

## Some properties defined by relative versions of star-covering properties II

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*Dedicated to the memory of Mikhail (Misha) Matveev*

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### ABSTRACT

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*In this paper we consider some recent relative versions of Menger property called set strongly star Menger and set star Menger properties and the corresponding Hurewicz-type properties. In particular, using [2], we “easily” prove that the set strong star Menger and set strong star Hurewicz properties are between countable compactness and the property of having countable extent. Also we show that the extent of a regular set star Menger or a set star Hurewicz space cannot exceed  $\mathfrak{c}$ . Moreover, we construct (1) a consistent example of a set star Menger (set star Hurewicz) space which is not set strongly star Menger (set strongly star Hurewicz) and show that (2) the product of a set star Menger (set star Hurewicz) space with a compact space need not be set star Menger (set star Hurewicz). In particular, (1) and (2) answer some questions posed by Kočinac, Konca and Singh in [17] and [23].*

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## 1. INTRODUCTION

Let  $\mathcal{U}$  be a cover of a space  $X$  and  $A$  be a subset of  $X$ ; the star of  $A$  with respect to  $\mathcal{U}$  is the set  $st(A, \mathcal{U}) = \bigcup\{U : U \in \mathcal{U} \text{ and } U \cap A \neq \emptyset\}$ . The star of a one-point set  $\{x\}$  with respect to a cover  $\mathcal{U}$  is denoted by  $st(x, \mathcal{U})$ .

Recall that a space  $X$  is star compact, briefly SC (strongly star compact, briefly SSC) if for every open cover  $\mathcal{U}$  of the space  $X$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  (resp., a finite subset  $F$  of  $X$ ) such that  $st(\bigcup \mathcal{V}, \mathcal{U}) = X$  (resp.,  $st(F, \mathcal{U}) = X$ ) (see [12] where another terminology is used, and [7]);  $X$  is star Lindelöf, briefly SL (strongly star Lindelöf, briefly SSL) if for every open cover  $\mathcal{U}$  of the space  $X$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  (resp., a countable subset  $C$  of  $X$ ) such that  $st(\bigcup \mathcal{V}, \mathcal{U}) = X$  (resp.,  $st(C, \mathcal{U}) = X$ ) (see [10] and [11], where different terminology is used).

In [18, 17] Kočinac, Konca and Singh introduced the following relative versions of SC, SSC, SL and SSL properties.

**Definition 1.1** ([17]). A space  $X$  is set star compact, briefly set SC (resp., set strongly star compact, briefly set SSC), if for every nonempty subset  $A$  of  $X$  and for every family  $\mathcal{U}$  of open sets in  $X$  such that  $\overline{A} \subseteq \bigcup \mathcal{U}$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  (resp., finite subset  $F$  of  $\overline{A}$ ) such that  $st(\bigcup \mathcal{V}, \mathcal{U}) \supset A$  (resp.,  $st(F, \mathcal{U}) \supset A$ ).<sup>1</sup>

Replacing “finite” with “countable” in Definition 1.1, one obtains the classes of set star Lindelöf (briefly set SL) and set strongly star Lindelöf (briefly set SSL) spaces (see [18]).

In the following CC means countably compact.

**Proposition 1.2** ([2, Proposition 2.2]). *In the class of Hausdorff spaces SSC, set SSC and CC are equivalent properties.*

We prove the following

**Proposition 1.3.** *In the class of regular spaces set SC and CC are equivalent properties.*

*Proof.* Of course, every CC space is set SC. Now, let  $X$  be a regular set SC space. By contradiction, assume there exists a closed and discrete subspace  $D = \{x_n : n \in \omega\}$  of  $X$ . By regularity, there exists a disjoint family  $\mathcal{U} = \{U_n : n \in \omega\}$  of open subsets of  $X$  such that  $x_n \in U_n$ , for every  $n \in \omega$ . Then  $D \subseteq \bigcup \mathcal{U}$  but for every finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$ , we have that  $D \not\subseteq st(\bigcup \mathcal{V}, \mathcal{U})$ ; a contradiction.  $\square$

For a space  $X$ ,  $e(X) = \sup\{|C| : C \text{ is a closed and discrete subset of } X\}$  and  $c(X) = \sup\{|\mathcal{A}| : \mathcal{A} \text{ is a cellular family of } X\}$  are, respectively, the extent and the cellularity of  $X$ . One says that a space  $X$  has the countable chain condition (briefly ccc) if  $c(X) = \omega$ .

<sup>1</sup>Recently, the properties of Definition 1.1 were studied in [2]. Note that in [2] there is a misprint in the statement of the definition of “relatively\* SSC” that the authors use to describe set SSC property: in particular, the authors write that the set “ $F$  is a finite subset of  $A$ ” instead of “ $F$  is a finite subset of  $\overline{A}$ ”.

**Proposition 1.4** ([2, Proposition 3.1]). *In the class of  $T_1$  spaces, set SSL spaces are exactly spaces having countable extent.*

**Proposition 1.5** ([2, Corollary 3.3]). *Every ccc space is set SL.*

Recall that a space  $X$  is Menger, briefly M, if for each sequence  $(\mathcal{U}_n : n \in \omega)$  of open covers of  $X$  there exists a sequence  $(\mathcal{V}_n : n \in \omega)$  such that  $\mathcal{V}_n, n \in \omega$ , is a finite subset of  $\mathcal{U}_n$  and  $X = \bigcup_{n \in \omega} \bigcup \mathcal{V}_n$ ;  $X$  is Hurewicz, briefly H, if for each sequence  $(\mathcal{U}_n : n \in \omega)$  of open covers of  $X$  there exists a sequence  $(\mathcal{V}_n : n \in \omega)$  such that  $\mathcal{V}_n, n \in \omega$ , is a finite subset of  $\mathcal{U}_n$  and for every  $x \in X$ ,  $x \in \bigcup \mathcal{V}_n$  for all but finitely many  $n \in \omega$ .

