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Partial actions on quotient spaces and globalization

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Abstract

Given a partial action of a topological group G on a topological space X we determine properties $\mathcal{P}$ which can be extended from X to its globalization. We treat the cases when $\mathcal{P}$ is any of the following: Hausdorff, regular, metrizable, second countable and having invariant metric. Further, for a normal subgroup H, we introduce and study a partial action of G/H on the orbit space $X/\sim_H$, applications to invariant metrics and inverse limits are presented.

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1. Introduction

Given an action $a : G \times Y \to Y$ of a group G on a space Y and an invariant subset X of Y (i.e. $a(g, x) \in X$, for any $g \in G$, $x \in X$), the restriction of a to $G \times X$ is an action of G over X. If X is not invariant, we get what is called a *partial action* on X, that is a collection of partial maps $\{\eta_g : g \in G\}$, on X satisfying $\eta_1 = \mathrm{id}_X$ and $\eta_g \circ \eta_h \subseteq \eta_{gh}$, for each $g, h \in G$. The notion

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of partial group action appeared in the context of C^*-algebras in [8], there C^*-algebraic crossed products by partial automorphisms played an important role to analyze and characterize their internal structure. After the work [8], partial actions have been spreading in several branches of mathematics, for a detailed account on partial actions the interested reader may consult [5] or [9]. A relevant question is if a partial action can be obtained by restriction of a corresponding collection of maps on some superspace. In the topological context, this is known as the globalization problem and was studied in [1] and independently in [12]. It was proven that for any partial action η of a topological group G on a topological space X there is a topological superspace Y of X and a continuous action μ of G on Y such that the restriction of μ to X is η. Such a space is called a globalization of X. It is also shown that there is a minimal globalization X_G called the enveloping space of X.

We shall mainly work with partial actions for which the partial maps have clopen domains, that is closed and open, this kind of partial actions were considered in [6] where the authors studied the ideal structure of the algebraic partial crossed product $\mathcal{L}_c(X) \rtimes G$, being $\mathcal{L}_c(X)$ the algebra consisting of all locally constant, compactly supported functions on X, while in [10] the authors showed that partial actions on the Cantor set by clopen subsets are exactly the ones for which the enveloping space is Hausdorff, also in [3] partial actions with clopen domains were relevant to introduce and study topological entropy for a partial action of $\mathbb{Z}$ on metric spaces, and in [11] the authors studied topological dynamics arising from partial actions on clopen subsets of a compact space.

Our work is organized in the following way: After the introduction, in Section 2 we present some notions, examples and results that will be useful during the work, especially Proposition 2.8 gives conditions for the enveloping space to be T_1, while Theorem 2.12 establishes that the globalization of a partial action is actually an orbit space. At the beginning of Section 3, we treat the question if a structural property $\mathcal{P}$ of a space X endowed with a partial action of a group G is inherited by the spaces $X/\sim_G$ and X_G (see equations (2.6) and (2.2) for the proper definitions of $X/\sim_G$ and X_G, respectively). To do that, we first show in Lemma 3.1 that the quotient map π_G defined in (2.7) is perfect, this allows us to present in Theorem 3.2 sufficient conditions in which an affirmative answer holds for when $\mathcal{P}$ is any properties of being Hausdorff, regular, metrizable and second countable. The second part of Section 3 deals with invariant metrics, there we give in Theorem 3.10 a condition for a space X with a partial action of a compact group so that it admits an invariant metric. It is important to remark that, in the classical case, the problem of finding characterizations of G-spaces having invariant metrics have been extensively studied, in particular it is known that if a space X with a global action admits an invariant metric, then the orbit space $X/\sim_G$ is metrizable provided that is T_1. However, this result does not hold for partial actions, where one needs to impose regularity conditions (see Remark 3.11 and Proposition 3.12, respectively). In Section 4 we take a partial action η of G on a space X, a normal subgroup H of G and we show in Theorem 4.1 how to construct a partial action $\eta_{G/H}$ of G/H

on the orbit space $X/\sim_H$, moreover, in the same theorem is shown that the orbit spaces $(X/\sim_H)/\sim_{G/H}$ and $X/\sim_G$ are homeomorphic. The structure of the partial action $\eta_{G/H}$ as well as its globalization are presented in Theorem 4.2. As an application for the construction of $\eta_{G/H}$ we treat in Proposition 4.9 partial actions on inverse limits, where we provide suitable conditions for which a space X is G-equivalent to an inverse limit $\varprojlim X_i$, and such that the partial action on X satisfies a compatibility relation with the partial actions associated to X_i.

Throughout the work, several examples are shown to clarify the notions and results.

2. Preliminaries

Let G be a group with identity element 1, X be a set, and $\eta : G \times X \to X$, $(g, x) \mapsto g \cdot x$ be a partially defined function, that is, a function whose domain, denoted by $G * X$, is contained in $G \times X$. We shall write $\exists g \cdot x$ to mean that (g, x) belongs to $G * X$. We say that η is a *partial action* of G on X if for each $g, h \in G$ and $x \in X$ the following assertions hold:

(PA1) If $\exists g \cdot x$, then $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$;

(PA2) If $\exists g \cdot (h \cdot x)$, then $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$;

(PA3) $\exists 1 \cdot x$ and $1 \cdot x = x$.

For $g \in G$, we set $X_g = \{x \in X \mid \exists g^{-1} \cdot x\}$. Then η induces a family of bijections $\{\eta_g \colon X_{g^{-1}} \ni x \mapsto g \cdot x \in X_g\}_{g \in G}$. We also denote this family by η. Notice that η *acts* (globally) on X if $\exists g \cdot x$, for all $(g, x) \in G \times X$, or equivalently, $X_g = X$, for any $g \in G$. The following result characterizes partial actions in terms of a family of bijections:

Proposition 2.1 ([16, Lemma 1.2])**.** *A partial action η of G on X is a family $\eta = \{\eta_g \colon X_{g^{-1}} \to X_g\}_{g \in G}$, where $X_g \subseteq X$, $\eta_g \colon X_{g^{-1}} \to X_g$ is bijective, for all $g \in G$, and:*

(i) $X_1 = X$ *and* $\eta_1 = \mathrm{id}_X$;

(ii) $\eta_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}$;

(iii) $\eta_g \eta_h \colon X_{h^{-1}} \cap X_{h^{-1}g^{-1}} \to X_g \cap X_{gh}$, *and* $\eta_g \eta_h = \eta_{gh}$ *in* $X_{h^{-1}} \cap X_{h^{-1}g^{-1}}$;

for all $g, h \in G$.

Definition 2.2. Let G be a topological group and X be a topological space. A *topological partial action* of G on X is a partial action $\eta = \{\eta_g \colon X_{g^{-1}} \to X_g\}_{g \in G}$ on the underlying set X such that X_g is open and η_g is a homeomorphism, for any $g \in G$. Moreover, we say that η is continuous if $\eta : G * X \to X$ is continuous, where $G \times X$ has the product topology and $G * X$ is endowed with the relative topology.

Throughout this paper G will denote a Hausdorff topological group, X a topological space and all partial actions will be topological.

Now we present an example of a continuous and topological partial action that will be useful in Section 3. We endow $\mathbb{Z}$ with the *p-adic topology* $\mathcal{T}_p$, where p is a prime number. For the reader's convenience we recall its construction here. See [17, Example 1.18] for details. The family $\mathcal{V} = \{p^k\mathbb{Z}\}_{k\in\mathbb{Z}^+}$ satisfies the conditions given in [17, Theorem 1.13], then $\mathcal{B} = \{m + p^k\mathbb{Z} : m \in \mathbb{Z},\ k \in \mathbb{Z}^+\}$ is a basis for the topology $\mathcal{T}_p$ and $(\mathbb{Z}, +, \mathcal{T}_p)$ is a topological group.

