

Fixed points of set-valued mappings in Menger probabilistic metric spaces endowed with an amorphous binary relation

GOPI PRASAD , SHEETAL DESHWAL AND RUPESH K. SRIVASTAV

Department of Mathematics, Dr. Shivanand Nautiyal Government Post Graduate College, Karanprayag, Uttarakhand, India-246444 (gopiprasad127@gmail.com, sheetald1990@gmail.com, rupeshk.srivastav@gmail.com)

Communicated by O. Valero

ABSTRACT

In this paper, we prove the existence of fixed point results for set-valued mappings in Menger probabilistic metric spaces equipped with an amorphous binary relation and a Hadžić-type t -norm. For the usability of such findings we present a Kelisky-Rivlin type result for a class of Bernstein type special operators introduced by Deo et. al. [Appl. Math. Comput. 201, (2008), 604-612] on the space $C([0, \frac{n}{n+1}])$. In this way, these investigations extend, modify and generalize some prominent recent fixed point results of the existing literature.

2020 MSC: 47H10; 54H25.

KEYWORDS: fixed point; set-valued mapping; Bernstein operator.

1. INTRODUCTION

An analog of Banach contraction principle [4] in the settings of metric spaces was investigated by Turinici [27, 28] which was later explored by several authors (see Ran and Reurings [23], Nieto and López [18], Samet and Turinici [24], Alam and Imdad [1] and Alam et al. [2] and this process is still on. Meanwhile, Jachymski [13] established an interesting metrical fixed point results by incorporating the notion of graphical contraction mapping besides presenting a

variant of the Kelisky and Rivlin theorem [15] concerning Bernstein operators via contraction principle on the space $C[0, 1]$ and there exist detailed generalization on this theme, see for instance ([3], [20]-[22]).

Among all these generalizations, we must recite Alam and Imdad [1] and Jachymski [13] in which the respective authors utilize relational and graphical variants of usual metrical definitions such as continuity, contractions and completeness to obtain some interesting generalizations of the contraction principle. Noticeably, Alam et al. [2] presented a refinements of the relation-theoretic contraction principle besides highlighting the importance of the notion of d -self-closedness utilized by Alam and Imdad [1] to such settings. In fact, the respective authors highlighted that the relation theoretic approach still remain a genuine improvement over graphical approach.

On another point of note, Dinevari and Frigon [7] extended some fixed point results of Jachymski [13] to set-valued mappings. The respective authors introduced the notion of set-valued G -contractions and presented fixed point theorems. They also presented a comparison between fixed point sets obtained from Picard iterations starting from different points. Recently, Kamran et al. [14] extended the results of Jachymski [13] to the setting of Menger probabilistic metric spaces. They introduced the class of probabilistic G -contraction for single-valued mappings and studied the existence of fixed points for such mappings. More recently, some fixed point results for probabilistic α -minimum Ciric type contraction are presented by Bhandari et al. [5].

Aim of this research work is to investigate a new fixed point theorem for set-valued mappings defined on a Menger probabilistic metric space equipped with an amorphous binary relation and establish a Kelisky-Rivlin type result for a class of Bernstein type special operators introduced by Deo et. al. [6] on the spaces of continuous functions defined on closed interval $[0, \frac{n}{n+1}]$ in the light of obtained results. In this way, we utilize the contractive assumption enjoying only on those elements which are associated with an amorphous binary relation instead of the entire space.

2. PRELIMINARIES

The notion of statistical metric spaces was introduced by Karl Menger in 1942. Later on the new theory of fundamental probabilistic structures was developed by many authors. In this section, we start by recalling some basic concepts from Menger probabilistic metric spaces. For more details on such spaces, we refer to [11, 12, 25, 26]. We provide some basic definitions which will work as a relevant necessary background for further presentations. We use notations \mathcal{R} for a non-empty binary relation, \mathbb{N}_0 for the the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

and \mathbb{R} for the set of real numbers thoroughly in this paper.

A mapping $\mathcal{F} : \mathbb{R} \rightarrow [0, 1]$ is called a distribution function if it satisfies the following conditions:

- (d_1) \mathcal{F} is nondecreasing;
- (d_2) \mathcal{F} is left continuous;
- (d_3) $\inf_{t \in \mathbb{R}} \mathcal{F}(t) = 0$ and $\sup_{t \in \mathbb{R}} \mathcal{F}(t) = 1$.

If, in addition, we have

- (d_4) $\mathcal{F}(0) = 0$, then \mathcal{F} is called a distance distribution function.

Let \mathcal{D}^+ be the set defined by

$$\mathcal{D}^+ = \left\{ \mathcal{F} : \mathbb{R} \rightarrow [0, 1] : \mathcal{F} \text{ is distance distribution function, } \lim_{t \rightarrow +\infty} \mathcal{F}(t) = 1 \right\}.$$

The element $\delta_0 \in \mathcal{D}^+$ defined by

$$\delta_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0, \end{cases}$$

is the Dirac distribution function.

Definition 2.1 ([12]). A mapping $\mathcal{T} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a triangular norm (briefly t-norm) if for every $u, v, w \in [0, 1]$, we have

- (t_1) $\mathcal{T}(u, v) = \mathcal{T}(v, u)$;
- (t_2) $\mathcal{T}(u, \mathcal{T}(v, u)) = \mathcal{T}(\mathcal{T}(u, v), w)$;
- (t_3) $\mathcal{T}(u, v) \leq \mathcal{T}(u, w)$ if $v \leq w$;
- (t_4) $\mathcal{T}(u, 1) = u$.

The commutativity (t_1), the monotonicity (t_3), and the boundary condition (t_4) imply that for each t-norm \mathcal{T} and for each $u \in [0, 1]$, we have the following boundary conditions:

$$\mathcal{T}(u, 1) = \mathcal{T}(1, u) = u \text{ and } \mathcal{T}(u, 0) = \mathcal{T}(0, u) = 0.$$

Typical examples of t-norms are $\mathcal{T}_M(u, v) = \min\{a, b\}$ and $\mathcal{T}_P(u, v) = uv$.

Definition 2.2 ([12]). A t-norm \mathcal{T} is said to be of \mathcal{H} -type if the family of functions $\{\mathcal{T}^n\}_{n \in \mathbb{N}}$ is equicontinuous at $t = 1$, where $\mathcal{T}^n : [0, 1] \rightarrow [0, 1]$ is recursively defined by

$$\mathcal{T}^1(t) = \mathcal{T}(t, t), \quad \mathcal{T}^{n+1}(t) = \mathcal{T}(\mathcal{T}^n(t), t); \quad t \in [0, 1], n = 1, 2, \dots$$

A trivial example of a t-norm of \mathcal{H} -type is $\mathcal{T}_M = \min$, but there exist t-norms of \mathcal{H} -type with $\mathcal{T} \neq \mathcal{T}_M$.

