

# Hemi metric spaces and Banach fixed point theorem

VILDAN OZTURK <sup>a</sup>  AND STOJAN RADENOVIC <sup>b</sup> 

<sup>a</sup> Department of Mathematics, Faculty Of Polatli Science And Letters, University Of Ankara Haci Bayram Veli, Ankara, Turkey. ([vildan.ozturk@hbv.edu.tr](mailto:vildan.ozturk@hbv.edu.tr))

<sup>b</sup> Faculty of Mechanical Engineering, University of Belgrade, 11120 Belgrade, Serbia. ([radens@beote1.net](mailto:radens@beote1.net))

Communicated by S. Romaguera

## ABSTRACT

---

*In this work, we will define a new type metric with degree  $m$  and  $m+1$  points which is called  $m$ -hemi metric as a generalization of 2-metric spaces. We will give and prove some topological properties. Also, Banach contraction mapping principle was proved and an application to Fredholm integral equation were given in hemi metric spaces.*

---

2020 MSC: 47H10; 54H25; 54E35.

KEYWORDS: *hemi-metric; fixed point; Banach contraction.*

## 1. HEMI METRIC SPACES

Metric spaces are used in many sciences besides mathematics. Metric fixed point theory started with the Banach contraction principle in 1922. In cases where this principle is insufficient, some generalized metric spaces and generalized contraction principles were defined and many fixed point theorems were proved. One of them is quasi-metric by defined without symmetry axiom [2] and the other one is semi-metric by defined without triangular inequality [31]. The concept of partial metric spaces are defined as the distance from  $x$  to  $x$  may not be zero in the usual metric [24, 25].  $b$ -metric, modular metric,  $F$ -metric are some another generalizations of metric in the literature and most of fixed point and common fixed point theorems were proved in these spaces (see more [1, 4, 5, 14, 17, 18, 23, 28]).

Gähler [15] introduced the notion of 2-metric based on the geometry of more than two points. Gähler [15] and later Lahiri et. al [21] gave some topological properties of 2-metric spaces. Mustafa and Sims [26] defined another notion of generalized metric space based on three point geometry which is named G-metric. In [3], Branciari suggested a new generalization of the metric notion by replacing the triangle inequality by a more general one involving four points which is named rectangular metric. George et. al [16] defined rectangular b-metric by considering the concepts of b-metric and rectangular metric together. Choi et.al. [7] gave the concept of g-metric with degree n is a distance of n + 1 points, generalizing the ordinary distance between two points and G-metric between three points. Moreover, authors proved some fixed point theorems in these spaces [8, 6, 12, 13, 19, 20, 22, 26, 27, 29].

Deza and Rosenberg [11] introduced the notion of  $m$ -hemi-metric on a set with at least  $m+2$  elements for integer  $m$ . Authors considered generalizations of the notion of metric in the direction of distances between three or more elements. Also see [9, 10].

In this work, we give the notion of the hemi-metric as a new type of metric and some topological properties. Further, we introduce fixed point theorems for Banach contraction and a generalized contraction principle in hemi-metric space.

**Definition 1.1** ([11]). Let  $m \in \mathbb{Z}^+$  and  $H$  a set with at least  $m + 2$  elements.  $d^h : H^{m+1} \rightarrow \mathbb{R}$  is called  $m$ -hemi-metric if for all  $h_1, h_2, \dots, h_{m+2} \in H$ ,

- i.  $d^h(h_1, h_2, \dots, h_{m+1}) \geq 0$  (non-negativity),
- ii.  $d^h(h_1, h_2, \dots, h_{m+1}) = 0 \Leftrightarrow$  if for any  $h_i, h_k \in H$ ,  $h_i = h_k$  (zero conditioned),
- iii.  $d^h(h_1, h_2, \dots, h_{m+1}) = d^h(h_{\pi(1)}, h_{\pi(2)}, \dots, h_{\pi(m+1)})$ , for any permutation  $\pi$  of  $\{1, 2, \dots, m + 1\}$  (totally symmetry)
- iv.  $d^h(h_1, h_2, \dots, h_{m+1}) \leq \sum_{i=1}^{m+1} d^h(h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_{m+2})$  ( $m$ -simplex inequality).

Then  $(H, d^h)$  is called  $m$ -hemi metric space.

The concept of  $m$ -hemi-metric as an  $m$ -ary generalization of the concept of semi-metric. A significant appropriate case of the  $m$ -hemi-metric is the following case obtained for  $m = 2$ .

A function  $d^h : H^3 \rightarrow \mathbb{R}$  is called a 2-metric if  $d^h$  satisfies (i), (ii), (iii) and the following tetrahedron inequality,

$$d^h(h_1, h_2, h_3) \leq d^h(h_1, h_2, h_4) + d^h(h_1, h_3, h_4) + d^h(h_2, h_3, h_4)$$

**Lemma 1.2** ([30]). Let  $d^h$  is an  $m$ -hemi-metric on  $H$ , then  $\frac{d^h}{1+d^h}$  is an  $m$ -hemi-metric on  $H$ .

