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Some topological and cardinal properties of the N_τ^φ-nucleus of a space X

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Abstract

In this paper, we study the behavior of some topological and cardinal properties of topological spaces under the influence of the N_τ^φ-kernel of a space X. It has been proved that the N_τ^φ-kernel of a space X preserves the density and the network π- weight of normal spaces. Besides, shown that the N-compact kernel of a space X preserves the Souslin properties, the weight, the density, and the π-network weight of normal spaces.

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1. Introduction

The study of the influence of normal and seminormal functors to the topological and cardinal properties of topological spaces has been developed fastly in recent investigations (see [3], [4], [6], [8], [9], [10], [11], [12], [14], [16], [18], [19]). The influence of normal and seminormal functors to the cardinal properties such as density, local density, weight, π-weight, caliber, Shanin number, Souslin number, Hewitt-Nachbin number, weak density, local weak density, network weight, network π-weight, extent, spread of topological spaces has been well studied. In [6] it had been studied the connection between the following spaces: the final compact, pseudocompact, extremally disconnected, $\aleph$-space

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and their hyperspaces. Especially, related results obtained for some normal functors: the functors $\exp_n$, $\exp_\omega$, exp preserve the final compactness, the pseudocompactness, the extremal disconnectedness and $\aleph$-spaces.

In [10] it had been checked that if a seminormal functor has an invariant extension then its extension preserves a point, the empty set, intersection, and is a monomorphic functor. If this functor has a finite degree then its extension is continuous and hence a seminormal functor in Tych. If the functor is of an infinite degree then continuity may be lost.

The concept of complete linked system of a topological space was first introduced by A. V. Ivanov in [8]. Some cardinal invariants of the space of the complete linked systems were investigated by Talaat Makhmud and F. G. Mukhamadiev in [10, 13, 14, 15, 18]. As well as, in [13] it was proved that the Shanin number of the space of the complete linked systems does not exceed the Shanin number of the space itself. T. K. Yuldashev and F. G. Mukhamadiev [18] obtained more general results.

In this paper, we study the behavior of the density and the network π- weight of topological spaces under the influence of the N_τ^φ-kernel of a space X. It has been proved that the N_τ^φ-kernel of a space X preserves the density and the network π- weight of topological spaces. Besides, shown that the N-compact kernel of a space X preserves the Souslin properties, the weight, the density, and the π-network weight of topological spaces.

All spaces are assumed to be normal, τ means an infinite cardinal number, the cardinality of a countable set is denoted by ω.

2. Preliminaries

A system η of closed subsets of a space X is said to be linked if any two elements of η intersect.

Condition 2.1. Any neighborhood OF of the set F contains an element $G \in \eta$.

A complete linked system (CLS) is a linked system η of closed subsets of a space X such that for any closed set $F \subset X$, Condition 2.1 implies $F \in \eta$ (see [8]).

The space of complete linked systems of a space X is the set NX of all complete linked systems endowed with the topology whose open base is formed by sets of the form

$$E = O(U_1, U_2, \ldots, U_n)\langle V_1, V_2, \ldots, V_s\rangle = \{\mathfrak{M} \in NX : \textit{ for any } i = 1, 2, \ldots, n \textit{ there exists } F_i \in \mathfrak{M} \textit{ such that } F_i \subset U_i \textit{ and for any } j = 1, 2, \ldots, s \textit{ and } F \in \mathfrak{M}, \textit{ we have } F \cap V_j \neq \varnothing\},$$

where $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_s$ are nonempty open sets in X.

In [15] results for the density and the network π-weight of the space NX were proved.

Let $\mathfrak{M}$ be a complete linked system (CLS) of a compactum X. An CLS is called a compact complete coupled system, if the system $\mathfrak{M}$ contains at least one compact element. A compact complete linked system $\mathfrak{M}$ of a compact X is denoted by CCLS.

The N-compact kernel of a topological space X is the subspace

$$N_cX = \{\mathfrak{M} \in NX : \mathfrak{M} \ is \ CCLS\}$$

of the space NX.

