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The extension of two-Lipschitz operators

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Abstract

The paper deals with some further results concerning the class of two-Lipschitz operators. We prove first an isometric isomorphism identification of two-Lipschitz operators and Lipschitz operators. After defining and characterizing the adjoint of a two-Lipschitz operator, we prove a Schauder type theorem on the compactness of the adjoint. We study the extension of two-Lipschitz operators from the cartesian product of two complemented subspaces of a Banach space to the cartesian product of whole spaces. Also, we show that every two-Lipschitz functional defined on the cartesian product of two pointed metric spaces admits an extension with the same two-Lipschitz norm under some requirements on domaine spaces.

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1. Introduction and notation

Sánchez Pérez in [14] introduced the definition of real two-Lipschitz maps acting in a cartesian product of two pointed metric spaces (called Lipschitz bi-forms) which possess a continuous bi-linearization between Banach spaces. A detailed and systematic study of these mappings with values in a Banach space is given recently in [10], where the authors introduce the new concept of two-Lipschitz operator ideals between pointed metric spaces and Banach spaces.

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A number of class of linear operators have been fruitfully generalized to the Lipschitz setting in recent years by several authors (see [1], [2], [3], [4], [7] and the references therein). Note that the concept of a two-Lipschitz mapping was firstly introduced by Dubei et al. [9] as those mappings that are Lipschitz in each variable.

In the present paper we go further in this direction and prove some results concerning this class of non-linear mappings. After the introductory one, in Section 2, by using the linearization of Lipschitz operators and the bi-linearization of two-Lipschitz operators, we obtain an important canonical identification of two-Lipschitz operators and Lipschitz operators. We define the adjoint of a two-Lipschitz operator which is the key for proving the Schauder type theorem for compact two-Lipschitz operators. In Section 3, we present some theorems on the extension of two-Lipschitz operators when they are defined on the cartesian product of two Banach spaces or the product of two pointed metric spaces.

Throughout the paper we will use standard notation and concepts of Banach space theory and the theory of Lipschitz functions. From now on, unless otherwise stated, X and Y will denote pointed metric spaces with base point 0 that is, 0 is any arbitrary fixed point of X and the metric will be denoted by d. We denote by $B_X = \{x \in X : d(x,0) \leq 1\}$. Also, E and F denote Banach spaces over the same field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) with dual spaces E^* and F^*. A Banach space E is a pointed metric space with distinguished point 0 (the null vector) and metric $d(x,x') = \|x - x'\|$. A map $T : X \longrightarrow E$ is called Lipschitz if there is a constant $C > 0$ such that for any pair of points $x, x' \in X$, we have $\|T(x) - T(x')\| \leq Cd(x,x')$. We will consider Lipschitz maps that map 0 to 0, and then the infimum of all constants $C > 0$ as above determines a complete norm on the space $Lip_0(X,E)$ of all such maps. Note that the elements of $Lip_0(X,E)$ are known as Lipschitz operators. The space $Lip_0(X,\mathbb{K})$ is called the Lipschitz dual of X and it will be denoted by $X^{\#}$. It is worth mentioning that the Banach space of all linear operators $\mathcal{L}(E,F)$ is a subspace of $Lip_0(E,F)$ and, so, E^* is a subspace of $E^{\#}$. A molecule on X is a finitely supported function $m : X \longrightarrow \mathbb{R}$ that satisfies $\sum\limits_{x\in X} m(x) = 0$. The set $\mathcal{M}(X)$ of molecules is a vector space. Note that molecules have the form $m = \sum\limits_{j=1}^{n} \lambda_j m_{x_j x'_j}$, where $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ with $\lambda_j \in \mathbb{R}$ and $x_j, x'_j \in X$ and χ_A is the characteristic function of the set A. The completion of $\mathcal{M}(X)$, when endowed with the norm

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{j=1}^{n} |\lambda_j|\, d(x_j, x'_j),\ m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}\right\},$$

where the infimum is taken over all representations of the molecule m, is denoted Æ(X) and called Arens–Ells space associated to X (see [5]). Consider the canonical Lipschitz injection map $\delta_X : X \longrightarrow$Æ$(X)$ defined by $\delta_X(x) = m_{x0}$. It is well-known that δ_X isometrically embeds X in Æ(X). If we consider