Example 2.3. Let X be a disconnected topological space, $U \subseteq X$ be a proper clopen set, $f : U \to U$ be a homeomorphism and $n \in \mathbb{Z}$. We set $f^0 = \mathrm{id}_U$, if $n \in \mathbb{Z}^+$ we write f^n as n-times the composition of f with itself, and if $n < 0$ then $f^n = (f^{-1})^{-n}$. We define a partial action η of $\mathbb{Z}$ on X by setting $\mathbb{Z} * X = (\mathbb{Z} \times U) \cup (\{0\} \times X)$ and

$$\eta : \mathbb{Z} * X \to X, (n, a) \mapsto \begin{cases} f^n(a), \text{ if } a \in U, \\ a, \text{ if } n = 0 \text{ and } a \notin U. \end{cases} \tag{2.1}$$

Suppose there is a prime number p such that $f^p = \mathrm{id}_U$, and consider $\mathbb{Z}$ with the p-adic topology. Since U is open, then η is a topological partial action. To show that η is continuous, take $(n, x) \in \mathbb{Z} * X$ and let $V \subseteq X$ be an open set such that $\eta(n, x) \in V$. There are two cases to consider:

Case 1: If $x \in U$, then $\eta(n, x) = f^n(x) \in V$. Since $V \cap U$ is open in U, there is an open set $Z \subseteq U$ such that $f^n(Z) \subseteq V \cap U$. First, we suppose that p does not divide $|n|$, then we affirm that the open set $W = [(n + p\mathbb{Z}) \times Z] \cap (\mathbb{Z} * X) \subseteq \mathbb{Z} * X$ satisfies $\eta(W) \subseteq V$. Indeed, given $(t, y) \in W$ we have $y \in Z \subseteq U$ and there is $m \in \mathbb{Z}$ such that $t = n + pm$. Note that $t \neq 0$ since p does not divide $|n|$. Further, since $y \in U$ we get $(n, y) \in \mathbb{Z} * X$, and $(pm, f^n(y)) \in \mathbb{Z} * X$, then the following equalities are valid:

$$\eta(t, y) = f^t(y) = f^{pm}(f^n(y)) = f^n(y) \in V.$$

Now, if p divides $|n|$ we let $i = \max\{k \in \mathbb{Z}^+ : p^k \text{ divides } |n|\}$. Consider the open set $W = [(n + p^{i+1}\mathbb{Z}) \times Z] \cap (\mathbb{Z} * X)$. Then for $(t, y) \in W$, by the maximality of i there is $m \in \mathbb{Z}$ such that $t = n + p^{i+1}m$, $y \in Z \subseteq U$, and $t \neq 0$. Since $y \in U$, we get the following:

$$\eta(t, y) = f^{n+p^{i+1}m}(y) = f^{p^{i+1}m}(f^n(y)) = f^n(y) \in V.$$

We conclude that $\eta(W) \subseteq V$.

Case 2: If $x \notin U$, by (2.1) we have $n = 0$ and $\eta(n, x) = x \in V$. Since U is closed, then $Z = V \cap (X \setminus U)$ is an open subset of X containing x. Observe that $(n, x) = (0, x) \in W = (p\mathbb{Z} \times Z) \cap (\mathbb{Z} * X)$. Moreover, $\eta(W) \subseteq V$ because, if $(t, y) \in W$, then $y \notin U$ and $t = 0$, from this we get $\eta(t, y) = \eta(0, y) = y \in V$, showing that η is continuous.

2.1. **On the enveloping space.** Partial actions can be induced from global ones as the following example shows:

Example 2.4. (Induced partial action) Let $u \colon G \times Y \to Y$ be a continuous action of G on a topological space Y and $X \subseteq Y$ an open set. For each $g \in G$, set $X_g = X \cap u_g(X)$ and let $\eta_g = u_g \restriction X_{g^{-1}}$. Then $\eta \colon G * X \ni (g, x) \mapsto$

$\eta_g(x) \in X$ is a continuous and topological partial action of G on X. In this case we say that η is *induced* by u.

Remark 2.5. Given a continuous global action η of G on X, its induced partial action on an open (resp. closed) subset Y of X has open (resp. closed) domain in $G \times Y$.

An important problem on partial actions is whether they can be induced by global actions. In the topological sense, this turns out to be affirmative and a proof was presented in [1, Theorem 1.1] and independently in [12, Section 3.1]. For the reader's convenience, we recall their construction. Let η be a partial action of G on X. Define an equivalence relation on $G \times X$ as follows:

$$(g, x)R(h, y) \iff x \in X_{g^{-1}h} \text{ and } \eta_{h^{-1}g}(x) = y, \tag{2.2}$$

and denote by $[g, x]$ the equivalence class of (g, x). Consider $X_G = (G \times X)/R$ with the quotient topology, then the following map:

$$\mu\colon G \times X_G \ni (g, [h, x]) \mapsto [gh, x] \in X_G, \tag{2.3}$$

is a well defined action, and the map

$$\iota\colon X \ni x \mapsto [1, x] \in X_G, \tag{2.4}$$

is injective.

Definition 2.6. Let η be a partial action of G on X. The action μ defined in (2.3) is called the enveloping action of η and X_G is the enveloping space or globalization of X.

In the next result we summarize some basic results about the enveloping space and the enveloping action. See [1, Theorem 1.1], [12, Theorem 3.13] and [12, Proposition 3.9].

Proposition 2.7. *Let η be a partial action of G on X. Then the following assertions hold:*

(i) *The maps μ and ι are continuous,*
(ii) *If η is continuous and $G * X$ is open in $G \times X$, then ι is an open map,*
(iii) *The quotient map*

$$q : G \times X \ni (g, x) \mapsto [g, x] \in X_G, \tag{2.5}$$

is continuous and open.

Now we provide conditions for X_G to be T_1.

Proposition 2.8. *Let η be a continuous partial action of G on X. Consider the following assertions:*

(i) *$G * X$ is closed;*
(ii) *For any $x \in X$ the set $G^x = \{g \in G \mid \exists g \cdot x\}$ is closed;*
(iii) *X_G is T_1.*

Then (i)$\Rightarrow$ (ii), and (ii)$\Rightarrow$(iii) provided that X is Hausdorff.

Proof. (i)$\Rightarrow$ (ii): Let $x \in X$ and $(g_\lambda)_{\lambda\in\Lambda}$ be a net in G^x such that $\lim g_\lambda = g$, for some $g \in G$. Then $(g_\lambda, x)_{\lambda\in\Lambda} \to (g,x) \in \overline{G * X} = G * X$, thus $g \in G^x$ and G^x is closed.

For the rest of the proof we assume that X is Hausdorff.