In [13, 14, 1] star versions of Menger and Hurewicz properties called star Menger, strongly star Menger, star Hurewicz and strongly star Hurewicz properties (Definitions 2.1 and 4.1 below) were introduced and recently in [17] Kočinac, Konca and Singh considered some relative versions of them called, respectively, set star Menger, set strongly star Menger, set star Hurewicz and set strongly star Hurewicz properties.

In this paper we study the previous set properties. In particular, using [2], we easily prove that set strongly star Menger and set strongly star Hurewicz properties are between countable compactness and property of having countable extent. Also we show that the extent of a regular set star Menger or a set star Hurewicz space cannot exceed  $\mathfrak{c}$  and use this result to give a Tychonoff star Menger (star Hurewicz) space which is not set star Menger (set star Hurewicz). In fact, the constructed example (Example 2.12) is even star compact and then it gives a positive answer to the following question.

**Question 1.6** ([23]). *Does there exist a Tychonoff star compact space which is not set star compact?*

Moreover, we give a consistent answer (Example 2.24) to the following question.

**Question 1.7** ([17]). *Does there exist a Tychonoff set star Menger space which is not set strongly star Menger?*

Further, we answer in the negative (Example 3.10) to the following.

**Question 1.8** ([17]). *Is the product of a set star Menger space with a compact space a set star Menger space?*

In fact Example 3.10 shows even more: it proves that set star compact and set star Lindelöf properties are not preserved in the product with compact spaces. Then, the same example answers in the negative the following two questions.

**Question 1.9.** *Is the product of a set star Hurewicz space with a compact space a set star Hurewicz space?*

**Question 1.10** ([23]). *Is the product of a set star compact space with a compact space a set star compact space?*

Moreover we give partial answers to the following questions.

**Question 1.11** ([17]). *Is the product of a set strongly star Menger space with a compact space a set strongly star Menger space?*

**Question 1.12.** *Is the product of a set strongly star Hurewicz space with a compact space a set strongly star Hurewicz space?*

No separation axiom will be assumed a priori. Recall that a family of sets is almost disjoint if the intersection of any two distinct elements is finite. Let  $\mathcal{A}$  be an almost disjoint family of infinite subsets of  $\omega$ . Put  $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$  and topologize  $\Psi(\mathcal{A})$  as follows: the points of  $\omega$  are isolated and a basic neighbourhood of a point  $a \in \mathcal{A}$  takes the form  $\{a\} \cup (a \setminus F)$ , where  $F$  is a finite set.  $\Psi(\mathcal{A})$  is called Isbell-Mrówka or  $\Psi$ -space (see [8]). Recall that for  $f, g \in \omega^\omega$ ,  $f \leq^* g$  means that  $f(n) \leq g(n)$  for all but finitely many  $n$  (and  $f \leq g$  means that  $f(n) \leq g(n)$  for all  $n \in \omega$ ). A subset  $B \subseteq \omega^\omega$  is bounded if there is  $g \in \omega^\omega$  such that  $f \leq^* g$  for every  $f \in B$ .  $D \subseteq \omega^\omega$  is cofinal if for each  $g \in \omega^\omega$  there is  $f \in D$  such that  $g \leq^* f$ . The minimal cardinality of an unbounded subset of  $\omega^\omega$  is denoted by  $\mathfrak{b}$ , and the minimal cardinality of a cofinal subset of  $\omega^\omega$  is denoted by  $\mathfrak{d}$ . The value of  $\mathfrak{d}$  does not change if one considers the relation  $\leq$  instead of  $\leq^*$  [6, Theorem 3.6].

2. ON SET STAR Menger AND SET STRONGLY STAR Menger PROPERTIES.

In [13], Kočinac introduced the following star versions of Menger property.

**Definition 2.1** ([13]). A space  $X$  is

- star Menger (briefly, SM) if for each sequence  $(\mathcal{U}_n : n \in \omega)$  of open covers of  $X$  there exists a sequence  $(\mathcal{V}_n : n \in \omega)$  such that  $\mathcal{V}_n, n \in \omega$ , is a finite subset of  $\mathcal{U}_n$  and  $X = \bigcup_{n \in \omega} st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ ;
- strongly star Menger (briefly, SSM) if for each sequence  $(\mathcal{U}_n : n \in \omega)$  of open covers of  $X$  there exists a sequence  $(F_n : n \in \omega)$  such that  $F_n, n \in \omega$ , is a finite subset of  $X$  and  $X = \bigcup_{n \in \omega} st(F_n, \mathcal{U}_n)$ .

The following result gives a characterization of the SSM property in terms of a relative version of it.

