2n}, \frac{1}{2m}\}$ and $d(x,y) = 4$ otherwise.

Then it holds that:

(1): d is a b-metric on X with coefficient $s = \frac{8}{3}$;

(2): d is not a metric on X;
(3): d is not continuous in each variable because we have that
$\lim_{n\to\infty} d(0, \frac{1}{2n}) = \lim_{n\to\infty} \frac{1}{2n} = 0$ and
$\lim_{n\to\infty} d(1, \frac{1}{2n}) = 4 \neq 1 = d(1,0)$.
(4): $0 \in B(1,2)$, but $B(0,r) \nsubseteq B(1,2)$ for every $r > 0$, so $B(1,2)$ is not an open set in topology induced by b-metric d.

The convergence in b-metric spaces is defined in [16] in the following way:

Definition 1.4. Let (X,d) be a b-metric space.

(1) The sequence $\{x_n\}$ converges to $x \in X \Longleftrightarrow lim_{n\to\infty} d(x_n,x) = 0$;
(2) The sequence $\{x_n\}$ is Cauchy $\Longleftrightarrow lim_{n,m\to\infty} d(x_n,x_m) = 0$.
We say that (X,d) is complete if and only if any Cauchy sequence in X is convergent.

Finally, Hussain et al. [14] have recently introduced the concept of wt-distance in generalized b-metric spaces, proved that it is a generalization of w-distance in [15] and also proved some fixed point theorems in a partially ordered b-metric space by using wt-distance. See also [18, 22].

Definition 1.5. Let (X,d) be a b-metric space with constant $l \geq 1$. Then a function $p : X \times X \longrightarrow [0,\infty)$ is called wt-distance on X if the following conditions are satisfied:

(a) $p(x,z) \leq l[p(x,y) + p(y,z)]\ \forall x,y,z \in X$;
(b) $\forall x \in X$, $p(x,\cdot) : X \longrightarrow [0,\infty)$ is l-lower semicontinuous;
(c) $\forall \epsilon > 0 \exists \delta > 0$ so that $p(z,x) \leq \delta \wedge p(z,y) \leq \delta \Longrightarrow d(x,y) \leq \epsilon$.

Let us recall that a real-valued function f defined on a b-metric space X is said to be l-lower semicontinuous at a point x_0 in X if either $\liminf_{x_n\to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n\to x_0} lf(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$.

In the same paper the following lemma has been proved and it is often used as a tool in order to prove fixed point theorems in terms of wt-distance.

Lemma 1.6 ([14])**.** *Let $(X,d,l \geq 1)$ be a b-metric space and p be a wt-distance on X.*

(i) *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} p(x_n,x) = \lim\limits_{n\to\infty} p(x_n,y) = 0$. Then $x = y$. In particular, if $p(z,x) = p(z,y) = 0$, then $x = y$.*
(ii) *If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to 0, then $\{y_n\}$ converges to y.*
(iii) *Let $\{x_n\}$ be a sequence in X such that for each $\varepsilon > 0$, there exists $N_\varepsilon \in N$ such that $m > n > N_\varepsilon$ implies $p(x_n,x_m) < \varepsilon$ (or $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$), then $\{x_n\}$ is a Cauchy sequence.*
(iv) *if $p(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then the sequence $\{x_n\}$ Cauchy.*

We define now the family of functions Ψ that we will use throughout our paper and after that we will introduce the notion of weak (ψ, M_p)-contractive mapping.

In this present paper, we introduce the concept of Ćirić-generalized contractions via $wt-$distance and then we will prove some new fixed point results for these mappings, which generalize and improve fixed point theorems by L. B. Ćirić in [9, 8, 10] and also, B. E. Rhoades in [23]. Some examples illustrate usefulness of the new results. At the end, we will give some applications to nonlinear fractional differential equations. In this paper, using the concept of weak (ψ, M_p)-contractive mappings, we will prove some new fixed point theorems which generalize fixed point theorems by Ćirić [8].

2. Main results

In [13] and [5] is given the following result:

Theorem 2.1. *Let $(X, d, l \geq 1)$ be a complete b-metric space and define the sequence $\{x_n\}$ in X by the recursion*

$$x_n = Tx_{n-1} = T^n x_0.$$

Let $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 [d(y, Tx) + d(x, Ty)]$$

for all $x, y \in X$, where $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$. Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is a unique fixed point.