$T \in Lip_0(X, E)$, then T always factors through Æ(X) as

$$T = T_L \circ \delta_X : X \longrightarrow \text{Æ}(X) \longrightarrow E.$$

The map $T_L :$ Æ$(X) \longrightarrow E$ is the unique continuous linear operator (referred to as the linearization of T) that satisfies $T = T_L \circ \delta_X$ and $\|T_L\| = Lip(T)$ and we get the isometric isomorphism identification

$$Lip_0(X, E) = \mathcal{L}(\text{Æ}(X), E), \tag{1.1}$$

through the correspondence $T \longleftrightarrow T_L$. If we take $E = \mathbb{K}$ in (1.1), then the spaces $X^{\#}$ and Æ$(X)^*$ are isometrically isomorphic (see [16, Theorem 2.2.2]).

Let $T : X \times Y \longrightarrow E$ be a map from the cartesian product of two pointed metric spaces X and Y to a Banach space E. We will say that T is a two-Lipschitz operator if there is a constant $C > 0$ such that for every pair of elements $x, x' \in X$ and $y, y' \in Y$,

$$\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| \leq C d(x, x') d(y, y') \tag{1.2}$$

and $T(x, 0) = T(0, y) = 0$. The Banach space of all these mappings is denoted by $BLip_0(X, Y; E)$. Where the norm of T is given by

$$BLip(T) = \sup_{x \neq x', y \neq y'} \frac{\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\|}{d(x, x') d(y, y')}, \tag{1.3}$$

Let E, F, G be Banach spaces, we will denote by $\mathcal{L}(E, F; G)$ the Banach space of all bilinear operators from $E \times F$ into G. Every bilinear operator $T \in \mathcal{L}(E, F; G)$ is two-Lipschitz with $BLip(T) = \|T\|$. In the case $E = \mathbb{K}$, we write $BLip_0(X, Y)$ for the space of all two-Lipchitz functionals from $X \times Y$.

The main tools that we will use is the bi-linearization theorem for two-Lipschitz operators which is proved in [10].

Theorem 1.1 ([10, Theorem 2.6]). *For every two-Lipschitz operator $T \in BLip_0(X, Y; E)$ there exists a unique bilinear operator $T_B :$Æ$(X) \times$Æ$(Y) \longrightarrow E$ satisfying*

$$T_B(m_{xx'}, m_{yy'}) = T(x, y) - T(x, y') - T(x', y) + T(x', y') \tag{1.4}$$

and

$$T = T_B \circ (\delta_X, \delta_Y) : X \times Y \overset{(\delta_X, \delta_Y)}{\longrightarrow} \text{Æ}(X) \times \text{Æ}(Y) \overset{T_B}{\longrightarrow} E. \tag{1.5}$$

Furthermore, $BLip(T) = \|T_B\|$. The bilinear operator T_B is called bi-linearization of the two-Lipschitz operator T.

2. Further results on two-Lipschitz operators

The next proposition follows immediately from the definition of two-Lipschitz operators as in the bilinear case.

Proposition 2.1. *Let $T \in BLip_0(X, Y; E)$. Then,*

$$\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| \leq BLip(T) d(x, x') d(y, y'), \tag{2.1}$$

for all $x, x' \in X$ and $y, y' \in Y$. Moreover $BLip(T)$ can be calculated also by the formula

$$BLip(T) = \inf \{C > 0 : C \text{ satisfies } (1.2)\}. \tag{2.2}$$

Example 2.2. (a) Let E and F be Banach spaces. The evaluation mapping on $Lip_0(E, F)$,

$$\psi : Lip_0(E, F) \times E \longrightarrow F, \quad \psi(T, x) = T(x)$$

is a two-Lipschitz operator. In order to see this, for all $T, S \in Lip_0(E, F)$ and $x, x' \in E$ we have $\psi(T, 0) = \psi(0, x) = 0$ and

$$\begin{aligned} & \|\psi(T, x) - \psi(T, x') - \psi(S, x) + \psi(S, x')\| \\ = \quad & \|(T - S)(x) - (T - S)(x')\| \leq Lip(T - S)\, \|x - x'\|. \end{aligned}$$

which means that ψ is in $BLip_0(Lip_0(E, F), E; F)$ and $BLip(\psi) \leq 1$.