It is clear that in the case of a compact X we have $N_cX = NX$.

Let X be a T_1-space, φ be a cardinal-valued function, and τ be a cardinal number.

Definition 2.2. The N_τ^φ-nucleus of the space X is the space

$$N_\tau^\varphi X = \{\mathfrak{M} \in NX : \text{there exists } F \in \mathfrak{M} \text{ such that } \varphi(F) \leq \tau\} \text{ (see [14]).}$$

Definition 2.3. Assume that $\mathfrak{M} \in N_\tau^\varphi$-basement of the CLS $\mathfrak{M}$ is the family

$$\mathfrak{F}_\tau^\varphi(\mathfrak{M}) = \{F \in \mathfrak{M} : \ \varphi(F) \leq \tau\} \text{ (see [14]).}$$

Definition 2.4. A topological space X is said to be N_τ^φ-nuclear if $N_\tau^\varphi X = NX$ (see [14]).

As φ, we take a density function d. Let $\tau = \omega$.

From the above definition implies that any space X such that $\tau = d(X)$ is N_τ^d-nuclear; in particular, any separable space X is N_ω^φ-nuclear.

In [18] it was proved that the N_ω^d-nucleus of the space X is everywhere dense in the space NX, i.e., $\left[N_\omega^d X\right]_{NX} = NX$.

Denote by $\pi w(X)$ the π-weight of the topological space X:

$$\pi w(X) = \min\{|\beta|, \ where \ \beta \ is \ \pi\text{-}base \ space \ of \ X\}.$$

The *density* of a space X is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X; this cardinal number is denoted by $d(X)$. If $d(X \leq \omega$, then we say that the space X is *separable*.

Denote by $n\pi w(X)$ the network π-weight of the topological space X:

$$n\pi w(X) = \min\{|\beta|, \ where \ \beta \ is \ \pi\text{-}network \ space \ of \ X\}.$$

In [14] it was proved the following theorem:

Theorem 2.5. *Let X be an infinite T_1-space. Then*

(1) $d(X) = d\left(N_\omega^d X\right)$;
(2) $\pi w(X) = \pi w\left(N_\omega^d X\right)$.

We will use the concepts taken from [7]. The space is said to satisfy the second axiom of countability, if $w(X) \leq \omega$, where $w(X)$ is the weight of the topological space X:

$$w(X) = \min\{|\beta|, \ where \ \beta \ is \ base \ space \ of \ X\}.$$

The Souslin number $c(X)$ of the space X is defined as the smallest of all cardinal numbers $\tau \geq \omega$, and the cardinality of each system of pairwise disjoint non-empty open subsets of the space X does not exceed τ. If $c(X) \leq \omega$, we say that the space X has the Souslin property.

3. Some topological and cardinal properties of the N_τ^φ-nucleus of a space X

A topological space $X$is called locally τ-dense at the point $x \in X$, if τ is the smallest cardinal number, such that x has a neighbourhood of density τ in X (see [20]-[17]). Local density at a point x is denoted by $ld\,(x)$. Local density of the space X is defined as follows: $ld\,(X) = \sup\{ld\,(x): \ x \in X\}$.

Theorem 3.1. *Let X be an infinite compact space. Then we have*

(1) $w\,(NX) = w\left(N_\omega^{ld}X\right) = w\,(X)$;
(2) $c\,(NX) = c\left(N_\omega^{ld}X\right) \geq c\,(X)$.

Proof. 1) Being that $X \subset N_\omega^{ld}X \subset NX$ and $w\,(X) = w\,(NX)$ (see [5]), hence we have that $w\,(X) \leq w\left(N_\omega^{ld}X\right) \leq w\,(NX) = w\,(X)$. Therefore $w\,(X) = w\left(N_\omega^{ld}X\right) = w\,(NX)$.