Definition 2.3 ([12]). A Menger probabilistic metric space or Menger PM-space is a triple $(\mathcal{X}, \mathcal{F}, \mathcal{T})$, where \mathcal{X} is a nonempty set, $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{D}^+$, and $\mathcal{T} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a t-norm such that for every $u, v, w \in \mathcal{X}$, we have

- (PM1) $\mathcal{F}(u, v) = \delta_0 \Leftrightarrow u = v$;
- (PM2) $\mathcal{F}(u, v) = \mathcal{F}(v, u)$;
- (PM3) $\mathcal{F}(u, w)(t + s) \geq \mathcal{T}(\mathcal{F}(u, v)(t), \mathcal{F}(v, w)(s))$ for all $t, s \geq 0$.

Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space. For $\varepsilon > 0$ and $\delta \in (0, 1]$, the (ε, δ) -neighborhood of $u \in \mathcal{X}$ is denoted by $N_u(\varepsilon, \delta)$ and is defined by

$$N_u(\varepsilon, \delta) = \{y \in \mathcal{X} : \mathcal{F}(u, v)(\varepsilon) > 1 - \delta\}.$$

Furthermore, if $\sup_{0 < a < 1} \mathcal{T}(a, a) = 1$, then the family of neighborhoods

$$\{N_u(\varepsilon, \delta) : u \in \mathcal{X}, \varepsilon > 0, \delta \in (0, 1]\}$$

determines a Hausdorff topology for \mathcal{X} .

Definition 2.4 ([12]). Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space.

- (a) A sequence $\{u_n\} \subset \mathcal{X}$ converges to an element $u \in \mathcal{X}$ if for every $\varepsilon > 0$ and $\delta \in (0, 1]$, there exists $n_0 \in \mathbb{N}$ such that $u_n \in N_u(\varepsilon, \delta)$ for every $n \geq n_0$.
- (b) A sequence $\{u_n\} \subset \mathcal{X}$ is a Cauchy sequence if for every $\varepsilon > 0$ and $\delta \in (0, 1]$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F}(u_n, u_m)(\varepsilon) > 1 - \lambda$, whenever $n, m \geq n_0$.
- (c) A Menger PM -space is complete if every Cauchy sequence in \mathcal{X} converges to a point in \mathcal{X} .
- (d) A subset \mathcal{A} of \mathcal{X} is closed if every convergent sequence in \mathcal{A} converges to an element of \mathcal{A} .

Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space. For all $\lambda \in (0, 1]$, we define the mapping $d_\lambda : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ by

$$d_\lambda(u, v) = \inf\{t > 0 : \mathcal{F}(u, v)(t) > 1 - \lambda\} \quad \text{for all } u, v \in \mathcal{X}.$$

We denote

$$\mathcal{D}(u, v) = \sup\{d_\lambda(u, v) : \lambda \in (0, 1]\} \quad \text{for all } u, v \in \mathcal{X}.$$

The following lemma will be useful later.

Lemma 2.5 ([8, 10]). Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space. For every $\lambda \in (0, 1]$, we have

- (a) $d_\lambda(u, v) < t \iff \mathcal{F}(u, v)(t) > 1 - \lambda$;
- (b) $d_\lambda(u, v) = 0$ for all $\lambda \in (0, 1] \iff u = v$;
- (c) $d_\lambda(u, v) = d_\lambda(v, u)$ for all $u, v \in \mathcal{X}$;
- (d) if \mathcal{T} is of \mathcal{H} -type, then for each $\lambda \in (0, 1]$, there exists $\mu \in (0, \lambda]$ such that for each $m \in \mathbb{N}$,

$$d_\lambda(u_0, u_m) \leq \sum_{i=1}^m d_\mu(u_{i-1}, u_i) \quad \text{for all } u_0, u_1, \dots, u_m \in \mathcal{X}.$$

2.1. Some relation theoretic metrical notions.

Definition 2.6 ([17, 1]). Let \mathcal{X} be a non-empty set and $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$. Then

- (a) \mathcal{R} is a binary relation on \mathcal{X} and "u relates v under \mathcal{R} " if and only if $(u, v) \in \mathcal{R}$,
- (b) u and v are \mathcal{R} -comparative, if either $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$, and denoted by $[u, v] \in \mathcal{R}$,
- (c) $(u, v) \in \mathcal{R}^s$, if and only if $[u, v] \in \mathcal{R}$.

Definition 2.7 ([1]). Let \mathcal{X} be a non-empty set equipped with a binary relation \mathcal{R} on \mathcal{X} . Let $\{u_n\}$ be a sequence in \mathcal{X} . Then $\{u_n\}$ is an \mathcal{R} -preserving sequence if $(u_n, u_{n+1}) \in \mathcal{R}$, $n \in \mathbb{N}_0$.

Definition 2.8. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space equipped with a binary relation \mathcal{R} on \mathcal{X} . Let $\{u_n\}$ be an \mathcal{R} -preserving sequence on \mathcal{X} . Then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is an \mathcal{R} -complete if every \mathcal{R} -preserving Cauchy sequence converges to a point in \mathcal{X} .

Remark 2.9. Every \mathcal{R} -complete Menger PM -space is complete Menger PM -space and in respect to the universal relation these notions are the same.

The subsequent notion defined by Turinici [28] is an extension of d -self-closedness of a partial order relation " \preceq ".

Definition 2.10 ([1]). Let (\mathcal{X}, d) be a metric space. A binary relation \mathcal{R} on \mathcal{X} is called d -self-closed if for any \mathcal{R} -preserving sequence $\{u_n\}$ such that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.

Inspired by the above definition by Alam and Imdad [1] we define analogue of d -self-closedness in Menger PM -space.

Definition 2.11. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space and \mathcal{R} be a binary relation on a non-empty set \mathcal{X} . Then \mathcal{R} is $d_{\mathcal{F}}$ -self-closed if for any $\{u_n\} \subset \mathcal{X}$ is a sequence in $\mathcal{T}_N(f, \mathcal{R}, u_0)$, so that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$.

Definition 2.12 ([1]). Let \mathcal{X} be a non-empty set and \mathcal{R} a binary relation on \mathcal{X} . A subset E of \mathcal{X} is \mathcal{R} -connected if for each pair $u, v \in E$, there exists a path in \mathcal{R} from u to v .

Definition 2.13 ([16]). Let \mathcal{X} be a non-empty set and \mathcal{R} be a binary relation on \mathcal{X} . Then a subset E of \mathcal{X} is \mathcal{R} -directed if for each pair $u, v \in E$, there exists $w \in \mathcal{X}$ so that $(u, w) \in \mathcal{R}$ and $(v, w) \in \mathcal{R}$.