(b) Let X and Y be pointed metric spaces and let E be a Banach space. We define the mapping $K : X \times Y \longrightarrow (BLip_0(X, Y))^*$ by

$$K(x, y)(\phi) := \phi(x, y), \quad \phi \in BLip_0(X, Y).$$

The following calculations show that K is two-Lipschitz and $BLip(K) = 1$,

$$\begin{aligned} & \sup_{x \neq x', y \neq y'} \frac{\|K(x, y) - K(x, y') - K(x', y) + K(x', y')\|}{d_X(x, x')\, d_Y(y, y')} \\ = \quad & \sup_{x \neq x', y \neq y'} \sup_{BLip(\phi) \leq 1} \frac{|\phi(x, y) - \phi(x, y') - \phi(x', y) + \phi(x', y')|}{d_X(x, x')\, d_Y(y, y')} \\ = \quad & \sup_{BLip(\phi) \leq 1} \sup_{x \neq x', y \neq y'} \frac{|\phi(x, y) - \phi(x, y') - \phi(x', y) + \phi(x', y')|}{d_X(x, x')\, d_Y(y, y')} = 1. \end{aligned}$$

Let E, F, G be Banach spaces. By the following canonical correspondence

$$\psi : \mathcal{L}(E, F; G) \longrightarrow \mathcal{L}(E, \mathcal{L}(F, G)), \quad \psi(T)(x)(y) := T(x, y),$$

we obtain the isometric isomorphism identifications

$$\mathcal{L}(E, F; G) = \mathcal{L}(E, \mathcal{L}(F, G)). \tag{2.3}$$

(see [8, Page 11]). This result can be extended to the two-Lipschitz operators.

Proposition 2.3. *Let X and Y be pointed metric spaces and let E be a Banach space. Then we have the isometric isomorphism identification*

$$BLip_0(X, Y; E) = Lip_0(X, Lip_0(Y, E)). \tag{2.4}$$

Proof. First, if the Banach spaces E and F are isometrically isomorphic under the correspondence φ, we can write the isometric isomorphism identification

$$Lip_0(X, E) = Lip_0(X, F), \tag{2.5}$$

through the mapping $T \longmapsto \varphi \circ T$. Now, by the bi-linearization theorem (Theorem 1.1) we get the isometric isomorphism identification

$$BLip_0(X, Y; E) = \mathcal{L}(Æ(X), Æ(Y); E), \tag{2.6}$$

through the correspondence $T \longleftrightarrow T_B$. Finally, if we combine the identifications (1.1), (2.3), (2.5) and (2.6) we get the identification (2.4). $\square$

Remark 2.4. It follows directly from (2.4) that $BLip_0(X, Y; E)$ is a Banach space under the norm $BLip(\cdot)$, which is a simple proof of the result [10, Theorem 2.5].

Sawashima, in [15], defined the Lipschitz adjoint of a Lipschitz operator $T \in Lip_0(X, E)$. Also, the definition of adjoint of a bilinear operator is due to Ramanujan and Schock [12]. Let E, F, G be Banach spaces and let $S \in \mathcal{L}(E, F; G)$. The linear operator $S^* : G^* \longrightarrow \mathcal{L}(E, F; \mathbb{K})$ defined as $S^*(g^*)(x, y) := g^*(S(x, y))$, for every $g^* \in G^*$ and $(x, y) \in E \times F$ is called the adjoint of S and has the property that $\|S^*\| = \|S\|$. Combining the two notions, we get the following definition of the two-Lipschitz transpose.

Definition 2.5. Let X, Y be pointed metric spaces and let E be Banach space. We define the transpose of a two-Lipschitz operator $T \in BLip_0(X, Y; E)$ as a linear mapping $T^t : E^* \longrightarrow BLip_0(X, Y)$ that maps each $e^* \in E^*$ to $e^* \circ T$ that is

$$T^t(e^*)(x, y) = e^*(T(x, y)), \quad (x, y) \in X \times Y.$$

The next result give the important properties of the two-Lipschitz transpose, though its proof has no difficulty.