In this section we will generalize this result in the framework of wt-distance.

Let (X, d) be a b-metric space with constant $l \geq 1$ and wt-distance p. Consider

$$\begin{aligned} M_{p,l}(x, y) = q_1(x, y)p(x, y) + q_2(x, y)p(x, fx) + q_3(x, y)p(y, fy) \\ + q_4(x, y)[p(x, fy) + p(fx, y) - p(y, y)]. \end{aligned} \tag{2.1}$$

with

$$\sup\{q_1(x, y) + q_2(x, y) + q_3(x, y) + 2lq_4(x, y) : x, y \in X\} = k < \frac{1}{l}, \tag{2.2}$$

We now introduce the notion of weak $(\psi, M_{p,l})$-contractive mapping, where the function $\psi : [0, \infty)^5 \to [0, \infty)$ satisfying the condition $\psi^{-1}\{0\} = \{(0, 0, 0, 0, 0)\}$.

Definition 2.2. Let p be a $wt-$distance on a $b-$metric space (X, d) with constant $l \geq 1$ and $f : X \to X$ be a given mapping. We say that f is a weak $(\psi, M_{p,l})$-contractive mapping if

$$p(fx, fy) \leq M_{p,l}(x, y) - \psi\Big(p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy)}{2}, \frac{p(fx, y)}{2}\Big), \tag{2.3}$$

for all $x, y \in X$. In the case $p = d$, it is called (ψ, M_l)-contractive mapping.

Recently in [17] is introduced the notion of $(C; k)$ condition based on the well-known $(C; 1)$ condition introduced and studied by Ćirić in [10]. It is said that a map $f : X \to X$ on a metric space (X, d) satisfies the condition $(C; k)$ if there is a constant $k \geq 0$ such that for every sequence $x_n \in X$,

$x_n \to x_0 \in X \Rightarrow D(x_0) \leq k \cdot limsupD(x_n)$ where $D(x) = d(x, fx), x \in X$.

Obviously, this condition is more relaxing than continuity.

Theorem 2.3. *Let p be a wt-distance on a complete b-metric space (X,d) with constant $l \geq 1$ and let $f : X \to X$ be a weak $(\psi, M_{p,l})$-contractive mapping. If f satisfies the condition $(C;k)$, or for every $w \in X$ with $w \neq Tw$, we have $\inf\{p(x,w)+p(x,Tx) : x \in X\} > 0$, then f has a unique fixed point u and moreover, $p(u,u)=0$.*

Proof. Define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$, for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $u = x_{n_0}$ is a fixed point of f, so the proof is completed. From now on, we assume that

$$x_n \neq x_{n+1} \text{ for all } n. \tag{2.4}$$

Step 1. We first show that $\lim\limits_{n\to\infty} p(x_n, x_{n+1}) = 0$. Using (2.3) and Definition 2.2, we have

$$\begin{aligned}
p(x_n, x_{n+1}) &= p(fx_{n-1}, fx_n) \\
&\leq M_p(x_{n-1}, x_n) - \psi\Big(p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_n, fx_n), \\
&\qquad\qquad \frac{p(x_{n-1}, fx_n)}{2}, \frac{p(fx_{n-1}, x_n)}{2}\Big) \\
&\leq M_p(x_{n-1}, x_n) \\
&= q_1(x_{n-1}, x_n)p(x_{n-1}, x_n) + q_2(x_{n-1}, x_n)p(x_{n-1}, fx_{n-1}) \\
&\qquad + q_3(x_{n-1}, x_n)p(x_n, fx_n) + q_4(x_{n-1}, x_n) \\
&\qquad \big[p(x_{n-1}, fx_n) + p(fx_{n-1}, x_n) - p(x_n, x_n)\big] \\
&= q_1(x_{n-1}, x_n)p(x_{n-1}, x_n) + q_2(x_{n-1}, x_n)p(x_{n-1}, x_n) \\
&\qquad + q_3(x_{n-1}, x_n)p(x_n, x_{n+1}) \\
&\quad + q_4(x_{n-1}, x_n)\big[p(x_{n-1}, x_{n+1}) + p(x_n, x_n) - p(x_n, x_n)\big] \\
&\leq \Big(q_1(x_{n-1}, x_n) + q_2(x_{n-1}, x_n)\Big)p(x_{n-1}, x_n) \\
&\qquad + q_3(x_{n-1}, x_n)p(x_n, x_{n+1}) \\
&\quad + q_4(x_{n-1}, x_n) \cdot l \cdot \big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big],
\end{aligned} \tag{2.5}$$

for all $n \geq 1$.