**Proposition 2.2.** *The followings are equivalent for a space  $X$ :*

- (1)  $X$  is SSM;
- (2) *for each nonempty subset  $A$  of  $X$  and each sequence  $(\mathcal{U}_n : n \in \omega)$  of collection of open sets of  $X$  such that  $\overline{A} \subset \bigcup \mathcal{U}_n$  for every  $n \in \omega$ , there exists a sequence  $(F_n : n \in \omega)$  such that  $F_n, n \in \omega$ , is a finite subset of  $X$  and  $A \subset \bigcup_{n \in \omega} st(F_n, \mathcal{U}_n)$ .*

*Proof.* 2.  $\Rightarrow$  1. is obvious. Let  $A \subseteq X$  be a nonempty subset and  $(\mathcal{U}_n : n \in \omega)$  be a sequence of families of open sets of  $X$  such that  $\overline{A} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ . Define

$$\mathcal{U}'_n = \mathcal{U}_n \cup \{X \setminus \overline{A}\}$$

for all  $n \in \omega$ . Clearly, each  $\mathcal{U}'_n$  is an open cover for  $X$ . Since  $X$  is SSM, there is a sequence  $(F_n : n \in \omega)$  of finite subsets of  $X$  such that  $X = \bigcup_{n \in \omega} st(F_n, \mathcal{U}_n)$ . Fix  $x \in A$ . Then there exists  $n \in \omega$  such that  $x \in st(F_n, \mathcal{U}'_n)$ . Observe that

$$x \in st(F_n, \mathcal{U}'_n) \Leftrightarrow st(x, \mathcal{U}'_n) \cap F_n \neq \emptyset.$$

We also have that

$$st(x, \mathcal{U}'_n) = \bigcup \{U \in \mathcal{U}'_n : x \in U\} = \bigcup \{U \in \mathcal{U}_n : x \in U\} = st(x, \mathcal{U}_n).$$

So

$$x \in st(F_n, \mathcal{U}'_n) \Leftrightarrow st(x, \mathcal{U}'_n) \cap F_n \neq \emptyset \Leftrightarrow st(x, \mathcal{U}_n) \cap F_n \neq \emptyset \Leftrightarrow x \in st(F_n, \mathcal{U}_n).$$

Since  $x$  is an arbitrary point of  $A$ , we have that  $A \subset \bigcup_{n \in \omega} st(F_n, \mathcal{U}_n)$ .  $\square$

In [17] the following relative version of the SM and SSM properties were considered.

**Definition 2.3** ([17]). A space  $X$  is

- set star Menger (shortly, set SM) if for each nonempty subset  $A$  of  $X$  and for each sequence  $(\mathcal{U}_n : n \in \omega)$  of collection of open sets of  $X$  such that  $\overline{A} \subset \bigcup \mathcal{U}_n$  for every  $n \in \omega$ , there exists a sequence  $(\mathcal{V}_n : n \in \omega)$  such that  $\mathcal{V}_n, n \in \omega$ , is a finite subset of  $\mathcal{U}_n$  and  $A \subset \bigcup_{n \in \omega} st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ .
- set strongly star Menger (shortly, set SSM) if for each nonempty subset  $A$  of  $X$  and for each sequence  $(\mathcal{U}_n : n \in \omega)$  of collection of open sets of  $X$  such that  $\overline{A} \subset \bigcup \mathcal{U}_n$  for every  $n \in \omega$ , there exists a sequence  $(F_n : n \in \omega)$  such that  $F_n, n \in \omega$ , is a finite subset of  $\overline{A}$  and  $A \subset \bigcup_{n \in \omega} st(F_n, \mathcal{U}_n)$ .

The following result is easy to check.

**Proposition 2.4.** *A space  $X$  is set SSM iff every closed subspace of  $X$  is SSM.*

The previous result is not true for set SM spaces as the following example shows.

**Example 2.5.** A set SM space having a closed subspace which is not SM.

Consider the set SM space  $X$  of Example 3.10 below and its closed subspace  $A$ . Since  $A$  is a discrete subspace of uncountable cardinality, it is not SM.  $\triangle$

Recall that in [25] it is proved that the extent of a  $T_1$  SSM space can be arbitrarily big. Also

**Proposition 2.6** ([22, Corollary 2.2]). *Every closed and discrete subspace of a regular SSM space has cardinality less than  $\mathfrak{c}$ . Hence a SSM space has extent less or equal to  $\mathfrak{c}$ .*

It is well known that a CC space has countable extent. Since every CC space is set SSC (see [7, Theorem 2.1.4] and recall that CC property is hereditary with respect to closed sets) and every set SSL space has countable extent [2, Proposition 3.1] we have that

$$\text{CC} \Rightarrow \text{set SSH} \Rightarrow \text{set SSM} \Rightarrow \text{countable extent.}$$

Note that the previous implications can not be reversed. Indeed, of course, every Hurewicz non countably compact space is a set SSH non countably compact space: consider, for example, the discrete space  $\omega$ . For the converse of the other implications see Examples 4.4 and 2.17.

In [21] it was shown that the extent of a Tychonoff SSL space can be arbitrarily large (note that in [21] a SSL space is called a space with countable weak extent). In [22] the space constructed in [21] was used to prove that the extent of a Tychonoff SM (in fact SC) space can be arbitrarily large. Moreover in [22] the author shows the following.

**Theorem 2.7** ([22]). *If  $X$  is a regular SM space such that  $w(X) = \mathfrak{c}$ , then every closed and discrete subspace of  $X$  has cardinality less than  $\mathfrak{c}$ . Hence, we have  $e(X) \leq \mathfrak{c}$ .*

Now we show that the extent of a regular set SM space cannot exceed  $\mathfrak{c}$ .

