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On σ-compact Hattori spaces

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Abstract

We present several characterizations of σ-compact Hattori spaces, and reject some possible characterization candidates of the spaces.

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1. Introduction

Let $\mathbb{R}$ be the set of real numbers and A be a subset of $\mathbb{R}$.

In [6] Hattori introduced a topology $\tau(A)$ on $\mathbb{R}$ defined as follows:

(1) if $x \in A$ then $\{(x-\epsilon, x+\epsilon) : \epsilon > 0\}$ is a nbd open basis at x,
(2) if $x \in \mathbb{R} \setminus A$ then $\{[x, x+\epsilon) : \epsilon > 0\}$ is a nbd open basis at x.

Note that $\tau(\varnothing)$ (respectively, $\tau(\mathbb{R})$) is the Sorgenfrey topology τ_S (respectively, the Euclidean topology τ_E) on the reals.

The topological spaces $(\mathbb{R}, \tau(A)), A \subseteq \mathbb{R}$, are called *Hattori spaces* and denoted by $H(A)$ or H-spaces (if A is unimportant for a discussion). It is easy to see that the identity mapping of reals is a continuous bijection of any H-space onto the real line.

Let us recall ([2]) that every H-space is T_1, regular, hereditary Lindelöf and hereditary separable. However there are topological properties as the metrizability or the Cech-completeness which some H-spaces possess and other H-spaces do not possess. When the H-spaces possess these properties one can find in [8] and [1].

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Recall ([5]) that each compact subset of the Sorgenfrey line (the space $H(\varnothing)$) is countable. So the space $H(\varnothing)$ cannot be σ-compact unlike to the space $H(\mathbb{R})$ (the real line) which is evidently σ-compact.

The following natural problem was posed by F. Lin and J. Li.

Problem 1.1 ([9, Question 3.7]). *For what subsets A of $\mathbb{R}$ are the spaces $H(A)$ σ-compact?*

F. Lin and J. Li also noted

Proposition 1.2 ([9, Theorem 3.13]). *For an arbitrary subset A of $\mathbb{R}$, if $H(A)$ is σ-compact, then $\mathbb{R} \setminus A$ is countable and nowhere dense in $H(A)$.*

Proposition 1.3 ([9, Theorem 3.14]). *For an arbitrary subset A of $\mathbb{R}$, if $\mathbb{R} \setminus A$ is countable and scattered in $H(A)$, then $H(A)$ is σ-compact.*

In this note I present several characterizations of Hattori spaces to be σ-compact, and show that the implications of Propositions 1.2 and 1.3 are not invertible. Moreover, Proposition 1.3 (formulated as above) does not hold, its corrected version is presented in Corollary 2.3. The implication of Corollary 2.3 is also not invertible.

For standard notions we refer to [4].

2. Main results

First of all let us recall the following fact.

Lemma 2.1 ([2, Lemma 2.1]). *Let $A \subseteq \mathbb{R}$ and $B \subseteq A$ and $C \subseteq \mathbb{R} \setminus A$. Then*

(i) *$\tau(A)|_B = \tau_E|_B$, where τ_E is the Euclidean topology on $\mathbb{R}$, and*
(ii) *$\tau(A)|_C = \tau_S|_C$, where τ_S is the Sorgenfrey topology on $\mathbb{R}$.*

Proposition 2.2. *For an arbitrary subset A of $\mathbb{R}$, if $B = \mathbb{R} \setminus A$ is countable and it is a G_δ-subset of the real line (in particular, if B is countable and closed in the real line), then $H(A)$ is σ-compact.*

Proof. Let us note that on the real line our set A is an F_σ-set and hence it is σ-compact there (i. e. $A = \cup_{i=1}^{\infty} A_i$, where each A_i is a compact subset of the real line). So by Lemma 2.1 A is σ-compact in $H(A)$ too because for each positive integer i we have $\tau(A)|_{A_i} = \tau_E|_{A_i}$, where τ_E is the Euclidean topology on $\mathbb{R}$. Since B is countable we get that $H(A)$ is σ-compact. □

Since every scattered subset of the real line is a G_δ (see [7, Corollary 4]) we get the following.

Corollary 2.3. *For any subset A of $\mathbb{R}$, if $\mathbb{R} \setminus A$ is countable and scattered in the real line, then $H(A)$ is σ-compact.*

We continue with several characterizations of H-spaces to be σ-compact.

Theorem 2.4. *Let $A \subseteq \mathbb{R}$ and $B = \mathbb{R} \setminus A$. Then the following conditions are equivalent.*

(a) *There exist a σ-compact subset D and a closed subset C of the space $H(A)$ such that $B \subseteq C \subseteq D$.*

(b) *There exists a closed σ-compact subset C of the space $H(A)$ such that $B \subseteq C$.*
(c) *The closure $Cl_{H(\mathbb{R})}(B)$ of B in the real line is σ-compact in $H(A)$.*
(d) *The closure $Cl_{H(A)}(B)$ of B in the space $H(A)$ is σ-compact in $H(A)$.*
(e) *the space $H(A)$ is σ-compact.*

Proof. The following implications are obvious: $(e) => (a)$, $(a) => (b)$, $(c) => (b)$, $(b) => (d)$, $(e) => (c)$.