Thus,

$$p(x_n, x_{n+1}) \leq \frac{q_1(x_{n-1}, x_n) + q_2(x_{n-1}, x_n) + lq_4(x_{n-1}, x_n)}{1 - q_3(x_{n-1}, x_n) - lq_4(x_{n-1}, x_n)} p(x_{n-1}, x_n), \tag{2.6}$$

for all $n \geq 1$.

Therefore, $p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n)$ for all $n \geq 1$; indeed, from $k < 1$ and the relation (2.2), we get

$$q_1(x,y) + q_2(x,y) + lq_4(x,y) + kq_3(x,y) + klq_4(x,y) \leq k,$$

and hence

$$\frac{q_1(x,y) + q_2(x,y) + lq_4(x,y)}{1 - q_3(x,y) - lq_4(x,y)} \leq k, \tag{2.7}$$

for all $x, y \in X$. Now, from (2.6) and (2.2) we have

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1), \tag{2.8}$$

for all $n \geq 1$, and so

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.9}$$

Step 2. We will show that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.10}$$

For each $m, n \in \mathbb{N}$ with $m > n$, applying (1) of Definition 6 and (2.8), we get

$$\begin{array}{rcl} p(x_n, x_m) & \leq & l \cdot [p(x_n, x_{n+1}) + p(x_{n+1}, x_m)] \\ & \leq & l \cdot p(x_n, x_{n+1}) + l^2 \cdot [p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_m)] \\ & \leq & l \cdot p(x_n, x_{n+1}) + l^2 \cdot p(x_{n+1}, x_{n+2}) + ... + l^{m-n} p(x_{m-1}, x_m) \\ & \leq & l \cdot k^n \cdot p(x_0, x_1) + l^2 \cdot k^{n+1} p(x_0, x_1) + ... + l^{m-n} \cdot k^{m-1} \cdot p(x_0, x_1) \\ & = & l \cdot k^n \cdot p(x_0, x_1)[1 + lk + ... + (lk)^{m-n-1}] \\ & \leq & \frac{l \cdot k^n}{1-lk}(p(x_0, x_1)) \to 0, \qquad \text{as } n \to \infty. \end{array}$$

Therefore, by Lemma 1.6, the sequence $\{x_n\}$ is Cauchy in (X, d). Since X is a complete b-metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Step 3. Now, we show that u is a fixed point of f.

Case † Suppose that f obeys the condition $(C; k)$. It holds that

$$d(u, fu) \leq k \cdot \limsup_{n\to\infty} d(x_n, fx_n) = k \cdot \limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$$

since x_n is a Cauchy sequence, so we conclude that $u = fu$.

Case ‡ Now, suppose that $\inf\{p(x, w) + p(x, Tx) : x \in X\} > 0$ for every $w \in X$ with $w \neq Tw$. By (2.10), for each $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon$ implies $p(x_{N_\varepsilon}, x_n) < \varepsilon$. But, $x_n \to u$ and $p(x, .)$ is l-lower semi-continuous, so using Definition 2.2, we have

$$p(x_{N_\varepsilon}, u) \leq \liminf_{n\to\infty} l \cdot p(x_{N_\varepsilon}, x_n) \leq \varepsilon.$$

Putting $\varepsilon = 1/lk$ and $N_\varepsilon = n_k$, we have

$$\lim_{k\to\infty} p(x_{n_k}, u) = 0. \tag{2.11}$$

Assume that $u \neq fu$. Then

$$0 < \inf\{p(x, u) + p(x, fx) : x \in X\} \leq \inf\{p(x_n, u) + p(x_n, x_{n+1}) : n \in \mathbb{N}\}.$$

Using (2.9) and (2.11), we get $\inf\{p(x_n, u) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} = 0$, which is a contradiction. Thus, $u = fu$.