Theorem 2.6. *The transpose $T^t : E^* \longrightarrow BLip_0(X, Y)$ of the two-Lipschitz operator $T \in BLip_0(X, Y; E)$ is linear and bounded with $\|T^t\| = BLip(T)$.*

Proof. The linearity is clear. From the inequalities

$$\begin{aligned} & \left|T^t(e^*)\,(x, y) - T^t(e^*)\,(x, y') - T^t(e^*)\,(x', y) + T^t(e^*)\,(x', y')\right| \\ \leq \;& \|e^*\| \, \|T\,(x, y) - T\,(x, y') - T\,(x', y) + T\,(x', y')\| \\ \leq \;& \|e^*\| \, BLip(T) d\,(x, x')\, d\,(y, y')\,, \end{aligned}$$

it follows that $BLip(T^t(e^*)) \leq BLip(T)\,\|e^*\|$, for each $e^* \in E^*$. Which implies that the linear mapping T^t is bounded and $\|T^t\| \leq BLip(T)$. To have the reverse inequality, let $x, x' \in X$ and $y, y' \in Y$. Then,

$$\begin{aligned} & \|T\,(x, y) - T\,(x, y') - T\,(x', y) + T\,(x', y')\| \\ = \;& \sup_{\|e^*\| \leq 1} |e^*\,(T\,(x, y) - T\,(x, y') - T\,(x', y) + T\,(x', y'))| \\ \leq \;& \sup_{\|e^*\| \leq 1} BLip(T^t(e^*)) d\,(x, x')\, d\,(y, y') \\ = \;& \left\|T^t\right\| d\,(x, x')\, d\,(y, y')\,, \end{aligned}$$

from which we conclude that $BLip(T) \leq \|T^t\|$. $\square$

Recall the notion of compact two-Lipschitz operators. Let X and Y be pointed metric spaces and let E be a Banach space. For every $T \in BLip_0(X, Y;\, E)$ consider $Im_{BLip}(T)$ the bounded subset of E formed by all vectors of the form

$$\frac{T\,(x, y) - T\,(x, y') - T\,(x', y) + T\,(x', y')}{d(x, x')d(y, y')},$$

where $x, x' \in X, y, y' \in Y$ with $x \neq x'$ and $y \neq y'$. The mapping T is called compact if $Im_{BLip}(T)$ is relatively compact in E. (see [10, Definition 4.1]).

Example 2.7. Let non-zero Lipschitz functions $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$. The mapping $T : X \times Y \longrightarrow E$ defined by $T(x, y) = f(x)g(y)e$ is two-Lipschitz and $BLip(T) = Lip(f)Lip(g) \|e\|$. (see the paragraph after [10, Remark 2.7]). An easy computation shows that $Im_{BLip}(T)$ consists of all elements $v \in E$ of the form

$$v = \frac{(f(x) - f(x'))}{d(x, x')} \frac{(g(y) - g(y'))}{d(y, y')} e,$$

where $x \neq x' \in X$ and $y \neq y' \in Y$. It is a bounded set in the one-dimensional subspace generated by e and so, $Im_{BLip}(T)$ is relatively compact in E and then T is compact.

As a consequence of the bi-linearization theorem and [13, Proposition 8], we obtain the following characterization of compact two-Lipschitz operators into the Banach space c_0 of all null sequences in $\mathbb{K}$.

Theorem 2.8. *Let X and Y be pointed metric spaces and let $T \in BLip_0(X, Y; c_0)$. Then T is compact if and only if T can be written as $T(x, y) = (a_n(x, y))_n$ for a sequence $(a_n)_n$ of two-Lipschitz functionals on $X \times Y$ with $BLip(a_n) \longrightarrow 0$.*

Proof. By [10, Theorem 4.3] we have that T is compact if and only if $T_B : Æ(X) \times Æ(Y) \longrightarrow c_0$ is bilinear compact and by [13, Proposition 8] this is equivalent to T_B has the form $T_B(m, m') = (b_n(m, m'))_n$ with $(b_n)_n \subset \mathcal{L}(Æ(X), Æ(Y); \mathbb{K})$ and $\|b_n\| \longrightarrow 0$. In the other hand, by Theorem 1.1 there exists $(a_n)_n \subset BLip_0(X, Y)$ such that $(a_n)_B = b_n$ and

$$T(x, y) = T_B(m_{x0}, m_{y0}) = ((a_n)_B (m_{x0}, m_{y0}))_n = (a_n(x, y))_n,$$

for every $x \in X$ and $y \in Y$. Then we have that T is compact if and only if $T(x, y) = (a_n(x, y))_n$ with $BLip(a_n) = \|b_n\| \longrightarrow 0$. $\square$

In the following result we present a Schauder type theorem for compact two-Lipschitz operators. In order to prove this result we need the following lemma.