**Theorem 2.8.** *If  $X$  is a regular set SM space, then every closed and discrete subspace of  $X$  has cardinality less than  $\mathfrak{c}$ . Hence, we have  $e(X) \leq \mathfrak{c}$ .*

*Proof.* Fix  $Y$  a closed and discrete subspace of  $X$  and assume  $|Y| = \mathfrak{c}$ . Consider a family  $\mathcal{B}$  of open subsets of  $X$  such that for every  $y \in Y$  there exists  $B \in \mathcal{B}$  such that  $y \in B$  and  $\overline{B} \cap Y = \{y\}$  and suppose that  $|\mathcal{B}| = \mathfrak{c}$ . Denote by  $[\mathcal{B}]^{<\omega}$  the family of all finite subsets of  $\mathcal{B}$ , by  $P = ([\mathcal{B}]^{<\omega})^\omega$  the family of all the sequences of elements of  $[\mathcal{B}]^{<\omega}$  and introduce on  $P$  the partial order “ $\leq$ ” defined as follows: if  $(\mathcal{B}'_n)_{n \in \omega}, (\mathcal{B}''_n)_{n \in \omega} \in P$  then  $(\mathcal{B}'_n)_{n \in \omega} \leq (\mathcal{B}''_n)_{n \in \omega}$  means  $\mathcal{B}'_n \subseteq \mathcal{B}''_n$  for every  $n \in \omega$ . Let  $\{(\mathcal{B}_{\alpha,n})_{n \in \omega} : \alpha < \mathfrak{c}\}$  be a cofinal family in  $(P, \leq)$ . Take  $Z = \{y_\alpha : \alpha < \mathfrak{c}\}$  by choosing for every  $\alpha < \mathfrak{c}$  a point  $y_\alpha \in Y \setminus \bigcup_{n \in \omega} \bigcup \overline{\mathcal{B}_{\alpha,n}}$  and  $y_\alpha \neq y_\beta$  for  $\alpha \neq \beta$ . For every  $\alpha < \mathfrak{c}$  let  $\{V_n(y_\alpha) : n \in \omega\}$  be a sequence of open neighbourhoods of  $y_\alpha$  such that  $V_n(y_\alpha) \subseteq B$  for some  $B \in \mathcal{B}$  and every  $n \in \omega$  and  $V_n(y_\alpha) \cap \bigcup \mathcal{B}_{\alpha,n} = \emptyset$  for every  $n \in \omega$ . For every  $n \in \omega$  put  $\mathcal{U}_n = \{V_n(y_\alpha) : \alpha < \mathfrak{c}\}$ . Clearly  $Z = \overline{Z} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ . We will show that the subset  $Z$  and the sequence  $(\mathcal{U}_n : n \in \omega)$  do not satisfy the set SM property. Let  $(\mathcal{V}_n : n \in \omega)$  be any sequence of finite subsets of  $\mathcal{U}_n$  for every  $n \in \omega$ . Let  $(\mathcal{B}'_n : n \in \omega) \in P$  such that every member of  $\mathcal{V}_n$  is contained in a member of  $\mathcal{B}'_n$ . Since  $\{(\mathcal{B}_{\alpha,n})_{n \in \omega} : \alpha < \mathfrak{c}\}$  is a cofinal family in  $P$ , there exists  $\gamma < \mathfrak{c}$  such that  $\mathcal{B}'_n \subseteq \mathcal{B}_{\gamma,n}$  for every  $n \in \omega$ . Then  $V_n(y_\gamma) \cap \bigcup \mathcal{V}_n \subseteq V_n(y_\gamma) \cap \bigcup \mathcal{B}_{\gamma,n} = \emptyset$  for every  $n \in \omega$ . Since  $V_n(y_\gamma)$  is the only member of  $\mathcal{U}_n$  containing  $y_\gamma$ , we have  $y_\gamma \notin \bigcup_{n \in \omega} st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ .  $\square$

Example 2.10 below gives a consistent example of a SSM space which is not set SSM. In fact, such an example was already described in [17]; here we show

that it can be easily obtained from the next characterization and from the fact that set SSM spaces have countable extent.

**Theorem 2.9** ([4]). *The followings are equivalent:*

- (i)  $\Psi(\mathcal{A})$  is SSM
- (ii)  $|\mathcal{A}| < \mathfrak{d}$ .

**Example 2.10** ([17]). ( $\omega_1 < \mathfrak{d}$ ) There exists a SSM space which is not set SSM.

Assume  $\omega_1 < \mathfrak{d}$  and consider  $\Psi(\mathcal{A})$  with  $|\mathcal{A}| = \omega_1$ . By Theorem 2.9 and since  $e(\Psi(\mathcal{A})) > \omega$ , we have that  $\Psi(\mathcal{A})$  is a SSM not set SSM space.  $\triangle$

**Question 2.11.** *Does there exist a ZFC example of a SSM space which is not set SSM?*

Using Theorem 2.8 we can give a Tychonoff space distinguishing SM and set SM properties. In fact, the following example distinguishes SCness and set SCness too.

**Example 2.12.** A Tychonoff SC (hence SM) space which is not set SM (hence not set SC).

In [21], for each infinite cardinal  $\tau$  the following space  $X(\tau)$  was considered. Let  $Z = \{f_\alpha : \alpha < \tau\}$  where  $f_\alpha$  denotes the points in  $2^\tau$  with only the  $\alpha$ th coordinate equal to 1. Consider the set

$$X(\tau) = (2^\tau \times (\tau^+ + 1)) \setminus ((2^\tau \setminus Z) \times \{\tau^+\})$$

with the topology inherited from the product topology on  $2^\tau \times (\tau^+ + 1)$ . Denote  $X_0 = 2^\tau \times \tau^+$  and  $X_1 = Z \times \{\tau^+\}$ . Then  $X(\tau) = X_0 \cup X_1$ .  $X_1$  is a closed and discrete subspace of  $X(\tau)$  of cardinality  $\tau$ . So the extent of  $X(\tau)$  is  $\tau$ .