(ii)$\Rightarrow$ (iii): Take $(g,x) \in G \times X$, and let q be the quotient map defined in (2.5), then the following equalities are valid:

$$q^{-1}(\{[g,x]\}) = \bigcup_{l\in G}\{gl^{-1}\} \times \eta_l(\{x\} \cap X_{l^{-1}}) = \{(gl^{-1}, l\cdot x) \mid l \in G^x\}.$$

We prove that $q^{-1}(\{[g,x]\})$ is closed. For this let $(h,y) \in \overline{q^{-1}(\{[g,x]\})}$, then there exists a net $\{l_i\}_{i\in I}$ in G^x such that $(gl_i^{-1}, l_i\cdot x) \to (h,y)$, in particular $l_i \to h^{-1}g \in G^x$. We set $\eta^x : G^x \ni g \mapsto g\cdot x \in X$, using the fact that η^x is continuous one gets $l_i \cdot x \to (h^{-1}g)\cdot x$, and $y = (h^{-1}g)\cdot x$ due to the uniqueness of limits in Hausdorff spaces. From this we obtain the next valid assertion:

$$(h,y) = (g(h^{-1}g)^{-1}, (h^{-1}g)\cdot x) \in q^{-1}(\{[g,x]\}),$$

thus X_G is T_1. ☐

Remark 2.9. With respect to Proposition 2.8 we have the following comments:

- The space X_G is T_1 when G is discrete and X is Hausdorff.
- Part (ii) $\Rightarrow$ (i) does not necessarily hold. Indeed, for the partial action of $\mathbb{Z}_2 = \{1,-1\}$ on $X = [0,1]$ presented in [1, Example 1.4.] that is $\alpha_1 = \mathrm{id}_X$ and $\alpha_{-1} = \mathrm{id}_V$, where $V = (0,1]$. One has that $\mathbb{Z}_2^x$ is closed for any $x \in [0,1]$ while $\mathbb{Z}_2 * [0,1] = \{(1,0)\} \cup (\mathbb{Z}_2 \times V)$ is not closed in $\mathbb{Z}_2 \times [0,1]$.
- Also part (iii) $\Rightarrow$ (ii) does not hold in general, for this let $G = GL(2;\mathbb{R})$ be the general linear group of degree 2 acting partially on $\mathbb{R}$ as follows: For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, set $\mathbb{R}_{g^{-1}} = \{x \in \mathbb{R} : cx + d \neq 0\}$ and consider $\eta_g : \mathbb{R}_{g^{-1}} \ni x \mapsto \dfrac{ax+b}{cx+d} \in \mathbb{R}_g$. There is a homeomorphism from $\mathbb{R}_G$ to the space $\mathbb{C}$ of complex numbers, then $\mathbb{R}_G$ is Hausdorff but $G^0 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : d \neq 0\right\}$ is not closed in G.

Definition 2.10. Let X, Y be topological spaces and $\eta : G * X \to X, \rho : G * Y \to Y$ be partial actions. A continuous map $f : X \to Y$ is called a G-map if $(g, f(x)) \in G * Y$ and $f(\eta(g,x)) = \rho(g, f(x))$, for every $(g,x) \in G * X$. If f is a homeomorphism and f^{-1} is a G-map, then X and Y are called G-equivalent.

Proposition 2.11. *The following assertions hold:*

(i) *Let X and Y be topological spaces equipped with partial actions of G. If X and Y are G-equivalent, then X_G and Y_G are homeomorphic, as well as $G * X$ and $G * Y$,*

(ii) *Let $\beta : G \times Y \to Y$ be a continuous action of G on a space Y. Let $X \subseteq Y$ be an open set such that $G \cdot X = Y$ and $\eta : G * X \to X$ be the* *induced partial action of β on X (see Example 2.4). Then the spaces X_G and Y are G-equivalent.*

Proof. Part (i) is clear. For (ii), let $i : G \times X \to G \times Y$ be the inclusion and $\alpha : X_G \ni [g, x] \mapsto \beta(g, x) \in Y$, then the following diagram:

$$\begin{array}{ccc} G \times X & \xrightarrow{\ i\ } & G \times Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \beta} \\ X_G & \xrightarrow[\alpha]{} & Y, \end{array}$$

is commutative. By [12, Proposition 3.5] the map α is a well defined bijection, moreover, it is continuous because the map $\alpha \circ q$ is continuous. Since β is open, then the map α is a homeomorphism. Finally, notice that α is a G-map. □

2.2. **The orbit equivalence relation.** Given a partial action η of G on X the *orbit equivalence relation* $\sim_G$ on X is:

$$x \sim_G y \iff \exists g \in G^x \text{ such that } g \cdot x = y, \tag{2.6}$$

for each $x, y \in X$. The orbit space of X is $X/\sim_G$ endowed with the quotient topology. The elements of $X/\sim_G$ are the *orbits* $G^x \cdot x$, for each $x \in X$. It was shown in [15, Lemma 3.2] that the projection map

$$\pi_G : X \ni x \mapsto G^x \cdot x \in X/\sim_G, \tag{2.7}$$

is continuous and open.

It is known that globalizations of topological spaces endowed with a partial action can be seen as orbit spaces. Indeed, the following result was shown in [15, Theorem 3.3]:

Theorem 2.12. *Let η be a topological partial action of G on X, then the family $\hat{\eta} = \{\hat{\eta}_g : (G \times X)_{g^{-1}} \to (G \times X)_g\}_{g \in G}$, where $(G \times X)_g = G \times X_g$ and*

$$\hat{\eta}_g : G \times X_{g^{-1}} \ni (h, x) \mapsto (hg^{-1}, \eta_g(x)) \in G \times X_g,$$

is a topological partial action of G on $G \times X$, and the enveloping space X_G of η is the orbit space of $G \times X$ by $\hat{\eta}$.

Let η be a partial action of G on X, and H be a subgroup of G, then the family $\eta_H : \{\eta_h : X_{h^{-1}} \to X_h\}_{h \in H}$ is a partial action of H on X. The corresponding orbit equivalence relation of η_H is denoted by $\sim_H$.

For convenience, the orbits in the space $X_G/\sim_H$ will be denoted by $H[g, x]$, for any $[g, x] \in X_G$. We finish this section with the next lemma:

Lemma 2.13. *Let η be a continuous partial action of G on X with $G * X$ open. Then for each subgroup H of G, the next map:*

$$\varphi : X/\sim_H \ni H^x \cdot x \mapsto H[1, x] \in X_G/\sim_H, \tag{2.8}$$

is continuous, open and injective, hence φ is an embedding.

Proof. Observe that φ is well defined. Indeed, take $x, y \in X$ such that $x \sim_H y$ and let $h \in H^x$ such that $\eta_h(x) = y$. Thus, $[1, y] \overset{(2.2)}{=} [h, x] \overset{(2.3)}{=} \mu_h([1, x])$, therefore $[1, y] \sim_H [1, x]$, then φ is well defined. It is easy to check that φ is injective. To prove that φ is continuous, consider the corresponding projection maps $\pi_H : X \to X/\sim_H$ and $\Pi_H : X_G \to X_G/\sim_H$. Since the map ι defined in (2.4) is continuous and $\varphi \circ \pi_H = \Pi_H \circ \iota$, we conclude that φ is continuous. It remains to check that φ is open. Let $U \subseteq X/\sim_H$ be an open set, then $\varphi(U) = \Pi_H(\iota(\pi_H^{-1}(U)))$ is open because $\pi_H^{-1}(U)$ is open in X and the maps ι and Π_H are open thanks to Proposition 2.7 and [15, Lemma 3.2], respectively. Therefore φ is an open map. ☐

3. Properties preserved by the enveloping space

We recall that a continuous surjection $f : X \to Y$ is *perfect* if it is closed and $f^{-1}(\{y\})$ is compact, for all $y \in Y$.