Lemma 2.9. *Let X and Y be pointed metric spaces and let E be a Banach space. For every $T \in BLip_0(X, Y; E)$, we have $R \circ T^t = (T_B)^*$ where $R : BLip_0(X, Y) \longrightarrow \mathcal{L}(Æ(X), Æ(Y); \mathbb{K})$ is the isomorphic isometry given by $R(\phi) = \phi_B$ (by taking $E = \mathbb{K}$ in (2.6)).*

Proof. For any $m_1 = \sum\limits_{i=1}^{n} \alpha_i m_{x_i x_i'} \in \mathcal{M}(X)$, $m_2 = \sum\limits_{j=1}^{r} \beta_j m_{y_j y_j'} \in \mathcal{M}(Y)$ and $e^* \in E^*$ we have

$$\begin{aligned}
& R \circ T^t(e^*)(m_1, m_2) \\
= \; & \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j \left(T^t(e^*)\right)_B \left(m_{x_i x_i'}, m_{y_j y_j'}\right) \\
= \; & \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j e^* \left(T(x,y) - T(x,y') - T(x',y) + T(x',y')\right) \\
= \; & e^* \left(T_B(m_1, m_2)\right) \\
= \; & (T_B)^*(e^*)(m_1, m_2),
\end{aligned}$$

and the result follows. $\square$

Theorem 2.10. *The two-Lipschitz operator $T \in BLip_0(X,Y;E)$ is compact if and only if its transpose T^t is a compact linear operator.*

Proof. By [10, Theorem 4.3], T is two-Lipschitz compact if and only if its bilinearization T_B :Æ$(X)\times$Æ$(Y) \longrightarrow E$ is compact bilinear operator. This is equivalent to $(T_B)^* : E^* \longrightarrow \mathcal{L}($Æ$(X)$,Æ$(Y);\mathbb{K})$ is linear compact (see [12, Theorem 2.6]). On the other hand, by the previous lemma we have $R \circ T^t = (T_B)^*$ (then $T^t = R^{-1} \circ (T_B)^*$). By the ideal property concerning the class of linear compact operators, we have that $(T_B)^*$ is compact if and only if T^t is too. $\square$

3. The extension of two-Lipschitz operators

In the following we give a condition for a two-Lipschitz mapping to be extended from the cartesian product $M \times N$ of two closed subspaces of Banach spaces E and F to the cartesian product $E \times F$.

Recall that a closed subspace F of a Banach space E is complemented in E if there exists another closed subspace G such that $E = F \oplus G$. This definition is equivalent to say that F is the range of a continuous projection P on E. If $\lambda \geq 1$, we say that F is λ-complemented in E if $\|P\| \leq \lambda$.

Theorem 3.1. *Let E, F and G be Banach spaces. If M is an α-complemented subspace in E and N is an β-complemented subspace in F, then every two-Lipschitz operator $T : M \times N \longrightarrow G$ has a two-Lipschitz operator extension $\widetilde{T} : E \times F \longrightarrow G$ with*

$$BLip(T) \leq BLip(\widetilde{T}) \leq BLip(T)\alpha\beta.$$

Proof. Let $P : E \longrightarrow E$ and $P' : F \longrightarrow F$ be the continuous projections having range M and N respectively, with $\|P\| \leq \alpha$ and $\|P'\| \leq \beta$. Consider the linear operators $Q : E \longrightarrow M$ and $Q' : F \longrightarrow N$ defined by $Q(x) = P(x)$ and $Q'(y) = P'(y)$ for all $x \in E, y \in F$. It follow that $\|Q\| = \|P\| \leq \alpha$ and $\|Q'\| = \|P'\| \leq \beta$. Let us consider the mapping $\widetilde{T} = T \circ (Q, Q')$ where