**Lemma 1.3** ([30]). Let  $d^h$  is an  $m$ -hemi-metric on  $H$ , then  $\min\{1, d^h\}$  is an  $m$ -hemi-metric on  $H$ .

**Example 1.4.** Let  $H = \{0\} \cup \{\frac{1}{n} : n \geq 1, n \in \mathbb{N}\}$ . Define  $d^h : H^{m+1} \rightarrow \mathbb{R}$  ,  
 $d^h (h_1, h_2, \dots, h_{m+1}) = \begin{cases} 1, & h_i \neq h_j, \text{ for all } i, j, \\ 0, & h_i = h_j, \text{ for any } i, j \end{cases}$

Then  $d^h$  is an  $m$ -hemi-metric on  $H$ .

**Example 1.5.** Define  $d^h : H^{m+1} \rightarrow \mathbb{R}$  ,  $d^h (h_1, h_2, \dots, h_{m+1}) = \min\{1, p (h_i, h_j)\}$  with usual metric  $p (h_i, h_j) = |h_i - h_j|$  , then  $d^h$  is an  $m$ -hemi-metric on  $H$ .

**Example 1.6.** Define  $d^h : H^{m+1} \rightarrow \mathbb{R}$  ,  $d^h (h_1, h_2, \dots, h_{m+1}) = \sum_{i,j=1}^{m+1} |h_i - h_j|$  , then  $d^h$  is an  $m$ -hemi-metric on  $H$ .

**Example 1.7.** Define  $d^h : H^{m+1} \rightarrow \mathbb{R}$  ,  $d^h (h_1, h_2, \dots, h_{m+1}) = |\prod_{i,j=1}^{m+1} h_i - h_j|$  , then  $d^h$  is an  $m$ -hemi-metric on  $H$ .

**Example 1.8.** Let  $d^h : \mathbb{N}^{m+1} \rightarrow \mathbb{R}$  ,  
 $d^h (h_1, h_2, \dots, h_{m+1}) = \begin{cases} 0, & \text{any } h_i = h_k, \\ \max \{h_i\}, & \text{other} \end{cases}$  . Then  $d^h$  is an  $m$ -hemi-metric on  $\mathbb{N}$ .

**Definition 1.9.** Let  $(H, d^h)$  be an  $m$ -hemi-metric space. For  $h_0, h_1, \dots, h_m \in H$  and  $\varepsilon > 0$  ,

$$B (h_0, h_1, \dots, h_{m-1}, \varepsilon) = \{y \in H : d^h (h_0, h_1, \dots, h_{m-1}, y) < \varepsilon\}$$

is called  $h$ -open ball centered at  $h_0, h_1, \dots, h_{m-1}$  with radius  $\varepsilon$ .

The topology generated on  $H$  by taking the collection of all  $h$ -open balls as a subbasis, which we call the  $m$ -hemi-metric topology. It is denoted by  $\tau$ . Members of  $\tau$  are called  $h$ -open sets and their complements,  $h$ -closed sets.

**Lemma 1.10.** Let  $(H, d^h)$  be an  $m$ -hemi-metric space and  $A \subseteq H$ . Then  $A$  is an  $h$ -open set if and only if for every  $a \in A$  there are finite number of points  $h_1, h_2, \dots, h_m, h'_1, h'_2, \dots, h'_m$  and  $\varepsilon_1, \varepsilon_2 > 0$  such that  $a \in B (a, h_1, h_2, \dots, h_{m-1}, \varepsilon_1) \cap B (a, h'_1, h'_2, \dots, h'_{m-1}, \varepsilon_2) \subset A$ .

*Proof.* The sufficiency of the claim is evident from the fact that the intersection of the  $h$ -open balls  $B (a, h_1, h_2, \dots, h_{m-1}, \varepsilon_1) \cap B (a, h'_1, h'_2, \dots, h'_{m-1}, \varepsilon_2)$  is  $h$ -open, the sufficiency of the condition follows immediately.

Conversely let  $A$  be a  $h$ -open set and  $a \in A$ . Then there exists  $h$ -open balls  $B (h_1, h_2, \dots, h_m, \varepsilon_1)$  ,  $B (h'_1, h'_2, \dots, h'_m, \varepsilon_2)$  such that  $a \in B (h_1, h_2, \dots, h_m, \varepsilon_1) \cap B (h'_1, h'_2, \dots, h'_m, \varepsilon_2) \subset A$ .