Step 4. Moreover, we have

$$\begin{array}{rcl} p(u, u) = p(fu, fu) & \leq & M_{p,l}(u, u) - \psi\Big(p(u, u), p(u, fu), p(u, fu), \frac{p(u,fu)}{2}, \frac{p(fu,u)}{2}\Big) \\ & < & M_{p,l}(u, u) \\ & = & q_1(u, u)p(u, u) + q_2(u, u)p(u, u) + q_3(u, u)p(u, u) \\ & & + \, lq_4(u, u)\big[p(u, u) + p(u, u) - p(u, u)\big] \\ & \leq & \Big(q_1(u, u) + q_2(u, u) + q_3(u, u) + lq_4(u, u)\Big)p(u, u) \\ & \leq & k \cdot p(u, u), \end{array}$$

so we conclude that $p(u, u) = 0$.

Step 5. we shall show that u is unique. Let v be another fixed point of f. We shall prove that $u = v$.
We have

$$\begin{aligned}
p(u,v) &= p(fu,fv)\\
&\le M_{p,l}(u,v) - \psi\Big(p(u,v), p(u,fu), p(v,fv), \frac{p(u,fv)}{2}, \frac{p(fu,v)}{2}\Big)\\
&< M_p(u,v)\\
&= q_1(u,v)p(u,v) + q_2(u,v)p(u,u) + q_3(u,v)p(v,v)\\
&\quad + lq_4(u,v)\big[p(u,v) + p(u,v) - p(u,u)\big]\\
&= \big(q_1(u,v) + 2q_2(u,v)\big)p(u,v)\\
&\le kp(u,v)\\
&< p(u,v),
\end{aligned}$$

which is a contradiction. Therefore, $p(u,v) = 0$. From Step 4, $p(u,u) = 0$. So by Lemma 1.6, $u = v$.

□

The following example illustrates the previous statement.

Example 2.4. Let $X = [0,2]$ and d be a function $d : X \times X \to [0,+\infty)$ defined by $d(x,y) = (x-y)^2$. Then d is a b-metric with coefficient $l = 2$. We define a function $p : X \times X : [0,+\infty)$ on (X,d) with $p(x,y) = y^2$. Then p is a $wt-$distance on (X,d).

Let $f : X \to X$ be a mapping such that $f(x) = \frac{x}{10}$ for $x \in [0,1]$ and $f(x) = \frac{x}{20}$ for $x \in (1,2]$. Notice that f is not continuous and f is not a contraction with respect to b-metric d.

On the other hand, if we put $q_1 = q_2 = q_3 = q_4 = \frac{1}{20}$ and define a function $\psi : [0,\infty)^5 \to [0,\infty)$ with $\psi(a_1,a_2,a_3,a_4,a_5) = \frac{1}{100}a_1 + \frac{1}{20}a_2 + \frac{1}{20}a_3 + \frac{1}{10}a_4 + \frac{1}{50}a_5$, then we obtain

$M_{p,l}(x,y) = \frac{1}{20}y^2 + \frac{1}{20}(fx)^2 + \frac{1}{20}(fy)^2 + \frac{1}{20}(fy)^2 = \frac{1}{20}y^2 + \frac{1}{20}(fx)^2 + \frac{1}{10}(fy)^2$,

$\psi(y^2,(fx)^2,(fy)^2,\frac{(fy)^2}{2},\frac{y^2}{2}) = \frac{1}{100}y^2 + \frac{1}{20}(fx)^2 + \frac{1}{20}(fy)^2 + \frac{1}{20}(fy)^2 + \frac{1}{100}(y)^2$,

so we have that

$M_{p,l}(x,y) - \psi(y^2,(fx)^2,(fy)^2,\frac{(fy)^2}{2},\frac{(y)^2}{2}) = \frac{1}{20}y^2 - \frac{1}{50}y^2 = \frac{3}{100}y^2$.