In [22] it is proven that the space  $X(\mathfrak{c})$  is SC (hence SM). By Theorem 2.8,  $X(\mathfrak{c})$  it is not set SM.  $\triangle$

Recall the following result.

**Proposition 2.13** ([22]). *Every SL (SSL) space of cardinality less than  $\mathfrak{d}$  is SM (SSM).*

Now we prove that the set versions of the previous proposition holds.

**Proposition 2.14.** *Every set SL space of cardinality less than  $\mathfrak{d}$  is set SM.*

*Proof.* Let  $X$  be a set SL space of cardinality less than  $\mathfrak{d}$ . Let  $A \subseteq X$  and  $(\mathcal{U}_n : n \in \omega)$  be a sequence of families of open sets of  $X$  such that  $\bar{A} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ . For every  $n \in \omega$  there is a countable subfamily  $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$  of  $\mathcal{U}_n$  such that  $A \subseteq st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ . For every  $x \in A$  we choose a function  $f_x \in \omega^\omega$  such that  $st(x, \mathcal{U}_n) \cap V_{n, f_x(n)} \neq \emptyset$  for all  $n \in \omega$ . Since  $\{f_x : x \in A\}$  is not a cofinal family in  $(\omega^\omega, \leq)$ , there are some  $g \in \omega^\omega$  and  $n_x \in \omega$  for  $x \in A$  such that  $f_x(n_x) < g(n_x)$ . Let  $\mathcal{W}_n = \{V_{n,j} : j \leq g(n)\}$ . Then  $A \subseteq \bigcup_{n \in \omega} st(\bigcup \mathcal{W}_n, \mathcal{U}_n)$ .  $\square$

In a similar way we can prove the following.

**Proposition 2.15.** *Every set SSL space of cardinality less than  $\mathfrak{d}$  is set SSM.*

Then, by Proposition 1.4 we obtain

**Corollary 2.16.** *For every  $T_1$  space  $X$  of cardinality less than  $\mathfrak{d}$ , the following are equivalent:*

- (1)  $X$  is set SSM
- (2)  $e(X) = \omega$ .

**Example 2.17.** There is a Tychonoff space of cardinality  $\mathfrak{d}$  having countable extent which is not set SSM.

Let  $\mathbb{P}$  be the space of irrationals. Take any non-Menger subspace  $X \subset \mathbb{P}$  of cardinality  $\mathfrak{d}$  (for instance, consider the Baire space  $\omega^\omega$  which is homeomorphic to  $\mathbb{P}$  and take a cofinal subset of cardinality  $\mathfrak{d}$ . It is well known that any cofinal subset of  $\omega^\omega$  is not Menger). Of course,  $X$  is a paracompact space having countable extent. Since in the class of paracompact Hausdorff spaces we have that  $M \Leftrightarrow SM$  (see [13]), we have that  $X$  is not set SSM.  $\triangle$

By Corollary 2.16 and Example 2.17 we have:

**Corollary 2.18.** *The following statements are equivalent:*

- (1)  $\omega_1 < \mathfrak{d}$ ;
- (2) every  $T_1$  space of cardinality  $\omega_1$  having countable extent is set SSM.

Recall the following result.

**Theorem 2.19** ([22]). *The following statements are equivalent for regular spaces.*

- (1)  $\omega_1 = \mathfrak{d}$ ;
- (2) if  $X$  is a SSM space, then  $e(X) \leq \omega$ .

We can add some equivalent conditions to the ones in the previous theorem.

**Theorem 2.20.** *The following statements are equivalent for regular spaces.*

- (1)  $\omega_1 = \mathfrak{d}$ ;
- (2) if  $X$  is a SSM space, then  $e(X) \leq \omega$ ;
- (3) for spaces of cardinality less than  $\mathfrak{d}$ , set SSM and SSM are equivalent properties.
- (4) for spaces of cardinality less than  $\mathfrak{d}$ , set SSL and SSL are equivalent properties.
- (5) every closed subspace of a SSM space  $X$  such that  $|X| < \mathfrak{d}$  is SSM.

*Proof.* 1.  $\Leftrightarrow$  2. holds by Theorem 2.19. Now we prove 2.  $\Rightarrow$  3.. Let  $X$  be a space of cardinality less than  $\mathfrak{d}$ . By 2. and Corollary 2.16, we have that  $X$  is SSM iff  $X$  is set SSM. Now we prove 3.  $\Rightarrow$  1.. Assume  $\omega_1 < \mathfrak{d}$ . Consider a space  $\Psi(\mathcal{A})$ , with  $|\mathcal{A}| = \omega_1$ . By Theorem 2.9,  $\Psi(\mathcal{A})$  is SSM, and since  $e(\Psi(\mathcal{A})) > \omega$ ,  $\Psi(\mathcal{A})$  is not set SSM. 3.  $\Leftrightarrow$  4. is obvious. 3.  $\Leftrightarrow$  5. follows from Proposition 2.4.  $\square$



Of course, countable spaces are Menger, then set SSM and SSM.

**Corollary 2.21.** *For regular spaces  $X$  such that  $\omega < |X| < \mathfrak{d}$ , SSM and set SSM are not equivalent properties.*

**Corollary 2.22.** *Uncountable regular spaces in which SSM and set SSM are equivalent properties have cardinality  $\geq \mathfrak{d}$ .*

In [17, Example 5] Kočinac, Konca and Singh constructed a  $T_1$  set SM space which is not set SSM and posed the following question.