$(Q, Q')(x, y) := (Q(x), Q'(y))$, $x \in E, y \in F$. Then, using [10, Proposition 2.3] and taking into account that every linear operators is a Lipschitz mapping, we obtain $\widetilde{T} \in BLip_0(E, F; G)$ with $BLip(\widetilde{T}) \leq BLip(T)\alpha\beta$. Let us check that $\widetilde{T}$ is an extension of T to $E \times F$. If $x \in M, y \in N$ then $Q(x) = P(x) = x$ and $Q'(y) = P'(y) = y$, this implies that $\widetilde{T}(x, y) = T(x, y)$. Finally,

$$\begin{aligned} BLip(\widetilde{T}) &= \sup_{x \neq x' \in E, y \neq y' \in F} \frac{\left\| \widetilde{T}(x, y) - \widetilde{T}(x, y') - \widetilde{T}(x', y) + \widetilde{T}(x', y') \right\|}{d(x, x')d(y, y')} \\ &\geq \sup_{x \neq x' \in M, y \neq y' \in N} \frac{\| T(x, y) - T(x, y') - T(x', y) + T(x', y') \|}{d(x, x')d(y, y')} \\ &= BLip(T). \end{aligned}$$

□

The next corollary is a natural generalization of the bilinear extension result [11, Corollary 2] to the two-Lipschitz settings. Its proof is an easy consequence of the above theorem.

Corollary 3.2. *If M and N are complemented subspaces in the Banach spaces E and F respectively with projections of norm one, then every $T \in BLip_0(M, N; G)$ admits an extension $\widetilde{T} \in BLip_0(E, F; G)$ with $BLip(\widetilde{T}) = BLip(T)$.*

Let X, Y be pointed metric spaces and let X_0, Y_0 be subset of X and Y respectively. Suppose that the base point of X, Y belongs to X_0, Y_0 respectively. The following results deals with the extension of two-Lipschitz functionals $T : X_0 \times Y_0 \longrightarrow \mathbb{K}$ under some requirements on the pointed metric spaces X, Y, X_0, Y_0.

Recall that a metric space X is said to be 1-injective (or absolute Lipschitz retract in the terminology of [6]) if for every metric space Y and for every subset $Z \subset Y$, every Lipschitz function $f : Z \longrightarrow X$ can be extended to a Lipschitz function $\widetilde{f} : Y \longrightarrow X$ with $Lip(f) = Lip(\widetilde{f})$.

Theorem 3.3. *Suppose that $X^{\#}$ and $Y_0^{\#}$ are 1-injective. Then every two-Lipschitz functional T on $X_0 \times Y_0$ can be extended to $X \times Y$ with the same norm.*

Proof. Let $T \in BLip_0(X_0, Y_0)$. For fixed $s \in X_0$ consider $A_s \in Y_0^{\#}$ defined by $A_s(t) = T(s, t)$ for every $t \in Y_0$ (see [10, Proposition 2.2]). Define the mapping $\varphi : X_0 \longrightarrow Y_0^{\#}$ by $\varphi(s) = A_s$, then the following computation shows that $\varphi \in Lip_0(X_0, Y_0^{\#})$ and $Lip(\varphi) = BLip(T)$,

$$\begin{aligned} & \sup_{s \neq s'} \frac{Lip(\varphi(s) - \varphi(s'))}{d(s, s')} \\ = & \sup_{s \neq s'} \left(\sup_{t \neq t'} \frac{|(\varphi(s) - \varphi(s'))(t) - (\varphi(s) - \varphi(s'))(t')|}{d(s, s')d(t, t')} \right) \\ = & \sup_{s \neq s', t \neq t'} \frac{|T(s, t) - T(s', t) - T(s, t') + T(s', t')|}{d(s, s')d(t, t')}. \end{aligned}$$