We conclude that

$p(fx,fy) = (fy)^2 \le \frac{3}{100}y^2$,

so f is a weak $(\psi, M_{p,l})$-contractive mapping.

The condition $\inf\{p(x,w) + p(x,fx) : x \in X\} > 0$ is satisfied for every $w \neq 0$,

because $p(x,w) + p(x,fx) = w^2 + (fx)^2$, so we can apply Theorem 3.2 to the function f.

Corollary 2.5. *Let p be a wt-distance on a complete b-metric space (X,d) with constant $l \ge 1$. Let $f : X \to X$ be a self-mapping satisfying*

$$p(fx,fy) \le M_{p,l}(x,y) - \phi(M_{p,l}(x,y)),$$

for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is continuous, and $\phi^{-1}(0) = 0$. Suppose either $\inf\{p(x,w) + p(x,fx) : x \in X\} > 0$ for every $w \in X$ with $w \neq fw$, or the mapping f is continuous. Then f has a unique fixed point u and moreover $p(u,u) = 0$.

Taking $\psi(t_1,t_2,t_3,t_4,t_5) = 1 - k(\sum_{i=1}^{5} t_i)$ in Theorem 2.3, we obtain the following Corollary which is a generalization of Theorem 1 from [13].

Corollary 2.6. *Let p be a wt-distance on a complete b-metric space (X,d) with constant $l \geq 1$. Let $f : X \to X$ be a self-mapping satisfying*

$$p(fx,fy) \leq kM_{p,l}(x,y),$$

for all $x, y \in X$, where $k \in (0,1)$. Suppose either $\inf\{p(x,w) + p(x,fx) : x \in X\} > 0$ for every $w \in X$ with $w \neq fw$, or the mapping f is continuous. Then f has a unique fixed point u and moreover $p(u,u) = 0$.

Remark 2.7. Let (X,d) be a complete b-metric space with constant $l \geq 1$. Then, we have $\inf\{d(x,w) + d(x,fx) : x \in X\} > 0$ for every $w \in X$ with $w \neq fw$. Suppose not; i.e. there exists a $w \in X$ with $w \neq fw$ and $\inf\{d(x,w)+d(x,fx) : x \in X\} = 0$. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \{d(x_n,w) + d(x_n,fx_n)\} = 0.$$

Thus $\lim_{n\to\infty} d(x_n,w) = 0$ and $\lim_{n\to\infty} d(x_n,fx_n) = 0$. Then, by the triangular inequality, we get $\lim_{n\to\infty} d(fx_n,w) = 0$, and so $fx_n \to w$ as $n \to \infty$. Since f is a generalized (ψ,M,l)-contractive mapping, with $x = x_n$ and $y = w$, we conclude that

$$d(fx_n,fw) \leq \psi(M(x_n,w)),$$

and so $d(w,fw) \leq \psi(d(w,fw))$ as $n \to \infty$. From $\psi(t) < t$ for all $t > 0$, we have $d(w,fw) < d(w,fw)$, a contradiction. Therefore $fw = w$. Therefore we have the following results.

Corollary 2.8. *Let (X,d) be a complete b-metric space with constant $l \geq 1$ and let $f : X \to X$ be a (ψ,M,l)-contractive mapping. Then there exists a unique $u \in X$ such that $fu = u$.*

Proof. Taking $p = d$ in Theorem 2.3, and Remark 2.7, we can conclude the statement. □

Taking $\psi(t) = kt$ in Corollary 2.8, and from Remark 2.7, we obtain the following result (The Ćirić result [9]).

Corollary 2.9. *complete b-metric space with constant $l \geq 1$ and let $f : X \to X$ be a self-mapping satisfying*

$$d(fx,fy) \leq k\,M(x,y) \tag{2.12}$$

for all $x, y \in X$, for some $k \in [0,1)$. Then f has a unique fixed point.