2) It is known that if $A \subset X$, then $c\,(A) = c\,([A])$ (see [2]). Since the set $N_\omega^{ld}X$ is everywhere dense in the space NX, i.e., $\left[N_\omega^{ld}X\right]_{NX} = NX$. We get that $c\left(N_\omega^{ld}X\right) = c\left(\left[N_\omega^{ld}X\right]\right) = c\,(NX) \geq c\,(X)$. Theorem 3.1 is proved. □

Corollary 3.2. *For every infinite compact space X the following conditions are equivalent:*

(1) *The space X satisfies the second axiom of countability;*
(2) *The space $N_\omega^{ld}X$ satisfies the second axiom of countability;*
(3) *The space NX satisfies the second axiom of countability.*

Corollary 3.3. *For every infinite compact space X the following conditions are equivalent:*

(1) *The space X has the Souslin property;*
(2) *The space $N_\omega^{ld}X$ has the Souslin property;*
(3) *The space NX has the Souslin property.*

Problem 3.4. *When the local densities of spaces X and NX will be equal, i.e. $ld\,(X) = ld\,(NX)$?*

Now let's put $\mathrm{H}_\tau^{ld}\,(NX) = \left\{\beta \subset N_\omega^{ld}X : \ [\beta] = NX \text{ and } t_\beta\,(NX) \leq \tau\right\}$, where

$t_\beta(NX) = \min\{\tau \geq \omega : \text{from } \eta \in [\beta] \Rightarrow \exists \beta' \subset \beta : |\beta'| \leq \tau \text{ and } \eta \in [\beta']\}$.

Proposition 3.5. *Let X be a normal space such that $\mathrm{H}_\tau^{ld}\,(NX) \neq \varnothing$. Then $N_\omega^{ld}X = NX$.*

Proof. Let η arbitrary CLS in X. We fix the set $\beta \in \mathrm{H}_\tau^{ld}\,(NX)$. Being that $\eta \in [\beta]$ and $t_\beta\,(NX) \leq \tau$, there exists a subset $\beta_\eta \subset \beta$ such that, $|\beta_\eta| \leq \tau$ and $\eta \in [\beta_\eta]$. From the N_ω^{ld}-basement N_ω^{ld} of each CLS $\eta_i \in \beta_\eta = \{\eta_i : \ i \in \theta\}$ we take one element at a time F_i, then the resulting set $\mathrm{H}_1 = \{F_i : \ i \in \theta\}$ has a cardinality equal to τ. In each set $F_i \in \mathrm{H}_1$ we fix one countable everywhere locally dense set D_i in F_i, then we get a set $\mathrm{H}_2 = \{D_i : \ i \in \theta\}$. It's obvious that $|\mathrm{H}_2| \leq \tau$ and $|\bigcup \mathrm{H}_2| \leq \tau$. Let's put $\mathrm{H} = [\bigcup \mathrm{H}_2]$. Then it is obvious that $ld\,(\mathrm{H}) \leq \tau$. Now let's show that $\eta \in \mathrm{H}^+ = \{\mathfrak{M} \in NX : \mathfrak{M} \supset \mathrm{H}\}$. Assume it's not. Then there will be at least one element $F \in \eta$ such that $F \bigcap \mathrm{H} = \varnothing$. Since the space X is normal, there is a neighbourhood W of the set F such that $W \bigcap \mathrm{H} = \varnothing$. Since the set $O = O(W)\left\langle W' \right\rangle$ is a neighbourhood of point η in NX and point $\eta \in [\beta_\eta]$, where W' is an arbitrary open set in X. We have that $O(W)\left\langle W' \right\rangle \bigcap \beta_\eta \neq \varnothing$. Let $\eta_i \in O(W)\left\langle W' \right\rangle \bigcap \beta_\eta$. Then the selected element $F_i \in \mathrm{H}_1$ intersects with the set W, i.e. $V = W \cap F_i \neq \varnothing$. Since $[D_i] = F_i$ and set V is open in F_i, hence we have that $W \cap D_i \neq \varnothing$, all the more we have that $W \bigcap \mathrm{H} \neq \varnothing$. The obtained contradiction proves that $\eta \in \mathrm{H}^+$. By virtue of the arbitrariness of the point η in NX, we have that $N_\omega^{ld} X = NX$. Proposition 3.5 is proved. □

Now, by using Proposition 3.5 we can prove the following proposition as in [15].