**Question 2.23** ([17]). *Does there exist a Tychonoff set SM space which is not set SSM?*

Using Proposition 2.14 we can give a consistent answer to Question 2.23.

**Example 2.24.** ( $\omega_1 < \mathfrak{d}$ ) A Tychonoff set SM space which is not set SSM.

Assume  $\omega_1 < \mathfrak{d}$  and consider  $\Psi(\mathcal{A})$  with  $|\mathcal{A}| = \omega_1$ . Since  $\Psi(\mathcal{A})$  is separable, it is set SL hence, by Proposition 2.14, it is set SM. Since  $c(\Psi(\mathcal{A})) > \omega$ ,  $\Psi(\mathcal{A})$  is not set SSM.  $\triangle$

### 3. ON THE PRODUCT OF SET SM AND SET SSM WITH COMPACT SPACES

Recall that the product of a SC (SSC) space with a compact space is SC (SSC) ([9], [7]); further the product of a SL space with a compact space is SL [7] while the product of a SSL space with a compact space need not be SSL [7, Example 3.3.4]. In [13] Kočinac proved that the product of a SM space with a compact space is SM. Using [3, Lemma 2.3], Matveev noted that assuming  $\omega_1 < \mathfrak{d}$ , if  $X = \Psi(\mathcal{A})$  with  $|\mathcal{A}| = \omega_1$  and  $Y$  is a compact space such that  $c(Y) > \omega$ , then the product  $X \times Y$  is not SSL, hence not SSM; therefore he gave a consistent example of a not SSM space which is the product of a SSM space and a compact space. Then, it is natural to consider the following questions.

**Question 3.1** ([17]). *Is the product of a set SSM space with a compact space a set SSM space?*

**Question 3.2** ([17]). *Is the product of a set SM space with a compact space a set SM space?*

In the following we give a partial answer to Question 3.1 and a negative answer to Question 3.2. Note that we also show that set SSL property is preserved in the  $T_1$  product with compact spaces and that set SC and set SL properties are not preserved in the product with compact spaces. (For completeness, we note that, by Proposition 1.2, set SSC property is preserved in the Hausdorff product with compact spaces).

The following fact can be easily checked (we give the proof for sake of completeness). Recall that a map is perfect if it is continuous, closed, onto and each fiber is compact.

**Proposition 3.3.** *If  $f : X \rightarrow Y$  is a perfect map and  $A$  is an uncountable closed and discrete subspace of  $X$ , then  $f(A)$  is an uncountable closed and discrete subspace of  $Y$ .*

*Proof.* Let  $f$  and  $A$  be as in the hypothesis. Clearly  $f(A)$  is closed in  $Y$ . Note that, for every  $y \in f(A)$ ,  $f^{-1}(y) \cap A$  is a closed subset of the compact subspace  $f^{-1}(y)$  and then, since  $A$  is discrete, it is finite. Then  $A$  is countable, otherwise  $f(A)$  is countable. Now, fix  $y \in f(A)$  and say  $f^{-1}(y) \cap A = \{x_1, \dots, x_n\}$ . For every  $i = 1, \dots, n$  fix an open subset  $U_i$  of  $X$  such that  $A \cap U_i = \{x_i\}$  and put  $U = \bigcup_{i=1}^n U_i$ . Since  $A \setminus U$  is a closed subset of  $X$ , we have that  $f(A \setminus U) = f(A) \setminus \{y\}$  is a closed subset of  $Y$ , and then  $\{y\}$  is open in  $f(A)$  with the topology inherited from  $Y$ .  $\square$

By the previous proposition, we obtain the following result.

**Corollary 3.4.** *The product of a space having countable extent with a compact space has countable extent.*

*Proof.* Let  $X$  be a space with countable extent and  $Y$  be a compact space. The projection from  $X \times Y$  onto  $X$  is a perfect map. Then, by Proposition 3.3,  $e(X \times Y) = \omega$ .  $\square$

By Proposition 1.4, the previous result can be restated as follows.

**Proposition 3.5.** *The  $T_1$  product of a set SSL space with a compact space is set SSL.*

**Corollary 3.6.** *The product of a set SSM space with a compact space has countable extent.*

Then, by Corollary 2.16 we have

**Corollary 3.7.** *The  $T_1$  product of cardinality less than  $\mathfrak{d}$  of a set SSM space with a compact space is set SSM.*

Recall the following proposition.

**Proposition 3.8** ([2, Proposition 3.4]). *Let  $X$  be a space. If there exist a closed and discrete subspace  $D$  of  $X$  having uncountable cardinality and a disjoint family  $\mathcal{U} = \{O_\alpha : \alpha \in D\}$  of open neighbourhoods of points  $a \in D$ , then  $X$  is not set SL.*

Now we prove the following useful result.

**Proposition 3.9.** *If  $e(X) > \omega$  and  $c(Y) > \omega$ , where  $Y$  is  $T_1$ , then  $X \times Y$  is not set SL.*

*Proof.* Let  $S = \{s_\alpha : \alpha < \omega_1\}$  be a closed and discrete subset of  $X$ ,  $\mathcal{O} = \{O_\alpha : \alpha < \omega_1\}$  be a pairwise disjoint family of nonempty open subsets of  $Y$ . For every  $\alpha < \omega_1$ , fix  $t_\alpha \in O_\alpha$ . Put  $A = \{(s_\alpha, t_\alpha) : \alpha < \omega_1\}$ . It is obvious that  $A$  is an uncountable discrete subspace of  $X \times Y$ . Now we prove that  $A$  is closed. For every  $\alpha < \omega_1$  there exists an open set, say  $N_\alpha$ , such that  $N_\alpha \cap S = \{s_\alpha\}$ . Then

$(X \times Y) \setminus A = ((X \setminus S) \times Y) \cup \bigcup_{\alpha < \omega_1} (N_\alpha \times (Y \setminus \{t_\alpha\}))$ . Then, by Proposition 3.8,  $X \times Y$  is not set SL.  $\square$

**Example 3.10.** There exists a set SC (hence set SL and set SM) space  $X$  and a compact space  $Y$  with  $c(Y) > \omega$  such that  $X \times Y$  is not set SL (hence neither set SM nor set SC).