We proceed with the next lemma:

Lemma 3.1. *Let $\eta : G * X \to X$ be a continuous partial action such that $G * X$ is closed in $G \times X$ and G is compact, then the following assertions hold:*

(i) *η is closed;*
(ii) *The maps π_G and $\hat{\pi}_G$ are perfect, being $\hat{\pi}_G$ the quotient map induced by the partial action $\hat{\eta}$, defined in Theorem 2.12.*

Proof. (i) Let C be a nonempty closed subset of $G * X$ and $y \in \overline{\eta(C)}$, then there is a directed set Λ and a net $(g_\lambda, x_\lambda)_{\lambda\in\Lambda}$ in C such that $\lim g_\lambda \cdot x_\lambda = y$. Since G is compact, we can suppose that $\lim g_\lambda = g$, for some $g \in G$. Notice that $(g_\lambda^{-1}, g_\lambda \cdot x_\lambda)_{\lambda\in\Lambda}$ is a net in $G * X$ and $\lim(g_\lambda^{-1}, g_\lambda \cdot x_\lambda) = (g^{-1}, y)$, then $(g^{-1}, y) \in G * X$ because $G * X$ is a closed subset of $G \times X$. Now consider the net $(g_\lambda, x_\lambda)_{\lambda\in\Lambda} = (g_\lambda, g_\lambda^{-1} \cdot (g_\lambda \cdot x_\lambda))_{\lambda\in\Lambda}$ in C, then the next assertion is true:

$$(g, g^{-1} \cdot y) = \lim(g_\lambda, g_\lambda^{-1} \cdot (g_\lambda \cdot x_\lambda)) = \lim(g_\lambda, x_\lambda) \in C,$$

so $y = g \cdot (g^{-1} \cdot y) = \eta(g, g^{-1} \cdot y) \in \eta(C)$, which implies that η is a closed map. (ii) The map π_G is closed because of (i) above and the equality $\pi_G^{-1}(\pi_G(F)) = \eta((G\times F)\cap(G*X))$, for any closed subset F of X. To prove our assertion we need to check that $\pi_G^{-1}(\pi_G(x))$ is compact, for any $x \in X$. By Proposition 2.8 we have that G^x is a compact subset G, then $\pi_G^{-1}(\pi_G(x)) = G^x \cdot x = \eta(G^x \times \{x\})$ is a compact subset of X. To show that $\hat{\pi}_G$ is closed we have by [14, Proposition 2.6] that $\hat{\eta}$ is continuous, moreover, from [14, Corollary 3.3] we get that $G*(G\times X)$ is closed in $G \times (G \times X)$, then the result follows. ☐

Theorem 3.2. *Let G be a compact group and $\eta : G * X \to X$ be a continuous partial action such that $G * X$ is closed in $G \times X$. Let $\mathcal{P}$ be any of the properties: Hausdorff, regular, metrizable and second countable. Then the following statements hold:*

(i) *If X is $\mathcal{P}$, then $X/\sim_G$ is $\mathcal{P}$,*
(ii) *If $G \times X$ is $\mathcal{P}$, then X_G is $\mathcal{P}$.*

Proof. (i): This follows from ítem (ii) in Lemma 3.1 and [7, Theorem 5.2], while (ii) is a consequence of ítem (ii) in Lemma 3.1, ítem (i) above, and the last assertion in Theorem 2.12. □

Remark 3.3. We remark the following facts:

- (i) In general the assumption that $G * X$ is closed in $G \times X$ cannot be removed in part (ii) of Theorem 3.2. Indeed, for the Abadie's partial action of $\mathbb{Z}_2 = \{1, -1\}$ on $X = [0, 1]$ presented in Remark 2.9, we have by Proposition 2.8 that the space $X_{\mathbb{Z}_2}$ is T_1 but not Hausdorff.
- (ii) Also, the fact that X_G is Hausdorff does not imply that G is compact, for instance, in [10, Proposition 2.1] a characterization for X_G to be Hausdorff is presented in the case when G is countable and discrete.

We illustrate the previous theorem with some examples.

Example 3.4. Consider $X = \mathbb{R} \setminus \{0\}$ as a subspace of $\mathbb{R}$. A partial action of $\mathbb{Z}_3$ on X is defined as follows: Let $X_1 = (-\infty, 0)$ and $X_2 = (0, \infty)$. Note that X_1 and X_2 are clopen subsets of X such that $X = X_1 \cup X_2$. Set $\eta_0 = \mathrm{id}_X$, $\eta_2 : X_1 \ni x \mapsto -x \in X_2$ and $\eta_1 = \eta_2^{-1}$, moreover, let

$$\mathbb{Z}_3 * X = (\{0\} \times X) \cup (\{1\} \times X_2) \cup (\{2\} \times X_1),$$

then $\eta : \mathbb{Z}_3 * X \to X$ is a partial action of $\mathbb{Z}_3$ on X such that $\mathbb{Z}_3 * X$ is clopen in $\mathbb{Z}_3 \times X$, thus by Theorem 3.2 the enveloping space $X_{\mathbb{Z}_3}$ is metrizable.

Example 3.5. Let X be a disconnected space and $U \subseteq X$ be a non-empty clopen subset of X with $U \neq X$. Then $\eta : \mathbb{Z}_2 * X \to X$ is a partial action of $\mathbb{Z}_2$ on X, where $\mathbb{Z}_2 * X = (\{0\} \times X) \cup (\{1\} \times U)$, and $\eta(1, u) = u$ for any $u \in U$. Since $\mathbb{Z}_2 * X$ is closed in $\mathbb{Z}_2 \times X$ we conclude that $X_{\mathbb{Z}_2}$ is metrizable.

In view of (ii) in Remark 3.3 we give the next proposition:

Proposition 3.6. *Let G be a compact group, X be a compact Hausdorff space and $\eta : G * X \to X$ be a partial action. If X_G is Hausdorff, then $G * X$ is closed.*

Proof. Let $\{(g_\lambda, x_\lambda)\}_{\lambda \in \Lambda}$ be a net in $G * X$ such that $\lim(g_\lambda, x_\lambda) = (g, x) \in G \times X$. Since X_G is Hausdorff, we have by [1, Proposition 1.2] that the space $\mathrm{Graph}(\eta) = \{(g, x, y) \in G \times X \times X : (g, x) \in G * X,\ g \cdot x = y\}$ is a closed subset of $G \times X \times X$, and thus compact. Therefore we may assume that $(g_\lambda, x_\lambda, g_\lambda \cdot x_\lambda)_{\lambda \in \Lambda}$ converges to $(g, x, p) \in \mathrm{Graph}(\eta)$, for some $p \in X$. In particular, $(g, x) \in G * X$ and $G * X$ is closed. □

Having at hand Proposition 3.6 one may ask if its assumptions imply that if the orbit space $X/{\sim_G}$ is Hausdorff then $G * X$ is closed in $G \times X$. But this is not the case as Example 3.7 below shows:

Example 3.7. Consider again the partial action η of $\mathbb{Z}_2$ on $X = [0, 1]$ given in [1, Example 1.4.]. We observed in Remark 2.9 that $\mathbb{Z}_2 * X$ is not closed in $\mathbb{Z}_2 \times X$. Moreover, since $\eta(1, x) = x$ for any $x \in (0, 1]$, we have $\pi_{\mathbb{Z}_2} : X \to X/{\sim_{\mathbb{Z}_2}}$ is injective and thus a homeomorphism and $X/{\sim_{\mathbb{Z}_2}}$ is Hausdorff.

3.1. **Invariant metrics.** Let $\eta : G * X \ni (g, x) \mapsto g \cdot x \in X$ be a partial action of G on the metric space (X, ρ). We say that ρ is *η-invariant* if for any $g \in G$ and $x, y \in X_{g^{-1}}$, $\rho(g \cdot x, g \cdot y) = \rho(x, y)$.

Example 3.8. Let η be as in equation (2.1). Suppose that X is a metrizable space and ρ is a compatible metric on X. If f is an isometry, then ρ is a η-invariant metric in any of the following cases:

(i) $\mathbb{Z}$ is considered as a discrete space,
(ii) $\mathbb{Z}$ is endowed with the p-adic topology and $f^p = \mathrm{id}_U$, for some prime number p.