Proposition 3.6. *Let X be a normal space such that $\mathrm{H}_\tau^{ld}(NX) \neq \varnothing$, where $\tau = ld(NX)$. Then*

(1) $d(X) = d(NX)$;
(2) $n\pi w(X) = n\pi w(NX)$.

Now, let us recall the concept of weak tightness $t_c^*(X)$ of a space X in the sense of [1]. Weak tightness $t_c^*(X)$ of a topological space X is the smallest cardinal number τ such that if $M \subset X$ and $M \neq [M]$, then there exist a point $x \in X \backslash M$ and a set $M' \subset M$ for which $\left| M' \right| \leq \tau$ and $x \in \left[M' \right]$.

Proposition 3.7. *Let X be a normal space such that $t_c^*(NX) \leq \omega$. Then we have that $N_\omega^{ld} X = NX$.*

Proof. Under condition $t_c^*(NX) \leq \omega$, we prove that $\left[N_\omega^{ld} X \right] = N_\omega^{ld} X$. Suppose that $\left[N_\omega^{ld} X \right] \neq N_\omega^{ld} X$, then since $t_c^*(NX) \leq \omega$, there exist a point $\eta \in NX \backslash N_\omega^{ld} X$ and a set $\beta \subset N_\omega^{ld} X$ such that $|\beta| \leq \omega$ and $\eta \in [\beta]$. Let us a set $\beta = \{\eta_1, \eta_2, \ldots, \eta_n, \ldots\ n \in \mathbb{N}^+\}$ and in the N_ω^{ld}-basement $\mathfrak{F}_\omega^{ld}(\eta_i)$ of each CLS $\eta_i \in \beta$ we fix one set F_i. Then we obtain a countable system $\mathrm{H}_1 = \{F_1, F_2, \ldots, F_n, \ldots\ n \in \mathbb{N}^+\}$. In each set F_i of H_1 we fix a countable set D_i everywhere locally dense in F_i, then we obtain system $\mathrm{H}_2 = \{D_1, D_2, \ldots, D_n, \ldots\ n \in \mathbb{N}^+\}$. We put $\mathrm{H} = [\bigcup \mathrm{H}_2]$. It is obvious that $|\bigcup \mathrm{H}_2| \leq \omega$, therefore $ld(\mathrm{H}) \leq \omega$. Now, we show that CLS $\eta \in \mathrm{H}^+ = \{\mathfrak{M} \in NX : \mathfrak{M} \supset \mathrm{H}\}$. Let's assume the opposite. Then there exists a set $F \in \eta$ such that $F \bigcap \mathrm{H} = \varnothing$. Since the space X is normal, there exists a neighbourhood W of the set F such that $W \bigcap \mathrm{H} = \varnothing$. Since $\eta \in O = O(W)\left\langle W' \right\rangle$ (where W' is an arbitrary open set in X) and $\eta \in [\beta]$, there exists at least one element $\eta_1 \in \beta$ lying in $O = O(W)\left\langle W' \right\rangle$, hence $W \cap F_i \neq \varnothing$. Since D_i is everywhere locally dense in F_i and $W \cap F_i$ is open in F_i, we have that $W \cap D_i \neq \varnothing$, especially $W \bigcap \mathrm{H} \neq \varnothing$. The resulting contradiction proves that $\left[N_\omega^{ld} X \right] = N_\omega^{ld} X$. Since $\left[N_\omega^{ld} X \right] = NX$, we have that $N_\omega^{ld} X = NX$. Proposition 3.7 is proved. □

From the previous results we have the following proposition.

Proposition 3.8. *Let X be a normal space such that $t_c^*(NX) \leq \omega$. Then we have*

(1) $ld(X) = ld(NX)$;
(2) $n\pi w(X) = n\pi w(NX)$.