Definition 2.14 ([16]). Let \mathcal{X} be a non-empty set and \mathcal{R} be a binary relation on \mathcal{X} . For $u, v \in \mathcal{X}$, a path of length k ($k \in \mathbb{N}$), in \mathcal{R} from u to v is a finite sequence $\{w_0, w_1, w_2, \dots, w_k\} \subset \mathcal{X}$ satisfying the subsequent assumptions:

- (a) $w_0 = u$ and $w_k = v$,
- (b) $(w_i, w_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k - 1$).

Noticeably, a path of length k has $k + 1$ elements of \mathcal{X} , though they are not necessarily distinct. For some $u \in \mathcal{X}$, denote by $\mathcal{P}(u; k)$ the set of all $v \in \mathcal{X}$ such that there exists a path of length k from u to v , that is,

$$\mathcal{P}(u; k) = \{v \in \mathcal{X} : \text{there exists a path of length } k \text{ from } u \text{ to } v.\}$$

Now, let us consider nonempty set and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a mapping. Then a sequence $\{u_n\} \subset \mathcal{X}$ is called a trajectory of the mapping f , starting at u_1 , if $u_{n+1} \in fu_n$ for all $n \in \mathbb{N}$.

Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space. Let \mathcal{R} be a nonempty binary relation defined on \mathcal{X} . Then $\{u^i\}_{i=0}^n$ is an n -directed path from $u \in \mathcal{X}$ to $v \in \mathcal{X}$ if $u^0 = u, \quad u^n = v, \quad (u^{i-1}, u^i) \in \mathcal{R} \quad \text{and} \quad \mathcal{D}(u^{i-1}, u^i) < \infty \quad \text{for all } i = 1, 2, \dots, n.$

For $u \in \mathcal{X}$ and $n \in \mathbb{N}$, we denote

$$\mathcal{P}(u, n, \mathcal{R}) = \{v \in \mathcal{X} : \text{there is an } n\text{-directed path from } u \text{ to } v\}$$

and

$$\mathcal{P}(u, \mathcal{R}) = \bigcup \mathcal{P}(u, n, \mathcal{R}).$$

Let $u \in \mathcal{X}, n \in \mathbb{N}, v \in \mathcal{P}(u, n, \mathcal{R})$ and $w \in \mathcal{P}(u, \mathcal{R})$. We define

$$\rho_n(u, v) = \inf \left\{ \sum_{i=1}^n \mathcal{D}(u^{i-1}, u^i) : (u^i)_{i=0}^n \text{ is an } n\text{-directed path from } u \text{ to } v \right\}$$

and

$$\rho(u, w) = \inf \left\{ \sum_{i=1}^n \mathcal{D}(u^{i-1}, u^i) : (u^i)_{i=0}^n \text{ is an } n\text{-directed path from } u \text{ to } w \right. \\ \left. \text{for some } n \in \mathbb{N} \right\}.$$

We prove the following lemma.

Lemma 2.15. *Let $u \in \mathcal{X}, n \in \mathbb{N}, v \in \mathcal{P}(u, n, \mathcal{R})$ and $w \in \mathcal{P}(u, \mathcal{R})$. Then*

- (a) $\rho_n(u, v) \geq \rho_{n+m}(u, v)$ for every $m, n \in \mathbb{N}$,
- (b) $\rho(u, w) = \inf \{ \rho_k(u, w) : k \in \mathbb{N}, w \in \mathcal{P}(u, k, \mathcal{R}) \}$.

Proof. Let $m, n \in \mathbb{N}$. Let $(u^i)_{i=0}^n$ be an n -directed path from u to v , that is, $u^0 = u, \quad u^n = v, \quad (u^{i-1}, u^i) \in \mathcal{R} \quad \text{and} \quad \mathcal{D}(u^{i-1}, u^i) < \infty \quad \text{for all } i = 1, 2, \dots, n.$

Let

$$u^{n+i} = v \quad \text{for all } i = 1, 2, \dots, m.$$

As $\mathcal{D}(v, v) = 0$, then $(u^i)_{i=0}^{n+m}$ is an $n + m$ -directed path from u to v . Further, we have

$$\sum_{i=1}^n \mathcal{D}(u^{i-1}, u^i) = \sum_{i=1}^{n+m} \mathcal{D}(u^{i-1}, u^i) \geq \rho_{n+m}(u, v).$$

Thus the proof of (a) accomplished.

Now, let $k \in \mathbb{N}$ such that $w \in \mathcal{P}(u, k, \mathcal{R})$. For every $(u^i)_{i=0}^k$, a k -directed path from u to w , we have

$$\sum_{i=1}^k \mathcal{D}(u^{i-1}, u^i) \geq \rho(u, w). \quad \text{Then } \rho_k(u, w) \geq \rho(u, w).$$

This implies that

$$\rho(u, w) \leq \inf \{ \rho_k(u, w) : k \in \mathbb{N}, w \in \mathcal{P}(u, k, \mathcal{R}) \}.$$

Finally, let $n \in \mathbb{N}$ and $(u^i)_{i=1}^n$ be an n -directed path from u to w . We obtain

$$\sum_{i=1}^k \mathcal{D}(u^{i-1}, u^i) \geq \rho_N(u, w) \geq \inf\{\rho_k(u, w) : k \in \mathbb{N}, z \in \mathcal{P}(u, k, \mathcal{R})\}.$$

Then we deduce that

$$\rho(u, w) \geq \inf\{\rho_k(u, w) : k \in \mathbb{N}, w \in \mathcal{P}(u, k, \mathcal{R})\},$$

which yields (b). □

3. FIXED POINTS FOR SET-VALUED $f_{\mathcal{R}}$ -CONTRACTIONS

In this section, we establish fixed point results for a set-valued $f_{\mathcal{R}}$ -contraction in a Menger PM -spaces.