In the context of hyperspaces endowed with partial actions we give the next example:

Example 3.9. Let $\eta : G * X \ni (g, x) \mapsto g \cdot x \in X$ be a continuous partial action of G on a compact metric space (X, d). Denote by 2^X the hyperspace of nonempty compact subsets of X endowed with the Hausdorff metric d_H, which is defined by the next rule:

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N(B, \varepsilon) \text{ and } B \subseteq N(A, \varepsilon)\},$$

where $A, B \in 2^X$ and $N(A, \varepsilon) = \bigcup_{a \in A} B_d(a, \varepsilon)$. It follows by [13, Theorem 3.2] that $2^\eta : G * 2^X \ni (g, A) \mapsto \eta_g(A) \in 2^X$, is a continuous partial action of G on 2^X, where

$$G * 2^X = \{(g, A) \in G \times 2^X : (g, a) \in G * X \ (\forall a \in A)\}.$$

Suppose that d is η-invariant. We observe that d_H is 2^η-invariant. For this take $g \in G$ and $A, B \in 2^X$ for which $(g, A), (g, B) \in G * 2^X$. Let $\epsilon > 0$ with $A \subseteq N(B, \varepsilon)$ and $B \subseteq N(A, \varepsilon)$. Now, given $a \in A$ there exists $b \in B$ such that $a \in B_d(b, \varepsilon)$, then $d(g \cdot a, g \cdot b) = d(a, b) < \varepsilon$ and we have proven that $\eta_g(A) \subseteq N(\eta_g(B), \varepsilon)$. In a similar way one shows that $\eta_g(B) \subseteq N(\eta_g(A), \varepsilon)$, therefore, $d_H(\eta_g(A), \eta_g(B)) \leq \varepsilon$, and $d_H(\eta_g(A), \eta_g(B)) \leq d_H(A, B)$.

On the other hand, take $\varepsilon > 0$ such that $\eta_g(A) \subseteq N(\eta_g(B), \varepsilon)$ and $\eta_g(B) \subseteq N(\eta_g(A), \varepsilon)$. For $a \in A$ choose $b \in B$ such that $g \cdot a \in B_d(g \cdot b, \varepsilon)$, then $d(a, b) = d(g \cdot a, g \cdot b) < \varepsilon$ and $A \subseteq N(B, \varepsilon)$, again one verifies $B \subseteq N(A, \varepsilon)$ which implies $d_H(A, B) \leq d_H(\eta_g(A), \eta_g(B))$, hence $d_H(A, B) = d_H(\eta_g(A), \eta_g(B))$, as desired.

It follows from [2, Proposition 5] that there is a compatible η-invariant metric for X provided that η is global and G is countably compact. The following theorem is a version of this result for partial actions:

Theorem 3.10. *Let $\eta : G * X \to X$ be a continuous partial action, then X and X_G are metrizable by an invariant metric under any of the following conditions:*

(i) *G is countably compact and X_G is metrizable,*
(ii) *G is compact and first countable, X is metrizable and $G * X$ is closed.*

Moreover if (i) holds and $X_G / \sim G$ is T_1, then $X / \sim_G$ is metrizable.

Proof. In both cases it is enough to prove that X_G has a compatible μ-invariant metric ρ, where μ is given by (2.3). Indeed, since η is continuous we have by [12, Proposition 3.12] that $\iota : X \to \iota(X)$ is a homeomorphism, where ι is given by (2.4), thus one obtains an η-invariant metric for X by restricting ρ to $\iota(X)$.

(i) Since μ is continuous, the result follows from [2, Proposition 5].

(ii) In this case the space $G \times X$ is metrizable, thus X_G is metrizable thanks to Theorem 3.2 and again the result follows from [2, Proposition 5]. To show the last assertion, we observe that X_G admits an invariant metric, then the result follows from [4, Theorem 2.16] and Lemma 2.13. □

Remark 3.11. It is known that when G acts globally on a space X that admits an invariant metric, then the space $X/\!\sim_G$ is metrizable provided that it is T_1, however this does not hold for partial actions. For a concrete example take the partial action given in Remark 2.9 and use Theorem 2.12 and Remark 3.3.

The following result tells us that one needs to impose the regularity condition on $X/\sim_G$:

Proposition 3.12. *Let X be a second countable space endowed with a partial action of G, then the following assertions are equivalent:*

(i) *$X/\!\sim_G$ is metrizable;*
(ii) *$X/\!\sim_G$ is regular and T_1.*

Proof. Clearly (i) implies (ii). To see (ii) implies (i), notice that $X/\!\sim_G$ is second countable because the quotient map π_G is open. Therefore, by Urysohn's metrization Theorem, the space $X/\sim_G$ is metrizable. □

4. Partial actions on orbit spaces

Let η be a partial action of G on X and H be a normal subgroup of G. We shall construct a partial action of G/H on $X/\!\sim_H$. If η is a global action, then G/H acts globally on $X/\!\sim_H$ via

$$\eta_{gH}(H \cdot x) = H \cdot (g \cdot x), \tag{4.1}$$

for any $g \in G$ and $x \in X$.

For the case of partial action, we notice that mimicking the construction above does not yield to a partial action of G/H on $X/\sim_H$ because it is not natural how to define the set $G/H * (X/\!\sim_H)$. Indeed, the construction of such partial action is essentially more laborious than the global one, as we shall see in the next result:

Theorem 4.1. *Let η be a continuous partial action of G on X and H be a normal subgroup of G. Then there is a continuous partial action $\eta_{G/H}$ of G/H on $X/\sim_H$, such that the orbit spaces $(X/\sim_H)/\sim_{G/H}$ and $X/\sim_G$ are homeomorphic.*

Proof. Let μ be the globalization of η. Then μ is continuous and by (4.1) it induces a continuous action $\mu_{G/H}$ on $X_G/\!\sim_H$ as follows:

$$\mu_{gH} : X_G/\!\sim_H \ni H[t,x] \mapsto H[gt,x] \in X_G/\!\sim_H,$$

for each $gH \in G/H$. Now, let φ be defined by (2.8). By Example 2.4 and Lemma 2.13 the map $\mu_{G/H}$ induces a continuous partial action $\eta'_{G/H}$ of G/H on the open set $\mathrm{Im}(\varphi)$ of $X_G/\sim_H$, where $\eta'_{G/H} = \{\eta'_{gH} : X_{g^{-1}H} \to X_{gH}\}_{gH\in G/H}$,

$$X_{gH} = \mu_{gH}(\mathrm{Im}(\varphi)) \cap \mathrm{Im}(\varphi) \quad \text{and} \quad \eta'_{gH} = \mu_{gH} \restriction X_{g^{-1}H}. \tag{4.2}$$

Let $\Omega := X/\sim_H$, then one obtains a partial action $\eta_{G/H}$ of G/H on Ω by setting $\Omega_{gH} = \varphi^{-1}(X_{gH}), g \in G$ and

$$\eta_{gH} : \Omega_{g^{-1}H} \ni x \mapsto \varphi^{-1}(\eta'_{gH}(\varphi(x))) \in \Omega_{gH}. \tag{4.3}$$

Then

$$\eta_{gH}(x) = (\varphi^{-1} \circ \mu_{gH} \circ \varphi)(x), \tag{4.4}$$

for each $x \in \Omega_{g^{-1}H}$ and $g \in G$. The fact that $\eta_{G/H}$ is continuous is straightforward.

Let $\sim_{G/H}$ be the orbit equivalence relation in Ω induced by $\eta_{G/H}$. To finish the proof we show that the spaces $\Omega/\sim_{G/H}$ and $X/\sim_G$ are homeomorphic. Consider the next diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi_G} & X/\sim_G \\ {\scriptstyle \pi_H}\downarrow & & \uparrow{\scriptstyle \psi} \\ \Omega & \xrightarrow[\pi_{G/H}]{} & \Omega/\sim_{G/H}, \end{array}$$

where ψ satisfies

$$\psi(\pi_{G/H}(\pi_H(x))) = \pi_G(x), \tag{4.5}$$

for each $x \in X$. Let us prove that ψ is well defined. Take $x, y \in X$ such that $\pi_{G/H}(\pi_H(x)) = \pi_{G/H}(\pi_H(y))$. Then there is $g \in G$ with

$$\pi_H(y) = \eta_{gH}(\pi_H(x)) \stackrel{(4.4)}{=} \varphi^{-1}(\mu_{gH}(\varphi(\pi_H(x)))) = \varphi^{-1}(H[g,x]),$$

which implies $H[g,x] = H[1,y]$ and there is $h \in H$ such that $[hg,x] = [1,y]$, thus $\eta_{hg}(x) = y$ and $\pi_G(x) = \pi_G(y)$, which shows that ψ is well defined. Moreover, notice that ψ is continuous and surjective.