In the following two fixed point theorems, let Ψ be the family of functions $\psi : [0,\infty)^5 \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) ψ is continuous;
(Ψ_2) $\psi(a_1, a_2, a_3, a_4, a_5) = 0$ if and only if $a_i = 0$ for $0 \le i \le 5$;
(Ψ_3) if $a, b \in [0, \infty)$ are such that $a \le \psi(b, b, a, 0, a+b)$ or $a \le \psi(b, a, b, a+b, 0)$ or $a \le \psi(a, a, b, a+b, 0)$ or $a \le \psi(a, a, b, 0, a+b)$, then $a \le \lambda b$, where $\lambda \in (0, 1)$;
(Ψ_4) if $a, b \in [0, \infty)$ is such that $a \le \psi(a, 0, 0, a, b)$ or $a \le \psi(0, a, 0, a, b)$ or $a \le \psi(0, 0, a, a, b)$, then $a = 0$.

In the following, we generalize and improve fixed point theorem by B. E. Rhoades in [23] in the setting wt-distances.

Theorem 2.10. *Let p be a wt-distance on a complete b-metric space (X, d) with constant $l \ge 1$ such that $p(x, x) = 0$, for all $x \in X$. Suppose that $f : X \to X$ is a self-map such that*

$$p(fx, fy) \le \psi\Big(p(x, y), p(x, fx), p(y, fy), p(fx, y), p(x, fy)\Big), \tag{2.13}$$

for all $x, y \in X$ and $\psi \in \Psi$. If f satisfies the condition $(C; k)$, or for every $w \in X$ with $w \neq Tw$, we have $\inf\{p(x, w) + p(x, Tx) : x \in X\} > 0$, then f has a unique fixed point u.

Proof. Define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$, for all $n \ge 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $u = x_{n_0}$ is a fixed point of f. The proof is completed. From now on, we assume that

$$x_n \neq x_{n+1} \text{ for all } n. \tag{2.14}$$

Step 1. We shall show that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Using (2.13) and Definition 2.2, we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(fx_{n-1}, fx_n) \\ &\le \psi\Big(p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_n, fx_n), p(fx_{n-1}, x_n), p(x_{n-1}, fx_n)\Big) \\ &= \psi\Big(p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_n, x_n), p(x_{n-1}, x_{n+1})\Big) \\ &= \psi\Big(p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), 0, p(x_{n-1}, x_n) + p(x_n, x_{n+1})\Big), \end{aligned} \tag{2.15}$$

for all $n \ge 1$.

Now, from (ψ_3), we get, $p(x_n, x_{n+1}) \le \lambda p(x_{n-1}, x_n)$, for all $n \ge 1$. Thus, by induction, we have $p(x_n, x_{n+1}) \le \lambda^n p(x_0, x_1)$, for all $n \ge 1$, and so

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.16}$$

Step 2,3 Step2 and Step 3 are completely the same as in Theorem 2.3; so we delete these statements.

Step 4. we shall show that u is unique. Let v be another fixed point of f. We shall prove that $u = v$.

We have

$$\begin{aligned} p(u,v) &= p(fu,fv) \\ &\leq \psi\Big(p(u,v),p(u,fu),p(v,fv),p(fu,v),p(u,fv)\Big) \\ &= \psi\Big(p(u,v),0,0,p(u,v),p(u,v)\Big). \end{aligned}$$

Now, from (ψ_4), we have $p(u,v)=0$. On the other hand $p(u,u)=0$. So by Lemma 1.6, $u=v$.

□

In the following, we generalize and improve fixed point theorem by B. E. Rhoades in [23] in the setting wt-distances.

Theorem 2.11. *Let p be a wt-distance on a complete b-metric space (X,d) with constant $l \geq 1$ such that $p(x,x)=0$, for all $x \in X$. Suppose that $f : X \to X$ is a self-map such that*

$$\begin{aligned} p(fx,fy) \leq \psi\Big(&ap(x,y),(1-a)p(x,fx),(1-a)p(y,fy),(1-a)p(x,fx)p(y,fy), \\ &(1-a)p(fx,y)p(x,fy)\Big), \end{aligned} \tag{2.17}$$

for all $x,y \in X$, where $0 < a \leq 1$ and $\psi \in \Psi$. If f satisfies the condition $(C;k)$, or for every $w \in X$ with $w \neq Tw$, we have $\inf\{p(x,w)+p(x,Tx) : x \in X\} > 0$, then f has a unique fixed point u.