Definition 3.1. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued mapping with nonempty values. Then f is a set-valued $f_{\mathcal{R}}$ -contraction if there exists $\kappa \in (0, 1)$ such that

$$(u, v) \in \mathcal{R}, p \in fu \implies \text{there exists } q \in fv: (p, q) \in \mathcal{R}, \mathcal{F}(p, q)(\kappa t) \geq \mathcal{F}(u, v)(t) \text{ for all } t > 0.$$

Lemma 3.2. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued $f_{\mathcal{R}}$ -contraction with respect to some $\kappa \in (0, 1)$. We have

$$(u, v) \in \mathcal{R}, p \in fu \implies \text{there exists } q \in fv: (p, q) \in \mathcal{R}, d_{\lambda}(p, q) \leq \kappa d_{\lambda}(u, v), \text{ for all } \lambda \in (0, 1].$$

Proof. Let $(u, v) \in \mathcal{R}$ and $p \in fu$. Since f is a set-valued $f_{\mathcal{R}}$ -contraction with respect to $\kappa \in (0, 1)$, there exists $q \in fv$ such that

$$\mathcal{F}(p, q)(\kappa t) \geq \mathcal{F}(u, v)(t) \text{ for all } t > 0.$$

Let $\lambda \in (0, 1]$ be fixed. Let $s > 0$ be such that $\mathcal{F}(u, v)(s) > 1 - \lambda$. Then we have

$$\mathcal{F}(p, q)(\kappa s) > 1 - \lambda,$$

which implies that

$$s \geq \kappa^{-1} d_{\lambda}(p, q).$$

By the definition of the inf, we obtain the required result. □

Lemma 3.3. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM -space with \mathcal{T} a t -norm of \mathcal{H} -type and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued $f_{\mathcal{R}}$ -contraction. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Then, for every $u \in \mathcal{X}$ and $v \in \mathcal{P}(u, n, \mathcal{R})$, one has

$$\text{for all } u_1 \in fu, \text{ there exists } v_1 \in fv \cap \mathcal{P}(u_1, n, \mathcal{R}): \rho_n(u_1, v_1) \leq \kappa(\rho_n(u, v) + \varepsilon) \tag{3.1}$$

and inductively, for all $k = 1, 2, \dots$,

$$\text{for all } u_{k+1} \in fu_k, \text{ there exists } v_{k+1} \in fv_k \cap \mathcal{P}(u_{k+1}, n, \mathcal{R}):$$

$$\rho_n(u_{k+1}, v_{k+1}) \leq \kappa^{k+1}(\rho_n(u, v) + \varepsilon). \tag{3.2}$$

Proof. Let $\varepsilon > 0$ and $(u^i)_{i=0}^n$ be an n -directed path from $u \in \mathcal{X}$ to $v \in \mathcal{P}(u, N, \mathcal{R})$ such that

$$\sum_{i=1}^n \mathcal{D}(u^{i-1}, u^i) < \rho_n(u, v) + \varepsilon.$$

Let $u_1 \in fu$. Since f is a set-valued $f_{\mathcal{R}}$ -contraction, by Lemma 3.2 there exists $u_1^1 \in fu^1$ such that

$$(u_1, u_1^1) \in \mathcal{R} \quad \text{and} \quad \mathcal{D}(u_1, u_1^1) \leq \kappa \mathcal{D}(u, u^1) < \infty.$$

Again, there exists $u_1^2 \in fu^2$ such that

$$(u_1^1, u_1^2) \in \mathcal{R} \quad \text{and} \quad \mathcal{D}(u_1^1, u_1^2) \leq \kappa \mathcal{D}(u^1, u^2) < \infty.$$

Recursively, for $i = 3, 4, \dots, n$, there exists $u_1^i \in fu^i$ such that

$$(u_1^{i-1}, u_1^i) \in \mathcal{R} \quad \text{and} \quad \mathcal{D}(u_1^{i-1}, u_1^i) \leq \kappa \mathcal{D}(u^{i-1}, u^i) < \infty.$$

Now, if we take $v_1 = u_1^n$, we have $v_1 \in fv \cap \mathcal{P}(u_1, n, \mathcal{R})$ and

$$\begin{aligned} \rho_n(u_1, v_1) &\leq \sum_{i=1}^n \mathcal{D}(u_1^{i-1}, u_1^i) \\ &\leq \kappa \sum_{i=1}^n \mathcal{D}(u^{i-1}, u^i) \\ &\leq \kappa(\rho_n(u, v) + \varepsilon). \end{aligned}$$

Thus (3.1) is proved. Again following the above lines of proof if $u_2 \in fu_1$, there exists $v_2 \in fv_1 \cap \mathcal{P}(u_2, n, \mathcal{R})$ so that

$$\rho_n(u_2, v_2) \leq \kappa \sum_{i=1}^n \mathcal{D}(u_1^{i-1}, u_1^i) \leq \kappa^2(\rho_n(u, v) + \varepsilon).$$

Continuing this process, by induction, we obtain (3.2). \square

The following concepts are adaptations of those introduced in [7] to the case of Menger PM-spaces.

Definition 3.4. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM-space. Let $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued mapping with nonempty values.

(a) Let $n \in \mathbb{N}$. Then a sequence $\{u_n\} \subset \mathcal{X}$ is a \mathcal{R}_n -Picard trajectory from u_0 if $u_n \in \mathcal{P}(u_{n-1}, n, \mathcal{R}) \cap fu_{n-1}$ for all $n \in \mathbb{N}$. We symbolize by $\mathcal{T}_n(f, \mathcal{R}, u_0)$ the set of all such \mathcal{R}_n -Picard trajectories from u_0 .

(b) A sequence $\{u_n\} \subset \mathcal{X}$ is a \mathcal{R} -Picard trajectory from u_0 if $u_n \in \mathcal{P}(u_{n-1}, \mathcal{R}) \cap fu_{n-1}$ for all $n \geq 1$. We symbolize by $\mathcal{T}(f, \mathcal{R}, u_0)$ the set of all such \mathcal{R} -Picard trajectories from u_0 .

Definition 3.5. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM-space. Let $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued mapping with nonempty values.

(a) Let $n \in \mathbb{N}$. Then f is \mathcal{R}_n -Picard continuous from $u_0 \in \mathcal{X}$ if the limit of any convergent sequence $\{u_n\} \in \mathcal{T}_n(f, \mathcal{R}, u_0)$ is a fixed point of \mathcal{T} .

(b) f is \mathcal{R} -Picard continuous from $u_0 \in \mathcal{X}$ if the limit of any convergent sequence $\{u_n\} \in \mathcal{T}(f, \mathcal{R}, u_0)$ is a fixed point of \mathcal{T} .

Now, we present the first main result.

Theorem 3.6. *Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a complete Menger PM-space with \mathcal{T} a t -norm of \mathcal{H} -type and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued $f_{\mathcal{R}}$ -contraction. Suppose that for some $n \in \mathbb{N}$, we have*

- (a) *there exists $u_0 \in \mathcal{X}$ such that $\mathcal{P}(u_0, n, \mathcal{R}) \cap fu_0 \neq \emptyset$;*
 - (b) *either f is \mathcal{R}_n -Picard continuous from u_0 or \mathcal{R} is $d_{\mathcal{F}}$ -self-closed.*
- Then there exists a sequence $\{u_n\} \in \mathcal{T}_n(f, \mathcal{R}, u_0)$ converging to $u^* \in \mathcal{X}$, a fixed point of f .*