Let us prove that ψ is injective. Let $z_1, z_2 \in \Omega/\sim_{G/H}$ such that $\psi(z_1) = \psi(z_2)$, and let $x, y \in X$ with $\pi_{G/H}(\pi_H(x)) = z_1$ and $\pi_{G/H}(\pi_H(y)) = z_2$. Since $\pi_G(x) = \pi_G(y)$, there is $g \in G^x$ satisfying $\eta_g(x) = y$. To prove that $z_1 = z_2$ we need to find $t \in G$ for which $\eta_{tH}(\pi_H(x)) = \pi_H(y)$. We claim that $\eta_{gH}(\pi_H(x)) = \pi_H(y)$. In fact, by (4.4) we get

$$\eta_{gH}(\pi_H(x)) = \varphi^{-1}(\mu_{gH}(\varphi(\pi_H(x)))) = \varphi^{-1}(H[g,x]),$$

and $\varphi(\pi_H(y)) = H[1,y] = H[g,x]$, then $\eta_{gH}(\pi_H(x)) = \pi_H(y)$ and ψ is injective. Let $U \subseteq \Omega/\sim_{G/H}$ be an open set. Since π_G is open, $\pi_G(\pi_H^{-1}(\pi_{G/H}^{-1}(U))) \subseteq X/\sim_G$ is open. Thus $\psi(U)$ is open and $\psi : \Omega/\sim_{G/H} \to X_G/\sim_G$ is a homeomorphism. □

The following result describes explicitly the partial action $\eta_{G/H}$ and its globalization:

Theorem 4.2. *Let η be a continuous partial action of G on X, H be a normal subgroup of G and $\eta_{G/H}$ be the partial action of G/H on $X/\sim_H$ defined above. Then the following assertions hold:*

(i) *For $g \in G$ we have*

$$(X/\sim_H)_{gH} = \{\pi_H(x) \mid \exists h \in H \text{ such that } (hg^{-1}, x) \in G * X\},$$

(ii) *The domain of $\eta_{G/H}$ is*

$$\{(gH, \pi_H(x)) : (g, x) \in G \times X \ \wedge \ \exists h \in H \text{ such that } (hg, x) \in G * X\},$$

(iii) *We have*

$$\eta_{G/H} : G/H * X/\sim_H \ni (gH, \pi_H(x)) \mapsto \pi_H((hg) \cdot x) \in X/\sim_H, \tag{4.6}$$

where $h \in H$ is such that $(hg, x) \in G * X$,

(iv) *The globalization of $\eta_{G/H}$ is (G/H)-equivalent to $X_G/\sim_H$, where G/H acts on $X_G/\sim_H$ via $\mu_{G/H}$.*

Proof. (i) Take $g \in G$ and $x \in X$ such that $\pi_H(x) \in (X/\sim_H)_{gH}$. By (4.3) $\varphi(\pi_H(x)) = H[1, x] \in X_{gH}$ and (4.2) gives an element $y \in X$ such that $\mu_{gH}(H[1, y]) = H[1, x]$, that is, $H[g, y] = H[1, x]$ and $[h_0, x] = [g, y]$ for some $h_0 \in H$, therefore $(g^{-1}h_0, x) \in G * X$. Since H is normal in G we have $g^{-1}h_0 = hg^{-1}$ for some $h \in H$ and $(hg^{-1}, x) \in G * X$. Conversely, if $x \in X$ verifies $(h_0 g^{-1}, x) \in G * X$ for some $h_0 \in H$, then $h_0 g^{-1} = g^{-1}h$ for some $h \in H$ and we have $[h, x] = [g, y]$, where $y = (g^{-1}h) \cdot x$ and

$$\varphi(\pi_H(x)) = H[1, x] = H[1, (h^{-1}g) \cdot y] \overset{(2.2)}{=} H[h^{-1}g, y] = H[g, y] = \mu_{gH}(H[1, y]),$$

thus $\varphi(\pi_H(x)) \in \mu_{gH}(\mathrm{Im}\varphi)$ and $\pi_H(x) \in (X/\sim_H)_{gH}$ thanks to equations (4.2) and (4.3).

(ii) This is a consequence of part (i) and the fact that $(gH, \pi_H(x)) \in G/H * X/\sim_H$ if and only if $\pi_H(x) \in (X/\sim_H)_{g^{-1}H}$.

(iii) Take $(gH, \pi_H(x)) \in G/H * X/\sim_H$. There exists $h \in H$ such that $(hg, x) \in G * X$, then $[hg, x] = [1, (hg) \cdot x]$ and $\varphi(\pi_H((hg) \cdot x)) = H[hg, x] = H[g, x]$. Then follows by (4.4) that:

$$\eta_{G/H}(gH, \pi_H(x)) = \varphi^{-1}(H[g, x]) = \pi_H((hg) \cdot x),$$

as desired.

(iv) Observe that $\mathrm{Im}\varphi = \{H[1, x] \mid x \in X\}$, then $\mu_{G/H}[\mathrm{Im}\varphi] = X_G/\sim_H$, thus by (ii) of Proposition 2.11 the spaces $(\mathrm{Im}\varphi)_{G/H}$ and $X_G/\sim_H$ are homeomorphic. Now we must show that the spaces $\mathrm{Im}\varphi$ and $X/\sim_H$ are G/H-equivalent, but by (i) in Proposition 2.11 it is enough to show that φ is a (G/H)-map, and this follows from (4.3). □

Example 4.3. Consider the partial action $\eta : \mathbb{Z} * X \to X$ of Example 3.8 and let $m \in \mathbb{Z}^+$ be the least integer such that $f^m = \mathrm{id}_U$. If $H = m\mathbb{Z}$, then the induced quotient morphism π_H satisfies $\pi_H(x) = \{x\}$, for any $x \in X$, thus the spaces X and $X/\sim_H$ are homeomorphic. Now we determine $\eta_{\mathbb{Z}/H}$. Take $(k+H, \pi_H(x)) \in \mathbb{Z}/H * X/\sim_H$, then, if $k \in H$, by (4.6) we get

$$\eta_{\mathbb{Z}/H}(k+H, \pi_H(x)) = \eta_{\mathbb{Z}/H}(H, \pi_H(x)) = \pi_H(x).$$

Suppose $k \notin H$. By (ii) of Theorem 4.2, there is $h \in H$ such that $(h+k, x) \in \mathbb{Z} * X$ and $\eta_{\mathbb{Z}/H}(k+H, \pi_H(x)) = \pi_H(\eta(h+k, x))$, thanks to (4.6). Since $(h+k, x) \in \mathbb{Z} * X$ and $k \notin H$, the equality (2.1) implies $x \in U$. Then, (h, x) and $(k+h, x)$ belong to $\mathbb{Z}*X$, which gives $\eta(h+k, x) = \eta(k+h, x) = f^{k+h}(x) = f^k(x)$. We have shown that if $k \notin H$ with $(k+H, \pi_H(x)) \in \mathbb{Z}/H * X/\sim_H$, one gets

$$\eta_{\mathbb{Z}/H}(k+H, \pi_H(x)) = \pi_H(\eta(h+k, x)) = \pi_H(f^k(x)) = \pi_H(\eta(k, x)).$$