Proof. Define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$, for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $u = x_{n_0}$ is a fixed point of f. The proof is completed. From now on, we assume that

$$x_n \neq x_{n+1} \text{ for all } n. \tag{2.18}$$

Step 1. We shall show that $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$. Using (2.17) and Definition 2.2, we have

$$\begin{aligned} p(x_n,x_{n+1}) &= p(fx_{n-1},fx_n) \\ &\leq \psi\Big(ap(x_{n-1},x_n),(1-a)p(x_{n-1},fx_{n-1}),(1-a)p(x_n,fx_n) \\ &\quad ,(1-a)p(x_{n-1},fx_{n-1})p(x_n,fx_n),(1-a)p(fx_{n-1},x_n)p(x_{n-1},fx_n)\Big) \\ &= \psi\Big(ap(x_{n-1},x_n),(1-a)p(x_{n-1},x_n),(1-a)p(x_n,x_{n+1}) \\ &\quad ,(1-a)p(x_{n-1},x_n)p(x_n,x_{n+1}),(1-a)p(x_n,x_n)p(x_{n-1},x_{n+1})\Big) \\ &= \psi\Big(ap(x_{n-1},x_n),(1-a)p(x_{n-1},x_n),(1-a)p(x_n,x_{n+1}), \\ &\qquad (1-a)p(x_{n-1},x_n)p(x_n,x_{n+1}),0\Big), \end{aligned} \tag{2.19}$$

for all $n \geq 1$.

Now, from (ψ_3), we get, $p(x_n, x_{n+1}) \leq \lambda p(x_{n-1}, x_n)$, for all $n \geq 1$. Thus, by induction, we have $p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1)$, for all $n \geq 1$, and so

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.20}$$

Step 2,3 Step2 and Step 3 are completely the same as in Theorem 2.3; so we delete these statements.
Step 4. we shall show that u is unique. Let v be another fixed point of f. We shall prove that $u = v$.
We have

$$\begin{aligned} ap(u,v) \leq p(u,v) &= p(fu,fv) \\ &\leq \psi\Big(ap(u,v), (1-a)p(u,fu), (1-a)p(v,fv), \\ &\qquad (1-a)p(u,fu)p(v,fv), (1-a)p(fu,v)p(u,fv)\Big) \\ &= \psi\Big(ap(u,v), 0, 0, 0, (1-a)p(u,v)p(u,v)\Big). \end{aligned}$$

Now, from (ψ_4), we have $p(u,v) = 0$. On the other hand $p(u,u) = 0$. So by Lemma 1.6, $u = v$.

□

3. Application

In this section, we will give an application of Theorem 2.10 to establish the existence of solutions for a nonlinear fractional differential equation considered in.

Theorem 3.1. *Consider the nonlinear Fredholm integral equation*

$$x(t) = g(t) - \int_a^b K(t,s,x(s))ds, \tag{3.1}$$

where $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Let $X = C[a,b]$ be the set of all real continuous functions on $[a,b]$. Clearly, X with the b-metric $d : X \times X \to [0,+\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} (x(t) - y(t))^2$$

for all $x, y \in X$, is a b-metric space with coefficient $l = 2$.

Suppose that the following conditions hold:

(i) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = g(t) - \int_a^b K(t,u,x(u))ds, \qquad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(ii) for each $x, y \in X$ with $x \neq y$ and $t, u \in [a, b]$, we have

$$(K(t,u,fx(u)))^2 + (K(t,u,fy(u)))^2 \leq \frac{1}{b-a}\Big(\psi\Big((x(u))^2 + (y(u))^2, (x(u))^2 + (fx(u))^2, (y(u))^2 + (fy(u))^2, (fx(u))^2 + (y(u))^2, (x(u))^2 + (fy(u))^2\Big). \tag{3.2}$$

Let f be a continuous mapping.

Then the nonlinear integral equation (3.1) has a unique solution. Moreover, for each $x \in C[a, b]$, the Picard iteration (x_n) defined by

$$x_n(t) = g(t) - \int_a^b K(t, u, x_{n-1}(u))ds$$

for all $n \in \mathbb{N}$ converges to a unique solution of the nonlinear integral equation (3.1).