Proof. Since $\mathcal{P}(u_0, n, \mathcal{R}) \cap fu_0 \neq \emptyset$, choose $u_1 \in \mathcal{P}(u_0, n, \mathcal{R}) \cap fu_0$. Let $\varepsilon > 0$ by Lemma 2.5 (d) and Lemma 3.3, there exists $u_2 \in fu_1 \cap \mathcal{P}(u_1, n, \mathcal{R})$ such that

$$\mathcal{D}(u_1, u_2) \leq \rho_n(u_1, u_2) \leq \kappa(\rho_n(u_0, u_1) + \varepsilon).$$

Again, since $u_2 \in \mathcal{P}(u_1, n, \mathcal{R}) \cap fu_1$, there exists $u_3 \in \mathcal{P}(u_2, n, \mathcal{R}) \cap fu_2$ such that

$$\mathcal{D}(u_2, u_3) \leq \rho_n(u_2, u_3) \leq \kappa^2(\rho_n(u_0, u_1) + \varepsilon).$$

More generally, for $n \geq 2$, there exists $u_{n+1} \in \mathcal{P}(u_n, n, \mathcal{R}) \cap fu_n$ such that

$$\mathcal{D}(u_n, u_{n+1}) \leq \rho_n(u_n, u_{n+1}) \leq \kappa^n(\rho_n(u_0, u_1) + \varepsilon).$$

Then $\{u_n\} \in \mathcal{T}_n(f, \mathcal{R}, u_0)$ and for $m \geq 1$,

$$\begin{aligned} \mathcal{D}(u_n, u_{n+m}) &\leq \sum_{i=n}^{n+m-1} \mathcal{D}(u_i, u_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} \kappa^i (\rho_n(u_0, u_1) + \varepsilon) \\ &\leq \frac{\kappa^n}{1 - \kappa} (\rho_n(u_0, u_1) + \varepsilon). \end{aligned}$$

Now we prove $\{u_n\}$ is a Cauchy sequence in the Menger PM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$. Let $t > 0$ and $\delta \in (0, 1]$. From the above inequality, as $\kappa \in (0, 1)$, there exists some $p \in \mathbb{N}$ such that

$$d_{\delta}(u_n, u_{n+m}) \leq \mathcal{D}(u_n, u_{n+m}) < t \quad \text{for all } n, m \geq p.$$

Using Lemma 2.5 (a), we have

$$\mathcal{F}(u_n, u_{n+m})(t) > 1 - \delta \quad \text{for all } n, m \geq p,$$

which implies that $\{u_n\}$ is a Cauchy sequence. Since $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is complete and f is \mathcal{R}_n -Picard continuous from u_0 , there exists some $u^* \in \mathcal{X}$ such that $\{u_n\}$ converges to u^* , a fixed point of f .

Alternately, suppose that \mathcal{R} is $d_{\mathcal{F}}$ -self-closed. Then there exists $\{u_n\} \in \mathcal{T}_n(f, \mathcal{R}, u_0)$ that converges to some $u^* \in \mathcal{X}$. At first, observe that $\mathcal{D}(u_n, u^*) \rightarrow$

0, as $n \rightarrow \infty$. In fact, since $\{u_n\}$ converges to u^* with respect to the Menger PM -space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ for $\varepsilon > 0$ and $\delta \in (0, 1]$, there exists some $p \in \mathbb{N}$ such that $\mathcal{F}(u_n, u^*)(\varepsilon) > 1 - \delta$ for every $n \geq p$. By Lemma 2.5(a), we have $d_\delta(u_n, u^*) < \varepsilon$ for every $n \geq p$. Then we have $\mathcal{D}(u_n, u^*) \leq \varepsilon$ for every $n \geq p$. This proves that $\mathcal{D}(u_n, u^*) \rightarrow 0$, as $n \rightarrow \infty$. Now, since f is a set-valued $f_{\mathcal{R}}$ -contraction, by Lemma 3.2, for every $k \in \mathbb{N}$, there exists $v_{n_{k+1}} \in fu^*$ such that $(u_{n_{k+1}}, v_{n_{k+1}}) \in \mathcal{R}$ and $\mathcal{D}(u_{n_{k+1}}, v_{n_{k+1}}) \leq \kappa \mathcal{D}(u_{n_k}, u^*)$. By using the triangular inequality and the above expression, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{D}(v_{n_{k+1}}, u^*) &\leq \mathcal{D}(v_{n_{k+1}}, u_{n_{k+1}}) + \mathcal{D}(u_{n_{k+1}}, u^*) \\ &\leq \kappa \mathcal{D}(u_{n_k}, u^*) + \mathcal{D}(u_{n_{k+1}}, u^*). \end{aligned}$$

Letting $k \rightarrow \infty$ in the above inequality, we have $\mathcal{D}(v_{n_{k+1}}, u^*) \rightarrow 0$, as $k \rightarrow \infty$. Following the above lines of proof in the light of Lemma 2.5 (a), we have $\{v_{n_{k+1}}\}$ converges to u^* with respect to the Menger PM -space. Since fu^* is closed, we have $u^* \in fu^*$, that is, u^* is a fixed point of f . This completes the proof. \square

Taking $n = 1$ and $\mathcal{R} = \mathcal{X} \times \mathcal{X}$ in Theorem 3.6, without assuming \mathcal{R}_N -Picard continuity we have the following Nadler fixed point theorem in Menger PM -spaces.

Corollary 3.7. *Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a complete Menger PM -space with \mathcal{T} a t -norm of \mathcal{H} -type and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued mapping with nonempty closed values. Suppose that*

- (a) *there exists $(u_0, u_1) \in \mathcal{X} \times \mathcal{X}$ such that $u_1 \in fu_0$ and $\mathcal{D}(u_0, u_1) < \infty$;*
- (b) *there exists $\kappa \in (0, 1)$ such that*

$$(u, v) \in \mathcal{X} \times \mathcal{X}, p \in fu \implies \text{there exists } q \in fv: \mathcal{F}(p, q)(\kappa t) \geq \mathcal{F}(u, v)(t), t > 0.$$

Then f has a fixed point.