Let us show $(d) => (e)$. Since $B \subseteq Cl_{H(A)}(B)$ each point $x \in H(A) \setminus Cl_{H(A)}(B)$ has inside the set $H(A) \setminus Cl_{H(A)}(B)$ an open nbd which is an open interval of the real line. Since the space $H(A)$ is hereditarily Lindelöf the set $H(A) \setminus Cl_{H(A)}(B)$ is a σ-compact subset of $H(A)$ (see Lemma 2.1). Thus even $H(A)$ is σ-compact. □

Remark 2.5. Note that the set $Cl_{H(\mathbb{R})}(B)$ does not need to be σ-compact in the space $H(A)$ (it is of course closed there) even if it is compact in the real line, see Proposition 2.11.

Let us consider in the set of reals the standard Cantor set $\mathbb{C}$ on the closed interval $[0, 1]$ which can be defined as follows.

For any closed bounded interval $[a, b]$ of $\mathbb{R}$ put

$$F([a, b]) = \{[a, \frac{2}{3}a + \frac{1}{3}b], [\frac{1}{3}a + \frac{2}{3}b, b]\}.$$

Then for each $n \geq 0$ by induction define a family $\mathcal{C}_n$ of closed intervals:

$$\mathcal{C}_0 = \{[0, 1]\},\ \mathcal{C}_n = \{F([a, b]) : [a, b] \in \mathcal{C}_{n-1}\}.$$

The standard Cantor set $\mathbb{C}$ of the closed interval $[0, 1]$ is the intersection $\cap_{n=0}^{\infty}(\cup\mathcal{C}_n)$, where $\cup\mathcal{C}_n$ is the union of all closed intervals from the family $\mathcal{C}_n$.

Put now $B_1 = \{a : [a, b] \in \mathcal{C}_n, n \geq 0\}$, $B_2 = \{b : [a, b] \in \mathcal{C}_n, n \geq 0\}$ and $A_1 = \mathbb{R} \setminus B_1$, $A_2 = \mathbb{R} \setminus B_2$. We will use the notations below.

Remark 2.6. Let us note that on the real line (i.e. on the reals with the Euclidean topology) the set $\mathbb{C}$ is compact, the sets B_1 and B_2 (which are subsets of $\mathbb{C}$) are homeomorphic to the space of rational numbers $\mathbb{Q}$, the sets $\mathbb{C} \setminus B_1$ and $\mathbb{C} \setminus B_2$ are homeomorphic to the space of irrational numbers $\mathbb{P}$. Moreover, B_1 and B_2 are nowhere dense in the real line.

Remark 2.7. Let us note that a set $Y \subset \mathbb{R}$ is nowhere dense in the real line iff Y is nowhere dense in any H-space (see for example, [3, Lemma 3.3]).

Proposition 2.8. *For the space $H(A_1)$ the following is valid.*

(a) *The subspace B_1 of $H(A_1)$ is nowhere dense in $H(A_1)$ and it is homeomorphic to the space of rational numbers $\mathbb{Q}$.*
(b) *The subspace $Cl_{H(A_1)}(B_1)$ of $H(A_1)$ is homeomorphic to the standard Cantor set $\mathbb{C}$ on the real line, and the subspace $Cl_{H(A_1)}(B_1) \setminus B_1$ of $H(A_1)$ is homeomorphic to the space of irrational numbers $\mathbb{P}$.*
(c) *The space $H(A_1)$ is σ-compact.*

Proof. (a) and (b) are obvious. Theorem 2.4 and (b) prove (c). □

Corollary 2.9. *Proposition 1.3 is not invertible.*

Proof. Let us note that $H(A_1)$ is σ-compact but the subspace B_1 of $H(A_1)$ is not scattered. □

Corollary 2.10. *Proposition 2.2 is not invertible as well as Corollary 2.3.*

Proof. Let us note that $H(A_1)$ is σ-compact but B_1 is not a G_δ-subset of the Cantor set $\mathbb{C}$ in the real line and hence it is not a G_δ in the real line. □

Proposition 2.11. *For the space $H(A_2)$ the following is valid.*

(a) *The subspace B_2 of $H(A_2)$ is nowhere dense in $H(A_2)$ and it is homeomorphic to the space of natural numbers $\mathbb{N}$.*
(b) *The subspace $Cl_{H(A_2)}(B_2)$ of $H(A_2)$ is equal to the standard Cantor set $\mathbb{C}$ of $\mathbb{R}$, and it is not σ-compact. The subspace $Cl_{H(A_2)}(B_2) \setminus B_2$ of $H(A_2)$ is homeomorphic to the space of irrational numbers $\mathbb{P}$,*
(c) *The space $H(A_2)$ is not σ-compact.*

Proof. (a) is obvious.