Corollary 4.4. *Let G be a compact group, H be a closed normal subgroup of G, and $\eta : G*X \to X$ be a continuous partial action on a compact Hausdorff space X. If $G*X$ is closed in $G \times X$, then $G/H * X/\sim_H$ is closed in $G/H \times X/\sim_H$.*

Proof. Let $\eta'_{G/H}$ be the partial action defined (4.2). By construction we get that $\eta_{G/H}$ and $\eta'_{G/H}$ are G/H-equivalent, and thus it is enough to show that $G/H * \mathrm{Im}(\varphi)$ is closed in $G/H \times \mathrm{Im}(\varphi)$. Having at hand Remark 2.5 we only need to see that $\mathrm{Im}(\varphi)$ is closed in $X_G/\sim_H$. Now, by (ii) in Theorem 3.2 the enveloping space X_G is Hausdorff and since H is compact then the first ítem of Theorem 3.2 implies that $X_G/\sim_H$ is Hausdorff. Also $X/\sim_H$ is compact which implies that φ is a closed map, then $\mathrm{Im}(\varphi)$ is closed in $X_G/\sim_H$ and $G/H * \mathrm{Im}(\varphi)$ is closed in $G/H \times \mathrm{Im}(\varphi)$ which finishes the proof. ☐

The following lemma is clear:

Lemma 4.5. *Let G and H be topological groups and $\phi : G \to H$ be a group homomorphism. If $\{\eta_h\colon X_{h^{-1}} \to X_h\}_{h\in H}$ is a partial action of H on X, then the family $\{\eta_{\phi(g)}\colon U_{g^{-1}} \to U_g\}_{g\in G}$, where $U_g = X_{\phi(g)}, g \in G$, is a partial action of G on X such that*

$$G * X = (\phi \times \mathrm{id}_X)^{-1}(H * X) \text{ and } G * X \ni (g, x) \mapsto \eta(\phi(g), x) \in X. \quad (4.7)$$

Remark 4.6. Using $\eta_{G/H}$ and the canonical homomorphism $p_H : G \to G/H$, it follows by Theorem 4.1 and Lemma 4.5 that there is a partial action η^{p_H} of G on $X/\sim_H$ which by (4.7) has domain

$$G * (X/\sim_H) = \{(g, \pi_H(x)) \mid g \in G,\ x \in X,\ (gH, \pi_H(x)) \in G/H * X/\sim_H\}, \quad (4.8)$$

and

$$\eta^{p_H}(g, \pi_H(x)) = \eta_{G/H}(gH, \pi_H(x)). \quad (4.9)$$

From now on we always consider G acting partially on $X/\sim_H$ via η^{p_H}.

Let H_1, H_2 be subgroups of G such that $H_1 \subseteq H_2$. We define $\pi_{H_1,H_2} : X/\sim_{H_1} \to X/\sim_{H_2}$ as the only map such that

$$\pi_{H_2} = \pi_{H_1,H_2} \circ \pi_{H_1}, \quad (4.10)$$

in particular, for a subgroup H of G the map $\pi_{H,H}$ is the identity on $X/\sim_H$.

Proposition 4.7. *Let H, H_1 and H_2 be normal subgroups of G with $H_1 \subseteq H_2$. Then π_H and π_{H_1,H_2} are G-maps.*

Proof. First we show that π_H is a G-map. Take $(g,x) \in G * X$, by (ii) of Theorem (4.2) the pair $(gH, \pi_H(x))$ belongs to $G/H * X/\sim_H$ and follows by (4.6) that $\pi_H(\eta(g,x)) = \eta_{G/H}(gH, \pi_H(x))$. Hence $(g, \pi_H(x)) \in G * X/\sim_H$ and $\eta^{p_H}(g, \pi_H(x)) = \pi_H(\eta(g,x))$ which shows that π_H is a G-map. Now we show that π_{H_1,H_2} is a G-map. Suppose $(g, \pi_{H_1}(x)) \in G * X/\sim_{H_1}$. We need to show that $(g, \pi_{H_2}(x)) \in G * X/\sim_{H_2}$ and $\pi_{H_1,H_2}(\eta_{G/H_1}(gH_1, \pi_{H_1}(x))) = \eta_{G/H_2}(gH_2, \pi_{H_2}(x))$. We have $(gH_1, \pi_{H_1}(x)) \in G/H_1 * X/\sim_{H_1}$. Using (ii) of Theorem 4.2 there exists $h \in H_1 \subseteq H_2$ such that $(hg, x) \in G * X$, thus $(gH_2, \pi_{H_2}(x)) \in G/H_2 * X/\sim_{H_2}$ and $(g, \pi_{H_2}(x)) \in G * X/\sim_{H_2}$. It follows from (4.6) that

$$\eta^{p_{H_1}}(g, \pi_{H_1}(x)) = \eta_{G/H_1}(gH_1, \pi_{H_1}(x)) = \pi_{H_1}(\eta(hg, x)),$$

in a similar way $\eta^{p_{H_2}}(g, \pi_{H_2}(x)) = \eta_{G/H_2}(gH_2, \pi_{H_2}(x)) = \pi_{H_2}(\eta(hg, x))$. Therefore

$$\pi_{H_1,H_2}(g \cdot \pi_{H_1}(x)) = \pi_{H_1,H_2}(\pi_{H_1}(hg \cdot x)) = \pi_{H_2}(hg \cdot x) = g \cdot \pi_{H_2}(x),$$

and we conclude that π_{H_1,H_2} is a G-map. □

4.1. **Inverse limits.** As an application of Theorem 4.1 we extend [2, Theorem 9] to the context of partial actions. Let G be a compact group, let $(I, \leq)$ be a directed set and consider an inverse system $\{G_i; p_i^j; I\}$ in the category of topological groups such that $G = \varprojlim G_i$, where $\{p_i : G \to G_i\}_{i \in I}$ is the family of projections such that $p_i^j \circ p_j = p_i$ for $i, j \in I$ and $i \leq j$. Take $i \in I$, then $H_i = \ker(p_i) = p_i^{-1}(\{e_i\})$ is a closed normal subgroup of G thus is compact and $H_j \leq H_i$, for every $i, j \in I$ with $i \leq j$. Let η be a continuous partial action of G on X. Now, for $i \in I$ the group H_i acts partially on X via restriction, setting $X_i = X/\sim_{H_i}$ we denote by $\pi_i^j = \pi_{H_j,H_i} : X_j \to X_i$, $i \leq j$, the G-map defined in (4.10) and $\pi_i = \pi_{H_i} : X \to X_i$, the orbit equivalence map.

We proceed with the next lemma:

Lemma 4.8. *Following the notations above consider $i, j \in I$ with $i \leq j$. Let $\eta : G * X \to X$ be a continuous partial action such that $G * X$ is closed, then the family $\{\pi_i : X \to X_i\}_{i \in I}$ separates points of closed sets in X.*

Proof. The proof follows the lines of [2, Lemma 3], where it is shown that $\pi_i(x) \notin \pi_i(C)$ for any $x \in X$ and $C \subseteq X$ a closed subset such that $x \notin C$. On the other hand, the fact that $G * X$ is closed is used to guarantee that $H_i * X = (G * X) \cap (H_i \times X)$, is closed in $H_i \times X$, which implies that π_i is closed, for any $i \in I$. Then the family $\{\pi_i : X \to X_i\}_{i \in I}$ separates points of closed sets in X, as desired. □

Take $i, j, k \in I$ such that $i \leq j \leq k$. For $x \in X$, we have $\pi_i^k(H_k^x \cdot x) = (\pi_i^j \circ \pi_j^k)(H_k^x \cdot x)$, then $\mathcal{X} = \{X_i, \pi_i^j, I\}$ is an inverse system of spaces endowed with continuous partial actions of G.