Corollary 3.8. *Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a complete Menger PM -space with \mathcal{T} a t -norm of \mathcal{H} -type and $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued $f_{\mathcal{R}}$ -contraction. Suppose that*

- (a) *there exists $u_0 \in \mathcal{X}$ such that $\mathcal{P}(u_0, \mathcal{R}) \cap fu_0 \neq \emptyset$;*
- (b) *either f is \mathcal{R} -Picard continuous from u_0 or \mathcal{R} is $d_{\mathcal{F}}$ -self-closed.*

Then there exist $n \in \mathbb{N}$ and a sequence $\{u_n\} \in \mathcal{T}_n(f, \mathcal{R}, u_0)$ converging to $u^ \in \mathcal{X}$, a fixed point of f .*

Proof. From (a), there exists some $n \in \mathbb{N}$ such that $\mathcal{P}(u_0, n, \mathcal{R}) \cap fu_0 \neq \emptyset$. Since from (b) f is \mathcal{R} -Picard continuous from u_0 , then it is \mathcal{R}_n -Picard continuous from u_0 . Now, the result follows from Theorem 3.6. \square

Let us consider now the case of a single-valued mapping. From Definition 3.1, a single-valued mapping $f : \mathcal{X} \rightarrow \mathcal{X}$ is a $f_{\mathcal{R}}$ -contraction if there exists $\kappa \in (0, 1)$ such that

$$(u, v) \in \mathcal{R} \implies (fu, fv) \in \mathcal{R}, \mathcal{F}(fu, fv)(\kappa t) \geq \mathcal{F}(u, v)(t) \text{ for all } t > 0.$$

In this case, for a given $n \in \mathbb{N}$ and a given $u_0 \in \mathcal{X}$, the set of \mathcal{R}_n -Picard trajectories from u_0 is given by

$$\mathcal{T}_n(f, \mathcal{R}, u_0) = \{ \{u_n\}_{n \in \mathbb{N}} : u_n = f^n u_0 \}.$$

From Theorem 3.6, we obtain the following result concerning single-valued mappings.

Corollary 3.9. *Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a complete Menger PM-space with \mathcal{T} a t-norm of \mathcal{H} -type and $f : \mathcal{X} \rightarrow \mathcal{X}$ be a single-valued \mathcal{R} -contraction. Suppose that there exists $u_0 \in \mathcal{X}$ such that $f u_0 \in \mathcal{P}(u_0, \mathcal{R})$. Suppose that one of the following conditions is satisfied:*

- (a) *if $\{f^n u_0\}$ converges to some $u \in \mathcal{X}$, then $u = f u$;*
- (b) *if $\{f^n u_0\}$ converges to some $u \in \mathcal{X}$, then there exists a subsequence $\{f^{n_k} u_0\}$ of $\{f^n u_0\}$ such that $(f^{n_k} u_0, u) \in \mathcal{R}$ for every $k \in \mathbb{N}$.*

Then f has a fixed point.

Example 3.10. Let $\mathcal{X} = \mathbb{R}^+$ with the metric $d(u, v) = |u - v|$. Suppose that

$$\mathcal{F}(u, v)(t) = \frac{t}{t + d(u, v)},$$

then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete Menger PM-space with \mathcal{T} a t-norm of \mathcal{H} -type. Let the binary relation $\mathcal{R} = \{(u, v) : u, v \in [0, 1]\}$. Define $f : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ by

$$f u = \begin{cases} 2u - \frac{5}{3}, & \text{if } u > 1, \\ [0, \frac{u}{2}], & \text{if } 0 \leq u \leq 1. \end{cases}$$

Now, we prove that f is $f_{\mathcal{R}}$ -contraction, that is, for $(u, v) \in \mathcal{R}$, and $p \in f u$, there exists $q \in f v$, such that

$$\begin{aligned} \mathcal{F}(p, q)(\kappa t) &= \frac{\kappa t}{\kappa t + |\frac{u}{2} - \frac{v}{2}|} = \frac{t}{t + \frac{1}{2\kappa}|u - v|} \\ &\geq \frac{t}{t + |u - v|} = \mathcal{F}(u, v)(t), \end{aligned}$$

for all $t > 0$ and holds for all $\kappa > \frac{1}{2}$. This implies that f is set-valued $f_{\mathcal{R}}$ -contraction. Also if we take $\{u_n\}$ such that $u_n \leq 1$ for all $n \in \mathbb{N}_0$ then $u_n \in \mathcal{T}_N(f, \mathcal{R}, u_0)$, such that $u_n \rightarrow u$. Therefore by the definition of the \mathcal{R} , we have $u_n \in [0, 1]$ for all $n \in \mathbb{N}_0$ and there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. This implies that \mathcal{X} satisfies the assumption (b) of the Theorem 3.6. Also it is easy to check that $\mathcal{P}(u_0, n, \mathcal{R}) \cap f u_0 \neq \emptyset$; by taking $u_0 = 1 \in \mathcal{X}$. Therefore, Theorem 3.6 with $\kappa > \frac{1}{2}$ ensures the existence of a fixed point. Moreover, f has infinitely many fixed points.

Notice that, if we take $u = 1, v = \frac{5}{2}$ then $f u = [0, \frac{1}{2}]$, $f v = \frac{10}{3}$, we have

$$\mathcal{F}(p, q)(\kappa t) = \frac{\kappa t}{\kappa t + \frac{10}{3}} = \frac{t}{t + \frac{10}{3\kappa}} \geq \frac{t}{t + \frac{1}{2}}$$

implies that $\kappa \geq \frac{20}{3}$, this implies that there is no $\kappa < 1$ such that

$$\mathcal{F}(p, q)(\kappa t) \geq \mathcal{F}(u, v)(t) \text{ for all } t > 0 \text{ with } p \in f u \text{ and } q \in f v.$$

This implies that f does not satisfy the contraction assumption. However if $(u, v) \in \mathcal{R}$ then the assumption is satisfied for all $(u, v) \in \mathcal{R}$. This shows that the corresponding Theorem 1 and Corollary 1 of [8] is not applicable here which indicate the usability of such generalizations over the corresponding several prominent recent fixed point results on this settings.

4. APPLICATIONS: KELISKY-RIVLIN TYPE RESULT FOR BERNSTEIN OPERATORS

As an applications, we establish in this section a Kelisky-Rivlin type result for a class of Bernstein type of special operators introduced by Deo et. al. [6]. Firstly, we prove the result associated with the convergence of successive approximations for a family of operators.

Theorem 4.1. *Let E be a group with respect to a operation $+$. Let \mathcal{X} be a subset of E endowed with a metric d such that (\mathcal{X}, d) is complete. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a closed subset of \mathcal{X} such that \mathcal{X}_0 is a subgroup of E . Let $f : \mathcal{X} \rightarrow \mathcal{X}$ be a single-valued mapping such that*

$$(u, v) \in \mathcal{X} \times \mathcal{X}, u - v \in \mathcal{X}_0 \implies d(fu, fv) \leq \kappa d(u, v),$$

where $\kappa \in (0, 1)$ is a constant. Suppose that

$$u - fu \in \mathcal{X}_0 \quad \text{for all } u \in \mathcal{X}. \quad (4.1)$$

Then we have

(a) for every $u \in \mathcal{X}$, the Picard sequence $\{f^n u\}$ converges to a fixed point of f ,
 (b) for every $u \in \mathcal{X}$, $(u + \mathcal{X}_0) \cap \text{Fix } f = \{\lim_{n \rightarrow \infty} f^n u\}$, where $\text{Fix } f$ denotes the set of fixed points of f .