Proposition 3.9. *Let X be a normal space such that $t_c^*(NX) \leq ld(NX) = \tau$. Then we have that $N_\tau^{ld}X = NX$.*

Proof. We can prove that similar to the proof of Proposition 3.2, under condition $t_c^*(NX) \leq d(NX) = \tau$, a set

$$N_\tau^{ld}X = \{\eta \in NX : \ \exists F \in \eta : \ ld(F) \leq ld(NX) = \tau\}$$

is closed in NX. Therefore, noticing that $N_\omega^{ld}X \subset N_\tau^{ld}X$ we have that $\left[N_\tau^{ld}X\right] = NX$. Hence we have that $N_\tau^{ld}X = NX$, where $\tau = ld(NX)$. Proposition 3.9 is proved. □

Now, by Proposition 3.9 it is easy to prove the following proposition.

Proposition 3.10. *Let X be a normal space such that $t_c^*(NX) \leq ld(NX) = \tau$. Then we have that*

(1) $ld(X) = ld(NX)$;
(2) $n\pi w(X) = n\pi w(NX)$.

Corollary 3.11. *Let X be a normal space whose CLS NX is locally separable. If $t_c^*(NX) \leq \omega$, then X is locally separable.*

Corollary 3.12. *Let X be a normal space and there exists a set $\beta \subset N_\tau^d X$ is everywhere locally dense in NX such that $|\beta| = \tau$. Then we have*

(1) $ld(X) = ld(NX)$;
(2) $n\pi w(X) = n\pi w(NX)$.

Problem 3.13. *When the local densities of spaces X and $N_\tau^\varphi X$ will be equal, i.e. $ld(X) = ld(N_\tau^\varphi X)$?*

4. Some topological and cardinal properties of the N-compact metrizable nucleus of a space X

Now we study some topological and cardinal properties of the N-compact metrizable kernel of the topological space X.

Let $\mathfrak{M}$ be a complete linked system (CLS) of a compactum X. An CLS is called a compact metrizable complete coupled system, if the system $\mathfrak{M}$ contains at least one compact metrizable element. A compact metrizable complete linked system $\mathfrak{M}$ of a compact X is denoted by CMCLS.

The N-compact metrizable kernel of a topological space X is the subspace

$$N_{cm}X = \{\mathfrak{M} \in NX : \mathfrak{M} \ is \ CMCLS\}$$

of the space NX.

It is clear that in the case of a compact metrizable X we have $N_{cm}X = NX$.

Proposition 4.1. *Let X be an infinite T_1-space. Then the N-compact metrizable nucleus of the space X is everywhere dense in the space NX, i.e., $[N_{cm}X]_{NX} = NX$.*

Proof. Let $\mathtt{E} = O\,(U_1, U_2, \ldots, U_n)\,\langle V_1, V_2, \ldots, V_s\rangle$ be an arbitrary non-empty open basis element in NX. Let $S\,(\mathtt{E}) = \{W_1, W_2, \ldots, W_p\}$ be the pairwise trace of element $\mathtt{E}$ in X. In each set $W_i \in S\,(\mathtt{E})$ we fix one point x_i, then we get a set $\sigma = \{x_1, x_2, \ldots, x_p\}$. Now we put sets of the form $\Phi_i = \{x_j \in \sigma :\ x_j \in U_i\}$, where $i = 1, 2, \ldots, n$. Obviously, a system $\mu = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$ is a linked system in the space X. Let us set

$$\mathfrak{M} = \{F \in \exp X :\ \exists \Phi_i \in \mu :\ \Phi_i \subset F\}.$$

Then the system $\mathfrak{M}$ is a CLS and it is clear that this system is a CMCLS and $\mathfrak{M} \in \mathtt{E}$. By virtue of the arbitrariness of the open basis element $\mathtt{E}$ in NX, we have that the set $N_{cm}X$ intersects with any open set in NX, hence $[N_{cm}X]_{NX} = NX$. Proposition 4.1 is proved. $\square$