Consider the set  $X = \omega_1 \cup A$ , where  $A = \{a_\alpha : \alpha \in \omega_1\}$  is a set of cardinality  $\omega_1$ , topologized as follows:  $\omega_1$  has the usual order topology and is an open subspace of  $X$ ; a basic neighborhood of a point  $a_\alpha \in A$  takes the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.$$

In [2] it was proved that  $X$  is set SC, hence  $X$  is set SM. We have that  $e(X) > \omega$ . If  $Y$  is any compact space with  $c(Y) > \omega$ , by Proposition 3.9,  $X \times Y$  is not set SL.  $\triangle$

#### 4. ON SET STAR HUREWICZ AND SET STRONGLY STAR HUREWICZ PROPERTIES

Recall the following definitions.

**Definition 4.1** ([13, 1]). A space  $X$  is

- star Hurewicz (briefly, SH) if for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$  there exists a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  such that  $\mathcal{V}_n, n \in \omega$ , is a finite subset of  $\mathcal{U}_n$  and  $\forall x \in X, x \in st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$  for all but finitely many  $n \in \omega$ ;
- strongly star Hurewicz (briefly, SSH) if for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$  there exists a sequence  $(F_n : n \in \mathbb{N})$  such that  $F_n, n \in \omega$ , is a finite subset of  $X$  and  $\forall x \in X, x \in st(F_n, \mathcal{U}_n)$  for all but finitely many  $n \in \omega$ .

The following result is a characterization of SSH property in terms of a relative version of it. The proof is similar to the proof of Proposition 2.2.

**Proposition 4.2.** *The followings are equivalent for a space  $X$ :*

- (1)  $X$  is SSH;
- (2) for each nonempty subset  $A$  of  $X$  and for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of collection of open sets of  $X$  such that  $\bar{A} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ , there exists a sequence  $(F_n : n \in \mathbb{N})$  such that  $F_n, n \in \omega$ , is a finite subset of  $X$  and  $\forall x \in A, x \in st(F_n, \mathcal{U}_n)$  for all but finitely many  $n \in \omega$ .

**Definition 4.3** ([17]). A space  $X$  is

- set star Hurewicz (briefly, set SH) if for each nonempty subset  $A \subseteq X$  and for each sequence  $(\mathcal{U}_n : n \in \omega)$  of collection of open sets of  $X$  such that  $\bar{A} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ , there exists a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  such that  $\mathcal{V}_n, n \in \omega$ , is a finite subset of  $\mathcal{U}_n$  and  $\forall x \in A, x \in st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$  for all but finitely many  $n \in \omega$ .

- set strongly star Hurewicz (briefly, set SSH) if for each nonempty subset  $A \subseteq X$  and for each sequence  $(\mathcal{U}_n : n \in \omega)$  of collection of open sets of  $X$  such that  $\overline{A} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ , there exists a sequence  $(F_n : n \in \omega)$  such that  $F_n, n \in \omega$ , is a finite subset of  $\overline{A}$  and  $\forall x \in A, x \in st(F_n, \mathcal{U}_n)$  for all but finitely many  $n \in \omega$ .

**Example 4.4.** ( $\mathfrak{b} < \mathfrak{d}$ ) There is a Tychonoff set SSM space which is not set SSH.

Consider an unbounded subset  $X$  of the Baire space  $\omega^\omega$  of cardinality  $\mathfrak{b}$ . Then  $X$  is not Hurewicz and, by Corollary 2.16,  $X$  is set SSM. Since  $X$  is a paracompact space and in the class of paracompact Hausdorff spaces we have that  $H \Leftrightarrow SH$  (see [1]), we have that  $X$  is not set SSH.  $\triangle$

Recall the following characterization of SSH spaces.

**Theorem 4.5** ([4]). *The following properties are equivalent:*

- (i)  $\Psi(\mathcal{A})$  is SSH
- (ii)  $|\mathcal{A}| < \mathfrak{b}$ .

Then, we can “easily” give the following result (the same example was given in [24] using a longer proof).

**Example 4.6.** ( $\omega_1 < \mathfrak{b}$ ) There exists a SSH not set SSH space.

Assume  $\omega_1 < \mathfrak{b}$  and consider  $\Psi(\mathcal{A})$  with  $|\mathcal{A}| = \omega_1$ . Then, by Theorem 4.5 and since  $e(\Psi(\mathcal{A})) > \omega$ , we have that  $\Psi(\mathcal{A})$  is SSH not a set SSH space.  $\triangle$

**Question 4.7.** *Does there exist a ZFC example of a SSH not set SSH space?*

By Theorem 2.8 we can give the following.

**Theorem 4.8.** *If  $X$  is a regular set SH space, then every closed and discrete subspace of  $X$  has cardinality less than  $\mathfrak{c}$ . Hence, we have  $e(X) \leq \mathfrak{c}$ .*

In [24, Esemplio 2.4] it is given a Hausdorff SH space which is not set SH. Now we can provide the following example.