Proof. Let us consider $\mathcal{F} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{D}^+$ defined by

$$\mathcal{F}(u, v)(t) = \delta_0(t - d(u, v)) \quad \text{for all } u, v \in \mathcal{X}, t > 0,$$

where δ_0 is the Dirac distribution function. Consider the arbitrary binary relation $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ such that

$$\mathcal{R} = \{(u, v) \in \mathcal{X} \times \mathcal{X} : u - v \in \mathcal{X}_0\}.$$

Observe that by (5.1), we have

$$\begin{aligned} (u, v) \in \mathcal{R} &\implies u - v \in \mathcal{X}_0 \implies fu - fv = (fu - u) + (u - v) + (u - v) \in \mathcal{X}_0 \\ &\implies (fu, fv) \in \mathcal{R}. \end{aligned}$$

Then by the definition of δ_0 , we have

$$(u, v) \in \mathcal{R} \implies (fu, fv) \in \mathcal{R}, \mathcal{F}(fu, fv)(\kappa t) \geq \mathcal{F}(u, v)(t), \forall t > 0,$$

which implies that f is a single-valued $f_{\mathcal{R}}$ -contraction. Also a sequence $\{u_n\} \subset \mathcal{X}$ converges to $u \in \mathcal{X}$ with respect to d if and only if $\{u_n\}$ converges to u with respect to the Menger PM -space. Let $u_0 \in \mathcal{X}$ be an arbitrary point. By (4.1), we have $u_0 - fu_0 \in \mathcal{X}_0$, that is, $(u_0, fu_0) \in \mathcal{R}$, which implies that $fu_0 \in \mathcal{P}(u_0, \mathcal{R})$.

Now suppose that $\{f^n u_0\}$ converges to $u \in \mathcal{X}$ with respect to $(\mathcal{X}, \mathcal{F}, \mathcal{T}_M)$, that

is, Menger *PM*-space. Then $\{f^n u_0\}$ converges to u with respect to the metric d .

On the other hand, we have $f u_0 = (f u_0 - u_0) + u_0 \in \mathcal{X}_0$. Again, we have $f^2 u_0 = (f^2 u_0 - f u_0) + f u_0 \in \mathcal{X}_0$. Continuing in this process, we have $f^n u_0 \in \mathcal{X}_0$ for every $n \in \mathbb{N}$. As \mathcal{X}_0 is closed, then $u \in \mathcal{X}_0$. As a result, we have $(f^n u_0, u) \in \mathcal{R}$ for every $n \geq 1$. Finally, in the light of Theorem 3.6, the proof of (a) accomplished.

Now, to prove (b) let $u \in \mathcal{X}$ be any arbitrary point. From (a), we know that $\{f^n u\}$ converges with respect to the metric d to some $u^* \in \mathcal{X}_0$, a fixed point of f . Moreover, from the proof of (a), we have $f^n u_0 - u \in \mathcal{X}_0$ for all $n \in \mathbb{N}$. Since \mathcal{X}_0 is closed, we have $u^* - u \in \mathcal{X}_0$, that is, $u^* \in u + \mathcal{X}_0$. On the other hand, suppose that $u_1, u_2 \in (u + \mathcal{X}_0) \cap \text{Fix } f$, with $u_1 \neq u_2$. Since $u_1 - u, u_2 - u \in \mathcal{X}_0$, then $d(u_1, u_1) = d(f u_1, f u_2) \leq \kappa d(u_1, u_2)$, which is a contradiction. This completes the proof of (b). \square

Remark 4.2. Theorem 4.1 recovers Theorem 4.1 in [13], where \mathcal{X} was supposed to be a Banach space and f was supposed to be a linear operator.

The Bernstein operator on $f \in C([0, 1])$, the space of all continuous real functions on the interval $[0, 1]$, is defined by

$$(B_n f)(u) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} u^k (1-u)^{n-k}, \quad f \in C([0, 1]), u \in [0, 1], n = 1, 2, \dots$$

Kelisky and Rivlin [15] proved that each Bernstein operator B_n is a weak operator. Moreover, for any n and $f \in C([0, 1])$,

$$\lim_{j \rightarrow \infty} (B_n^j f)(u) = f(0) + (f(1) - f(0))u, \quad u \in [0, 1].$$

The proof given by Kelisky and Rivlin is based on linear algebra involving the Stirling numbers of the second kind, and eigenvalues and eigenvectors of some matrices. Jachymski [13] (see Theorem 4.1) presented a simple and elegant proof utilizing a fixed point theorem for linear operators on a Banach space.

Now, we are interested in establishing Kelisky and Rivlin type results for a class of Bernstein type of special operators introduced by Deo et. al. [6] as follows.

If $f(u)$ is a function defined on $[0, \frac{n}{n+1}]$

$$(V_n f)(u) = \sum_{k=0}^n p_{n,k}(u) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(u) = \left(1 + \frac{1}{n}\right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u\right)^{n-k}, \quad \text{for } \frac{n}{n+1} \geq u.$$

Let

$$\mathcal{X} = \left\{ f \in C\left(\left[0, \frac{n}{n+1}\right]\right) : f(0) \geq 0, f\left(\frac{n}{n+1}\right) \geq 0 \right\}.$$

Clearly, $V_n(\cdot) : \mathcal{X} \rightarrow \mathcal{X}$ is well defined.

We have the following result.

Theorem 4.3. *Let $n \in \mathbb{N}$. Then, for every $f \in \mathcal{X}$, the Picard sequence $\{V_n^j(f)\}_{j \in \mathbb{N}}$ converges to a fixed point of $V_n(\cdot)$. Moreover, for every $f \in \mathcal{X}$, we have*

$$\lim_{j \rightarrow \infty} \max_{u \in [0, \frac{n}{n+1}]} |V_n^j(f)(u) - \omega(u)| = 0,$$

where $\omega(u) = f(0)\left(\frac{n}{n+1} - u\right) + f\left(\frac{n}{n+1}\right)u$, $u \in [0, \frac{n}{n+1}]$.