In (b) let us show that the subspace $Cl_{H(A_2)}(B_2)$ of $H(A_2)$ is not σ-compact. Assume that the subspace $Cl_{H(A_2)}(B_2)$ of $H(A_2)$ is σ-compact, i.e. $Cl_{H(A_2)}(B_2) = \cup_{i=1}^{\infty} K_i$, where K_i is compact in $H(A_2)$. Note that for each i the set K_i is compact in the real line and the Cantor set $\mathbb{C}$ with the topology from the real line is the union $\cup_{i=1}^{\infty} K_i$. Hence there is an open interval (c, d) of the reals and some i such that $(c, d) \cap \mathbb{C} \subseteq K_i$. Moreover, there exist points $b_0, b_1, \dots$ of B_2 such that $b_1 < b_2 < \dots < b_0$ and the sequence $\{b_j\}_{j=1}^{\infty}$ tends to b_0 in the real line. Since at the points of B_2 the topology of $H(A_2)$ is the Sorgenfrey topology we get a contradiction with the compactness of K_i in the space $H(A_2)$.

Theorem 2.4 and (b) prove (c). □

Corollary 2.12. *Proposition 1.2 is not invertible.*

Proof. Let us note that B_2 is nowhere dense in $H(A_2)$ (see Remark 2.7) but the space $H(A_2)$ is not σ-compact. □

Corollary 2.13. *Proposition 1.3 does not hold.*

Proof. Let us note that B_2 is scattered in $H(A_2)$ and the space $H(A_2)$ is not σ-compact. □

3. Additional questions

The following is obvious.

(a) If a space X is σ-compact then a subset Y of X is σ-compact iff it is an F_σ-subset of X. In particular, a subset of the real line is σ-compact iff it is an F_σ-set.
(b) A subset of the Sorgenfrey line is σ-compact iff it is countable,

(c) A subset of the space $\mathbb{P}$ of irrational numbers is σ-compact iff it is homeomorphic to an F_σ-subset of the standard Cantor set $\mathbb{C}$ on the real line.

One can pose the following problem.

Problem 3.1. *Let $A \subseteq \mathbb{R}$. Describe the σ-compact subsets of $H(A)$.*

Let us note in advance that according to (a) if $H(A)$ is σ-compact then a subset of $H(A)$ is σ-compact iff it is an F_σ-subset of $H(A)$

Below we present some other answers to Problem 3.1 by the use of observations (b) and (c) and some known facts.

Proposition 3.2 ([8, Theorem 6] and [1, Theorem 2.8])**.** *$H(A)$ is homeomorphic to the Sorgenfrey line iff A is scattered.*

Corollary 3.3. *If A is scattered then a subset of $H(A)$ is σ-compact iff it is countable.*

Proposition 3.4 ([1, Proposition 3.6])**.** *$H(A)$ is homeomorphic to the space $\mathbb{P}$ of irrational numbers iff $\mathbb{R} \setminus A$ is dense in the real line and countable.*

Corollary 3.5. *If $\mathbb{R} \setminus A$ is dense in the real line and countable then a subset of $H(A)$ is σ-compact iff it is homeomorphic to an F_σ-subset of the standard Cantor set $\mathbb{C}$ on the real line.*

Since the space $H(A_2)$ from Proposition 2.11 is not σ-compact (as well as any subset of $H(A_2)$ containing some $[a, b]$ from $\mathcal{C}_n, n = 0, 1, 2, \ldots$) one can pose the following problem.

Problem 3.6. *What subsets of $H(A_2)$ are σ-compact?*

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References

[1] A. Bouziad, and E. Sukhacheva, On Hattori spaces, Comment. Math. Univ. Carolin. 58, no. 2 (2017), 213–223

[2] V. A. Chatyrko, and Y. Hattori, A poset of topologies on the set of real numbers, Comment. Math. Univ. Carolin. 54, no. 2 (2013), 189–196.

[3] V. A. Chatyrko, and V. Nyagahakwa, Sets with the Baire property in topologies formed from a given topology and ideals of sets, Questions and answers in General Topology 35 (2017) 59–76

[4] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.

[5] M. S. Espelie, and J. E. Joseph, Compact subspaces of the Sorgenfrey line, Math. Magazine 49 (1976), 250–251.

[6] Y. Hattori, Order and topological structures of poset of the formal balls on metric spaces, Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. 43 (2010), 13–26.

[7] V. Kannan, and M. Rajagopalan, On scattered spaces, Proc. Amer. Math. Soc. 43, no. 2 (1974), 402–408.
[8] J. Kulesza, Results on spaces between the Sorgenfrey topology and the usual topology on $\mathbb{R}$, Topol. Appl. 231 (2017), 266–275.
[9] F. Lin, and J. Li, Some topological properties of spaces between the Sorgenfrey and usual topologies on the real numbers, `arXiv:1807.06938v4`.