Since $Y_0^{\#}$ is a Lipschitz 1-injective metric space, there exists $\widetilde{\varphi} \in Lip_0(X, Y_0^{\#})$ that extends φ to X with $Lip(\varphi) = Lip(\widetilde{\varphi})$. Let us consider the mapping $T_1 : X \times Y_0 \longrightarrow \mathbb{K}$ with $T_1(x,t) = \widetilde{\varphi}(x)(t)$ for every $x \in X, t \in Y_0$. This mapping is two-Lipschitz since $\widetilde{\varphi}(x)(0) = \widetilde{\varphi}(0)(t) = 0$ and

$$\begin{aligned} BLip(T_1) &= \sup_{x \neq x', t \neq t'} \frac{|T_1(x,t) - T_1(x',t) - T_1(x,t') + T_1(x',t')|}{d(x,x')d(t,t')} \\ &= \sup_{x \neq x', t \neq t'} \frac{|(\widetilde{\varphi}(x) - \widetilde{\varphi}(x'))(t) - (\widetilde{\varphi}(x) - \widetilde{\varphi}(x'))(t')|}{d(x,x')d(t,t')} \\ &= \sup_{x \neq x'} \frac{Lip(\widetilde{\varphi}(x) - \widetilde{\varphi}(x'))}{d(x,x')} = Lip(\widetilde{\varphi}). \end{aligned}$$

It is easy to see that T_1 is an extension of T to $X \times Y_0$. Using a similar procedure for the spaces $X^{\#}$ and Y_0 we obtain the desired extension $\widetilde{T}$ of the two-Lipschitz T to $X \times Y$. Moreover, we have

$$BLip(\widetilde{T}) = BLip(T_1) = Lip(\widetilde{\varphi}) = Lip(\varphi) = BLip(T).$$

$\square$

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References

[1] D. Achour, E. Dahia, and M. A. S. Saleh. Multilinear mixing operators and Lipschitz mixing operator ideals, Operators and Matrices 12, no. 4 (2018), 903–931.

[2] D. Achour, E. Dahia, and P. Turco, The Lipschitz injective hull of Lipschitz operator ideals and applications, Banach J. Math. Anal. 14 (2020), 1241–1257.

[3] D. Achour, E. Dahia, and P. Turco, Lipschitz p-compact mappings, Monatshefte für Mathematik 189 (2019), 595–609.

[4] D. Achour, P. Rueda, E. A. Sánchez-Pérez, and R. Yahi, Lipschitz operator ideals and the approximation property, J. Math. Anal. Appl. 436 (2016), 217–236.

[5] R. F. Arens and J. Jr. Eells, On embedding uniform and topological spaces, Pacific J. Math 6 (1956), 397–403.

[6] Y. Benyamini and J. Lindenstrauss. Geometric nonlinear functional analysis, vol. 1, Amer. Math. Soc. Colloq. Publ., vol. 48, Amer. Math. Soc., Providence, RI, 2000.

[7] E. Dahia, and K. Hamidi, Lipschitz integral operators represented by vector measures, Appl. Gen. Topol. 22, no. 2 (2021), 367–383.

[8] A. Defant and K. Floret, Tensor norms and operator ideals, North-Holland, Amsterdam, 1992.

[9] M. Dubei, E. D. Tymchatynb, and A. Zagorodnyuk, Free Banach spaces and extension of Lipschitz maps, Topology 48 (2009), 203–212.

[10] K. Hamidi, E. Dahia, D. Achour, and A. Tallab, Two-Lipschitz operator ideals, J. Math. Anal. Appl. 491 (2020), Paper no. 124346.
[11] T. L. Hayden, The extension of bilinear functionals, Pacific J. Math. 22 (1967), 99ÔÇô108.
[12] M. S. Ramanujan, and E. Schock, Operator ideals and spaces of bilinear operators, Lin. Multilin. Alg. 18 (1985), 307–318.
[13] D. Ruch, Characterizations of compact bilinear maps, Lin. Multilin. Alg. 25 (1989), 297–307.
[14] E. A. Sánchez-Pérez, Product spaces generated by bilinear maps and duality, Czechoslovak Mathematical Journal 65 (2015), 801–817.
[15] I. Sawashima, Methods of Lipschitz duals, Lecture Notes in econom. and Math. systems, vol. 419, Springer Verlag, 1975, pp. 247–259.
[16] N. Weaver. Lipschitz Algebras, World Scientific Publishing Co., Singapore, 1999.