Proof. Let $\mathbb{E} = C\left(\left[0, \frac{n}{n+1}\right]\right)$. We endow \mathcal{X} with the metric defined by

$$d(U, V) = \max_{u \in [0, \frac{n}{n+1}]} |U(u) - V(u)|, \quad U, V \in \mathcal{X}.$$

Clearly, (\mathcal{X}, d) is a complete metric space. Let

$$\mathcal{X}_0 = \left\{ U \in \mathbb{E} : U(0) = U\left(\frac{n}{n+1}\right) = 0 \right\}.$$

Then $\mathcal{X}_0 \subset \mathcal{X}$ is a closed subgroup of \mathbb{E} . Let $f, g \in \mathcal{X}$ such that $f - g \in \mathcal{X}_0$. Let $u \in [0, \frac{n}{n+1}]$, then we have

$$\begin{aligned} |V_n(f)(u) - V_n(g)(u)| &= \left| \sum_{k=0}^n \left(f\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right) \right) \left(1 + \frac{1}{n}\right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u\right)^{n-k} \right| \\ &\leq \sum_{k=0}^n \left| (f - g)\left(\frac{k}{n}\right) \right| \left(1 + \frac{1}{n}\right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u\right)^{n-k} \\ &\leq \sum_{k=1}^{n-1} \left(1 + \frac{1}{n}\right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u\right)^{n-k} d(f, g). \end{aligned}$$

Note that

$$\sum_{k=0}^n \left(1 + \frac{1}{n}\right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u\right)^{n-k} = 1.$$

Then it is easy to observe that

$$\begin{aligned} \sum_{k=1}^{n-1} \left(1 + \frac{1}{n}\right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u\right)^{n-k} &\leq 1 - \left(1 + \frac{1}{n}\right)^n u^n - \left(1 + \frac{1}{n}\right)^n \left(\frac{n}{n+1} - u\right)^n \\ &\leq 1 - \frac{1}{2^{n-1}}. \end{aligned}$$

As a consequence, we have

$$f, g \in \mathcal{X}, f - g \in \mathcal{X}_0 \implies d(V_n(f), V_n(g)) \leq \left(1 - \frac{1}{2^{n-1}}\right) d(f, g).$$

Now, let $f \in \mathcal{X}$. For any $u \in [0, \frac{n}{n+1}]$, we have

$$f(u) - V_n(f) = \sum_{k=0}^n \left(f(u) - f\left(\frac{k}{n}\right) \right) \left(1 + \frac{1}{n} \right)^n \binom{n}{k} u^k \left(\frac{n}{n+1} - u \right)^{n-k}.$$

Observe that $f(0) - V_n(f)(0) = f(\frac{n}{n+1}) - V_n(f)(\frac{n}{n+1}) = 0$. Then, for every $f \in \mathcal{X}$, we have

$$f - V_n(f) \in \mathcal{X}_0.$$

By Theorem 4.1, we deduce that for every $f \in \mathcal{X}$, the Picard sequence $\{V_n^j(f)\}_{j \in \mathbb{N}}$ converges to a fixed point of $V_n(\cdot)$ and

$$(f + \mathcal{X}_0) \cap \text{Fix } V_n(\cdot) = \left\{ \lim_{j \rightarrow \infty} V_n^j(f) \right\}.$$

Let $f \in \mathcal{X}$. It is not difficult to observe that $\omega(u) = f(0)(\frac{n}{n+1} - u) + f(\frac{n}{n+1})u \in \text{Fix } V_n(\cdot)$. We have also

$$\omega(u) = f(u) + \theta(u),$$

where

$$\theta(u) = f(0)\left(\frac{n}{n+1} - u\right) + f\left(\frac{n}{n+1}\right)u - f(u).$$

Observe that $\theta(0) = \theta(\frac{n}{n+1}) = 0$, which implies that $\theta \in \mathcal{X}_0$. This completes the proof of Theorem 4.3. \square

ACKNOWLEDGEMENTS. *The referees have reviewed the paper very carefully. The authors express their deep thanks for the comments.*

REFERENCES

- [1] A. Alam and M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.* 17, no. 4 (2015), 693–702.
- [2] A. Alam, R. George and M. Imdad, Refinements to relation-theoretic contraction principle, *Axioms* 11, no. 7 (2022), 316.
- [3] H. Argoubi, M. Jleli and B. Samet, The study of fixed points for multivalued mappings in a Menger probabilistic metric space endowed with a graph, *Fixed Point Theory Appl.* 2015 (2015), 113.
- [4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* 3 (1922), 133–181.
- [5] S. K. Bhandari, D. Gopal and P. Konar, Probabilistic α -min Ciric type contraction results using a control function, *AIMS Mathematics* 5, no. 2 (2020), 1186–1198.
- [6] N. Deo, M. A. Noor and M. A. Siddiqui, On approximation by a class of new Bernstein type operators, *Appl. Math. Comput.* 201 (2008), 604–612.
- [7] T. Dinevari and M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, *J. Math. Anal. Appl.* 405 (2013), 507–517.
- [8] J.-X. Fang, Fixed point theorems of local contraction mappings on Menger spaces, *Appl. Math. Mech.* 12 (1991), 363–372.

- [9] J.-X. Fang, A note on fixed point theorems of Hadžić, *Fuzzy Sets Syst.* 48 (1992), 391–395.
- [10] J.-X. Fang, Common fixed point theorems of compatible and weakly compatible maps in Menger spaces, *Nonlinear Anal.* 71 (2009), 1833–1843.
- [11] O. Hadžić, Fixed point theorems for multivalued mappings in probabilistic metric spaces, *Fuzzy Sets Syst.* 88 (1997), 219–226.
- [12] O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic, Dordrecht (2001).
- [13] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.* 136 (2008), 1359–1373.
- [14] T. Kamran, M. Samreen and N. Shahzad, Probabilistic G -contractions, *Fixed Point Theory Appl.* 2013 (2013), 223.
- [15] R. P. Kelisky and T. J. Rivlin, Iterates of Bernstein polynomials, *Pac. J. Math.* 21 (1967), 511–520.
- [16] B. Kolman, R. C. Busby and S. Ross, *Discrete mathematical structures*, Third Edition, PHI Pvt. Ltd., New Delhi, 2000.
- [17] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, (1964).
- [18] J. J. Nieto and R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order* 22 (2005), 223–239 .
- [19] S. B. Jr. Nadler, Multivalued contraction mappings. *Pac. J. Math.* 30 (1969), 475–487.
- [20] G. Prasad, Coincidence points of relational Ψ -contractions and an application, *Afrika Mathematica* 32, no. 6-7 (2021), 1475–1490.
- [21] G. Prasad, Fixed points of Kannan contractive mappings in relational metric spaces, *J. Anal.* 29, no. 3 (2021), 669–684.
- [22] G. Prasad and H. Işık, On solution of boundary value problems via weak contractions, *J. Funct. Spaces* 2022 (2022), Article ID 6799205.
- [23] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.* 132, no. 5 (2004), 1435–1443.
- [24] B. Samet and M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.* 13, no. 2 (2012), 82–97.
- [25] B. Schweizer, A. Sklar and E. Thorp, The metrization of statistical metric spaces, *Pac. J. Math.* 10 (1960), 673–675.
- [26] B. Schweizer and Sklar, *Probabilistic Metric Spaces*, North-Holland, New York (1983).
- [27] M. Turinici, Fixed points for monotone iteratively local contractions, *Demonstr. Math.* 19, no. 1 (1986), 171–180.
- [28] M. Turinici, Ran and Reuring's theorems in ordered metric spaces, *J. Indian Math. Soc.* 78 (2011), 207–214.