**Example 4.9.** A Tychonoff SC (hence SH) space which is not set SH.

Consider the space  $X(\mathfrak{c})$  of Example 2.12.  $X(\mathfrak{c})$  is SC (hence SH) and, by Theorem 4.8, it is not set SH.  $\triangle$

Recall the following

**Proposition 4.10** ([5, Corollary 3.10]). *Every SL (SSL) space of cardinality less than  $\mathfrak{b}$  is SH (SSH).*

In analogy to Proposition 2.14 and Proposition 2.15, we can prove the following.

**Proposition 4.11.** *Every set SL (set SSL) space of cardinality less than  $\mathfrak{b}$  is set SH (set SSH).*

*Proof.* Let  $X$  be a set SL space of cardinality less than  $\mathfrak{b}$  (the proof is similar if  $X$  is set SSL). Let  $A \subseteq X$  and  $(\mathcal{U}_n : n \in \omega)$  be a sequence of families of open sets of  $X$  such that  $\overline{A} \subseteq \bigcup \mathcal{U}_n$  for every  $n \in \omega$ . For every  $n \in \omega$  there is a countable subfamily  $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$  of  $\mathcal{U}_n$  such that  $A \subseteq st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ . For every  $x \in A$  we choose a function  $f_x \in \omega^\omega$  such that  $st(x, \mathcal{U}_n) \cap V_{n, f_x(n)} \neq \emptyset$  for all  $n \in \omega$ . Since  $\{f_x : x \in A\}$  is a bounded family in  $(\omega^\omega, \leq^*)$ , there exists  $g \in \omega^\omega$  such that for every  $x \in A$  we have that  $f_x(n) \leq g(n)$  for every but finitely many  $n \in \omega$ . Let  $\mathcal{W}_n = \{V_{n,j} : j \leq g(n)\}$ . Then for every  $x \in A$  we have that  $x \in st(\bigcup \mathcal{W}_n, \mathcal{U}_n)$  for all but finitely many  $n \in \omega$ .  $\square$

Then, by Proposition 1.4, we have

**Corollary 4.12.** *For every  $T_1$  space  $X$  of cardinality less than  $\mathfrak{b}$ , the followings are equivalent:*

- (1)  $X$  is set SSH
- (2)  $e(X) = \omega$ .

By Corollary 2.16 and Corollary 4.12 we have the following

**Corollary 4.13.** *For spaces  $X$  such that  $|X| < \mathfrak{b}$ , the followings are equivalent:*

- (1)  $X$  is set SSM
- (2)  $X$  is set SSH
- (3)  $e(X) = \omega$ .

In [24] the authors give a  $T_1$  set SH space which is not set SSH. Now we provide the following

**Example 4.14.** ( $\omega_1 < \mathfrak{b}$ ) A Tychonoff set SH space which is not set SSH.

Assume  $\omega_1 < \mathfrak{b}$  and consider  $\Psi(\mathcal{A})$  with  $|\mathcal{A}| = \omega_1$ . Since  $\Psi(\mathcal{A})$  is separable, it is set SL hence, by Proposition 4.11, it is set SH. Since  $e(\Psi(\mathcal{A})) > \omega$ ,  $\Psi(\mathcal{A})$  is not set SSH.  $\triangle$

Using Example 3.10 we obtain:

**Proposition 4.15.** *Set SH property is not preserved in the product with compact spaces.*

By Corollary 3.4 we have that

**Proposition 4.16.** *The product of a set SSH space with a compact space has countable extent.*

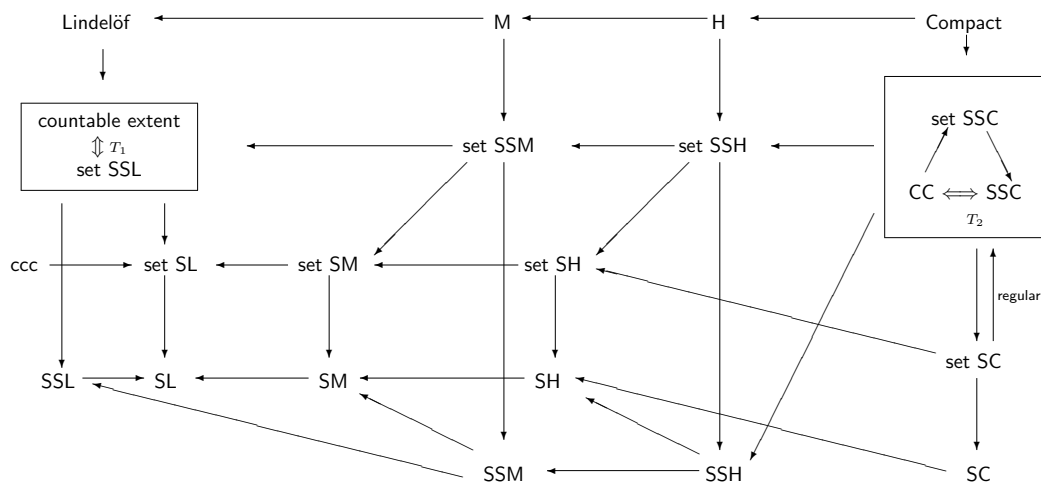
Then, by Corollary 4.13 we obtain

**Proposition 4.17.** *The  $T_1$  product of cardinality less than  $\mathfrak{b}$  of a set SSH space with a compact space is set SSH.*

The following question is open.

**Question 4.18.** *Is the product of a set SSH space with a compact space a set SSH space?*

We give the following useful diagram.



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