Proof. Let us define the function $p : X \times X \to [0, +\infty)$ by

$$p(x, y) = \sup_{t\in[a,b]} (x(t))^2 + \sup_{t\in[a,b]} (y(t))^2$$

for all $x, y \in X$. Clearly, p is a wt-distance on X and a ceiling distance of d. Here, we will show that f satisfies the contractive condition (2.13). Assume that $x, y \in X$ and $t \in [a, b]$. Then we get

$$\begin{aligned}
&((fx)(t))^2 + ((fy)(t))^2 \\
&\quad= \Big(g(t) - \int_a^b K(t,u,fx(u))du\Big)^2 + \Big(g(t) - \int_a^b K(t,u,fy(u))du\Big)^2 \\
&\quad\leq (g(t))^2 + \Big(\int_a^b K(t,u,fx(u))du\Big)^2 + (g(t))^2 + \Big(\int_a^b K(t,u,fy(u))du\Big)^2 \\
&\quad\leq 2(g(t))^2 + \int_a^b (K(t,u,fx(u)))^2du + \int_a^b (K(t,u,fy(u)))^2du \\
&\quad= 2(g(t))^2 + \int_a^b \Big((K(t,u,fx(u)))^2 + (K(t,u,fy(u)))^2\Big)du \\
&\quad\leq 2(g(t))^2 + \int_a^b \Big(\frac{A(u) - 2(g(t))^2}{b-a}\Big)du \\
&\quad\leq 2(g(t))^2 + \frac{1}{b-a}\int_a^b (A(u) - 2(g(t))^2)du \\
&\quad= A,
\end{aligned}$$

where

$$A(u) = \psi\Big((x(u))^2 + (y(u))^2, (x(u))^2 + (fx(u))^2, (y(u))^2 + (fy(u))^2, (fx(u))^2 + (y(u))^2, (x(u))^2 + (fy(u))^2\Big).$$

This implies

$$\sup_{t\in[a,b]}((fx)(t))^2+\sup_{t\in[a,b]}((fy)(t))^2\leq A,$$

and so

$$p(fx,fy)\leq A$$

for all $x,y\in X$. It follows that f satisfies condition (2.13). Therefore, all the conditions of Theorem 2.10 are satisfied, and thus f has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (3.1). $\square$

By using the similar method in the proof of Theorem 3.1, we can get the following result.

Theorem 3.2. *Consider the nonlinear Volterra integral equation*

$$x(t)=g(t)-\int_a^t K(t,u,x(u))du, \tag{3.3}$$

where $a,b\in\mathbb{R}$ with $a<b$, and $g:[a,b]\to\mathbb{R}$ and $K:[a,b]^2\times\mathbb{R}\to\mathbb{R}$ are given continuous mappings. Suppose that the following conditions hold:

(i) the mapping $f:C[a,b]\to C[a,b]$ defined by

$$(fx)(t)=g(t)-\int_a^t K(t,u,x(u))du,\qquad \textit{for all } x\in C[a,b] \textit{ and } t\in[a,b]$$

is a continuous mapping;

(ii) for each $x,y\in X$ with $x\neq y$ and $t,u\in[a,b]$, we have

$$\begin{aligned}\Big(K(t,u,fx(u))+K(t,u,fy(u))\Big)^2\leq \frac{1}{b-a}\Big(\psi\Big((x(u))^2+(y(u))^2,(x(u))^2+(fx(u))^2,\\ (y(u))^2+(fy(u))^2,(fx(u))^2+(y(u))^2,(x(u))^2+(fy(u))^2\Big)-2(g(t))^2\Big).\end{aligned} \tag{3.4}$$

Let f be a continuous mapping.

Then the nonlinear integral equation (3.3) has a unique solution. Moreover, for each $x\in C[a,b]$, the Picard iteration (x_n) defined by

$$x_n(t)=g(t)-\int_a^t K(t,u,x_{n-1}(u))du$$

for all $n\in\mathbb{N}$ converges to a unique solution of the nonlinear integral equation (3.3).

Remark 3.3. The theory of nonlinear fractional differential equations nowadays is a large subject of mathematics which found numerous applications of many branches such as physics, engineering, and other fields connected with real-world problems. Based on this fact, many authors studied various results in this theory (see [2, 3, 4]).

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