We finish this work with the next result:

Proposition 4.9. *Under the assumptions above, let* $\mathcal{X} = \{\varphi_i : \varprojlim X_i \to X_i\}_{i\in I}$ *be the family of projections associated to* $\varprojlim X_i$*. If* X *is Hausdorff and* $G * X$ *is closed in* $G \times X$*, then the following assertions hold:*

(i) *There is a partial action* θ *of* G *on* $\varprojlim X_i$ *such that* X *is* G*-equivalent to* $\varprojlim X_i$.

(ii) *The following diagram is commutative for any* $j \in I$

$$\begin{array}{ccc} G * \varprojlim X_i & \xrightarrow{\ \theta\ } & \varprojlim X_i \\ {\scriptstyle \mathrm{id}_G \times \varphi_j}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ G * X_j & \xrightarrow[\eta^{p_{H_j}}]{} & X_j, \end{array}$$

where $\eta^{p_{H_j}}$ *is the partial action of* G *on* X_j *given by* (4.9).

Proof. (i) It is not difficult to see that the family $\Pi = \{\pi_i : X \to X_i\}_{i\in I}$ is compatible with $\mathcal{X}$ then by the universal property of the inverse limit there exists a continuous map $\lambda : X \to \varprojlim X_i$, such that $\varphi_i \circ \lambda = \pi_i$, for any $i \in I$. We shall prove that λ is a homeomorphism. First, by Lemma 4.8, the family Π separates points of closed sets in X, further, by (i) in Theorem 3.2 each orbit space X_i is T_2, then the map λ is an embedding. Let $(x_i)_{i\in I} \in \varprojlim X_i$, since $H_i * X$ is closed in $H_i \times X$ and by Lemma 3.1 the map π_i is perfect, we have $A_i = \pi_i^{-1}(x_i)$ is a compact subset of X, for all $i \in I$. Now write $\mathcal{A} = \{A_i\}_{i\in I}$ and take $i, j \in I$ such that $i \le j$. For $y \in A_j$ we have $\pi_i(y) = \pi_i^j(\pi_j(y)) = \pi_i^j(x_j) = x_i$, and $A_j \subseteq A_i$, from this one concludes that $\mathcal{A}$ has the finite intersection property, therefore $\bigcap\limits_{i\in I} A_i \neq \varnothing$. Finally, if $y \in \bigcap\limits_{i\in I} A_i$, then $\pi_i(y) = x_i$, that is $(x_i)_{i\in I} = \lambda(y)$, therefore $\varprojlim X_i = \lambda(X)$ and λ is a homeomorphism. To define a partial action of G on $\varprojlim X_i$ we set

$$G * \varprojlim X_i = \left\{(g, x) \in G \times \varprojlim X_i \mid (g, \lambda^{-1}(x)) \in G * X\right\},$$

and

$$\theta : G * \varprojlim X_i \ni (g, x) \mapsto \lambda(g \cdot \lambda^{-1}(x)) \in \varprojlim X_i,$$

thus λ is a $G-$map which shows the first ítem.

(ii) Take $j \in I$. We first check that the map $\mathrm{id}_G \times \varphi_j$ is well defined, that is for $(g, x) \in G * \varprojlim X_i$ one has that $(g, x_j) \in G * X_j$, where $x = (x_i)_{i\in I}$. Indeed, if $(g, x) \in G * \varprojlim X_i$ we get that $(g, \lambda^{-1}(x)) \in G * X$ which by ítem (ii) in Theorem 4.2 implies $(gH_j, \pi_j(\lambda^{-1}(x))) \in G/H_j * X_j$ and thus $(g, x_j) = (g, \pi_j(\lambda^{-1}(x))) \in G * X_j$ thanks to (4.8), and $\mathrm{id}_G \times \varphi_j$ is well defined. To check that the diagram commutes observe that

$$\eta^{p_{H_j}} \circ (\mathrm{id}_G \times \varphi_j)(g, x) = \eta_{G/H_j}(gH_j, \pi_j(\lambda^{-1}(x))) = \pi_j((hg) \cdot \lambda^{-1}(x)),$$

where by (ii) of Theorem 4.2 the element $h \in H_j$ is such that $(hg, \lambda^{-1}(x)) \in G{*}X$. Since $\lambda^{-1}(x) \in X_{g^{-1}h^{-1}} \cap X_{g^{-1}}$ we get by ítem (ii) of Proposition 2.1 that $g \cdot \lambda^{-1}(x) \in X_{h^{-1}}$ thus $(hg) \cdot \lambda^{-1}(x) = h \cdot (g \cdot \lambda^{-1}(x))$ and $\pi_j((hg) \cdot \lambda^{-1}(x)) = \pi_j(g \cdot \lambda^{-1}(x))$. On the other hand $\varphi_j \circ \theta(g, x) = \varphi_j \lambda(g \cdot \lambda^{-1}(x)) = \pi_j(g \cdot \lambda^{-1}(x))$. Then $\eta^{p_{H_j}} \circ (\mathrm{id}_G \times \varphi_j) = \varphi_j \circ \theta$ which ends the proof. □

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References

[1] F. Abadie, Enveloping actions and Takai duality for partial actions. Journal of Funct. Anal. 197 (2003), 14–67.

[2] S. A. Antonyan, Equivariant embeddings into G-AR's, Glasnik Matematički 22 (42) (1987), 503–533.

[3] A. Baraviera, R. Exel, D. Gonçalves, F. B. Rodrigues and D. Royer, Entropy for partial actions of $\mathbb{Z}$, Proc. Amer. Math. Soc. 150 (2022), 1089–1103.

[4] S. De Neymet and R. Jiménez, Introducción a los grupos topológicos de transformaciones, Aportaciones matemáticas: Textos, 2005.

[5] M. Dokuchaev, Recent developments around partial actions, S ão Paulo J. Math. Sci. 13, no. 1 (2019), 195–247.

[6] M. Dokuchaev and R. Exel, The ideal structure of algebraic partial crossed products, Proc. Lond. Math. Soc. 115, no. 1 (2017), 91–134.

[7] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston, 1966.

[8] R. Exel, Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, J. Funct. Anal. 122, no. 3 (1994), 361–401.

[9] R. Exel, Partial dynamical systems, Fell bundles and applications, Mathematical surveys and monographs; volume 224, Providence, Rhode Island: American Mathematical Society, 2017.

[10] R. Exel, T. Giordano and D. Gonçalves, Enveloping algebras of partial actions as groupoid C^*-algebras, J. Operator Theory 65 (2011), 197–210.

[11] D. Gonçalves, J. Öinert and D. Royer, Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics, J. Algebra 420 (2014), 201–216.

[12] J. Kellendonk and M. V. Lawson, Partial Actions of Groups, International Journal of Algebra and Computation 14 (2004), 87–114.

[13] L. Martínez, H. Pinedo and E. Ramírez, Partial actions of groups on hyperspaces, Appl. Gen. Topol. 23, no. 2 (2022), 255–268.

[14] L. Martínez, H. Pinedo and A. Villamizar, Partial actions of groups on profinite spaces, Appl. Gen. Topol. 25, no. 1 (2024), 143–157.

[15] H. Pinedo and C. Uzcátegui, Polish globalization of Polish group partial actions, Math. Log. Quart. 63, no. 6 (2017), 481–490.

[16] J. C. Quigg and I. Raeburn, Characterizations of crossed products by partial actions, J. Operator Theory 37 (1997), 311–340.

[17] M. Tkačenko, L. Villegas, C. Hernández and O. Rendón, Grupos topológicos, SEP, 1997.