Since $\exp X \subset NX$, we have the following result:

Corollary 4.2. *$N_cX = NX$ if and only if X is a compact metrizable space.*

By Proposition 4.1 we have the following result:

Corollary 4.3. *A space $N_{cm}X$ is compact if and only if a space X is compact.*

Proposition 4.4. *For any infinite T_1-space X we always have*

(1) *$\pi w\,(N_{cm}X) = \pi w\,(X)$;*
(2) *$d\,(N_{cm}X) = d\,(X)$.*

Proof. 1) It is known that for any arbitrary space X and its everywhere dense subset Y we have $\pi w\,(X) = \pi w\,(Y)$. Since $N_{cm}X$ is an everywhere dense set in NX, we have that $\pi w\,(N_{cm}X) = \pi w\,(NX)$. Now, by equality $\pi w\,(X) = \pi w\,(NX)$ (see [13]) we have that $\pi w\,(N_{cm}X) = \pi w\,(X)$.

2) First, let's show that $d\,(N_{cm}X) \leq d\,(X)$. Let X_0 be an everywhere dense set in X such that $|X_0| = d\,(X) = \tau$. We put $\exp_{cm}(X_0, X) = \{\Phi \in \exp X :\ \Phi \subset X_0 \text{ and } \Phi \text{ - compact metrizable}\}$ and $\Sigma_{cm}(X_0, X) = \{\mu \subset \exp_{cm}(X_0, X) :\ \mu \text{ - CLS}\}$. Now, for each $\mu \in \Sigma_{cm}(X_0, X)$ linked by the system, we put sets of the form $\mathfrak{M}_\mu = \{F \in \exp X :\ \exists \Phi \in \mu :\ \Phi \subset F\}$, then we get a set

$$N_{cm}(X_0, X) = \{\mathfrak{M}_\mu :\ \mu \in \Sigma_{cm}(X_0, X)\}.$$

It is obvious that $|N_{cm}(X_0, X)| = \tau = d\,(X)$ and we have that $N_{cm}(X_0, X) \subset N_{cm}X$. Now we show that the set $N_{cm}(X_0, X)$ is an everywhere dense set in $N_{cm}X$. Let $\mathtt{E} = O\,(U_1, U_2, \ldots, U_n)\,\langle V_1, V_2, \ldots, V_s\rangle$ be an arbitrary non-empty open basic element in N_cX. Let $S\,(\mathtt{E}) = \{W_1, W_2, \ldots, W_p\}$ be the pairwise trace of the element $\mathtt{E}$ in X. Since X_0 is an everywhere dense set in X, we have that $X_0 \cap W_i \neq \varnothing$, where $i = 1, 2, \ldots, p$. Now in each intersection of the form $X_0 \cap W_i$, where $i = 1, 2, \ldots, p$ we fix one point x_i, then we get a set $\sigma = \{x_1, x_2, \ldots, x_p\}$. Now we put sets of the form $\Phi_i =$

$\{x_j \in \sigma : \ x_j \in U_i\}$, where $i = 1, 2, \ldots, n$. Obviously, $\mu = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$ is a linked system of closed subsets in X, and it is clear that $\mu \in \Sigma_{cm}(X_0, X)$, therefore, $\mathfrak{M}_\mu \in N_{cm}(X_0, X)$. By the construction of the system $\mathfrak{M}_\mu$, we have that $F \cap V_j \neq \varnothing$ for any $F \in \mathfrak{M}_\mu$ and for any $j = 1, 2, \ldots, s$. In addition, it is clear that $\Phi_i \subset U_i$, where $i = 1, 2, \ldots, n$. Therefore, we have that $\mathfrak{M}_\mu \in \mathtt{E} = O(U_1, U_2, \ldots, U_n)\langle V_1, V_2, \ldots, V_s\rangle$. So $N_{cm}(X_0, X) \cap O(U_1, U_2, \ldots, U_n)\langle V_1, V_2, \ldots, V_s\rangle \neq \varnothing$. By virtue of the arbitrariness of the basis element $\mathtt{E}$ in $N_{cm}X$, we have that the set $N_{cm}(X_0, X)$ intersects with every open set in $N_{cm}X$. Therefore, $N_{cm}(X_0, X)$ is an everywhere dense set in $N_{cm}X$. Since $d(N_{cm}X) \leq |N_{cm}(X_0, X)| = |X_0| = d(X)$, we have that $d(N_{cm}X) \leq d(X)$.

Now we will prove that $d(X) \leq d(N_{cm}X)$. Let $\mathfrak{B} = \{\mathfrak{M}_i : \ i \in \theta\}$ be an everywhere dense subset of $N_{cm}X$ such that $|\mathfrak{B}| = d(N_{cm}X) = \tau$. From each CMCLS $\mathfrak{M}_i \in \mathfrak{B}$ we fix one compact metrizable set F_i, then we get a system $H = \{F_i : \ i \in \theta\}$. Let us a set $X_0 = \bigcup H$. Then it is obvious that $|X_0| = \tau$. Now we will show that the set X_0 is an everywhere dense set in X. Let U be an arbitrary non-empty open set in X. Then set $\mathtt{E} = O(U)\langle X\rangle$ is open in $N_{cm}X$, hence a set $\mathtt{E}_{cm} = O(U)\langle X\rangle \cap N_{cm}X$ is an open subset of $N_{cm}X$. Therefore, there exists at least one element $\mathfrak{M}_i \in \mathfrak{B}$ such that $\mathfrak{M}_i \in \mathtt{E}_{cm}$, hence $F_i \cap U \neq \varnothing$, and even more so $X_0 \cap U \neq \varnothing$. So, due to the arbitrariness of the set U in X, we have that the set X_0 intersects with any open set in X, i.e. X_0 is an everywhere dense subset of X. Therefore, $d(X) \leq |X_0| = \tau = d(N_{cm}X)$. So, finally, we have that $d(X) = d(N_{cm}X)$. Proposition 4.4 is proved. □

Corollary 4.5. *T_1-space X is separable if and only if space $N_{cm}X$ is separable.*

Now we put $N^n_{cm}X = \underbrace{N_{cm}N_{cm}N_{cm}\ldots N_{cm}}_{n}X$, hence, we have the following results.

Corollary 4.6. *Let X be an infinite T_1-space and n any natural number. Then we have*

(1) $\pi w(X) = \pi w(N^n_{cm}X)$;
(2) $d(X) = d(N^n_{cm}X)$.

Corollary 4.7. *Let X be an infinite T_1-space. Then the following properties are equivalent:*

(1) *X is separable;*
(2) *$N_{cm}X$ is separable;*
(3) *$N^n_{cm}X$ is separable for any $n \in N$.*

Since $X \subset N_{cm}X$ and $[N_{cm}X] = NX$, it is easy to prove the following proposition.

Proposition 4.8. *For any infinite T_1-space X we always have:*

(1) $c(N_{cm}X) = c(X)$;
(2) $w(N_{cm}X) = w(X)$.

Corollary 4.9. *For every infinite compact space X the following conditions are equivalent:*

(1) *The space X has the Souslin property;*
(2) *The space $N_\omega^{ld}X$ has the Souslin property;*
(3) *The space NX has the Souslin property;*
(4) *The space $N_{cm}X$ has the Souslin property.*

Corollary 4.10. *For every infinite compact space X the following conditions are equivalent:*

(1) *The space X satisfies the second axiom of countability;*
(2) *The space $N_\omega^{ld}X$ satisfies the second axiom of countability;*
(3) *The space NX satisfies the second axiom of countability;*
(4) *The space $N_{cm}X$ satisfies the second axiom of countability.*

Problem 4.11. *When the local densities of spaces X and $N_{cm}X$ will be equal, i.e. $ld\,(X) = ld\,(N_{cm}X)$?*

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