Since  $a \in B (h_1, h_2, \dots, h_m, \varepsilon_1)$  and  $a \in B (h'_1, h'_2, \dots, h'_m, \varepsilon_2)$ , then  $d^h (h_1, h_2, \dots, h_m, a) = p_1 < \varepsilon_1$  and  $d^h (h'_1, h'_2, \dots, h'_m, a) = p_2 < \varepsilon_2$ . Choose  $t_i < \frac{\varepsilon_i - p_i}{2}$  for  $i = 1, 2$ . Then, we have

$$\begin{aligned} a &\in B (a, h_1, h_2, \dots, h_{m-1}, t_1) \cap B (a, h'_1, h'_2, \dots, h'_{m-1}, t_2) \\ &\subset B (h_1, h_2, \dots, h_m, \varepsilon_1) \cap B (h'_1, h'_2, \dots, h'_m, \varepsilon_2) \subset A. \end{aligned}$$

This completes the proof. □

**Definition 1.11.** If  $A$  is a subset of an  $m$ -hemi-metric space  $(H, d^h)$ , we define the  $h$ -closure of  $A$ , denoted by  $A^-$ , as the  $h$ -closure of  $A$  with respect to the topology  $\tau$ .

**Definition 1.12.** Let  $(H, d^h)$  be an  $m$ -hemi-metric space and  $\{h_n\}$  be a sequence in  $H$ .

- i.  $\{h_n\}$  is named  $h$ -convergent to a  $y \in H$  if and only if

$$\lim_{n_1, n_2, \dots, n_m \rightarrow \infty} d^h(h_{n_1}, h_{n_2}, \dots, h_{n_m}, y) = 0.$$

Therefore  $\{h_n\}$  is  $h$ -convergent to  $y$  if and only if it converges to  $y$  with respect to the topology  $\tau$ .

- ii.  $\{h_n\}$  is named  $h$ -Cauchy sequence if and only if

$$\lim_{n, m \rightarrow \infty} d^h(h_{n_0}, h_{n_1}, \dots, h_{n_m}) = 0.$$

- iii.  $(H, d^h)$  is called  $h$ -complete if every  $h$ -Cauchy sequence  $h$ -convergent in  $H$ .

- iv. A mapping  $f$  is called  $h$ -continuous on  $H$  if  $fh_n \rightarrow fy$  when  $h_n \rightarrow y$ .

**Proposition 1.13.** Let  $(H, d^h)$  be a  $m$ -hemi-metric space and  $\{h_n\}$  be a sequence in  $H$ .

- i.  $\{h_n\}$  convergence to a  $y \in H$  if for all  $\varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$  such that  $n_1, \dots, n_m \geq n_0 \Rightarrow d^h(y, h_{n_1}, \dots, h_{n_m}) < \varepsilon$ .
- ii. If  $\{h_n\}$  is a  $h$ -Cauchy sequence, then for all  $\varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$  such that  $n_0, n_1, \dots, n_m \geq n_0 \Rightarrow d^h(h_{n_0}, h_{n_1}, \dots, h_{n_m}) < \varepsilon$ .

**Example 1.14.** Let  $H = \{0\} \cup \{\frac{1}{n} : n \geq 1, n \in \mathbb{N}\}$ . Define  $d^h : H^{m+1} \rightarrow \mathbb{R}$ ,

$$d^h(h_1, h_2, \dots, h_{m+1}) = \begin{cases} 1, & h_i \neq h_j, \text{ for all } i, j, \\ 0, & h_i = h_j, \text{ for any } i, j \end{cases}$$

Then the sequence  $\{\frac{1}{n+1}\} \subset H$  converges to 0. But it is not  $h$ -Cauchy sequence.

**Lemma 1.15.** The limit is unique in  $m$ -hemi-metric space.

*Proof.* Let  $(H, d^h)$  be an  $m$ -hemi-metric space and  $\{h_n\}$  be a  $h$ -convergent sequence in  $H$ . Assume  $x, y \in H$  are two limit of  $\{h_n\}$ . Thus we get for all  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $n_1, \dots, n_m \geq n_0 \Rightarrow d^s(x, h_{n_1}, \dots, h_{n_m}) < \frac{\varepsilon}{(m+1)}$  and there exists  $n'_0 \in \mathbb{N}$  such that  $n_1, \dots, n_m \geq n'_0 \Rightarrow d^h(y, h_{n_1}, \dots, h_{n_m}) < \frac{\varepsilon}{(m+1)}$ .

Let  $N = \max\{n_0, n'_0\}$ . For  $m > N$  and using  $m$ -simplex inequality for all distinct element  $h_{n_1}, \dots, h_{n_{m-1}}, h_{n_m} \in H$ ,

$$\begin{aligned} d^h(x, y, h_{n_1}, \dots, h_{n_{m-1}}) &\leq [d^h(y, h_{n_1}, \dots, h_{n_{m-1}}, w_{n_m}) + d^h(x, h_{n_1}, \dots, h_{n_{m-1}}, h_{n_m}) \\ &\quad + d^h(x, y, h_{n_2}, \dots, h_{n_{m-1}}, h_{n_m}) + \dots \\ &\quad + d^h(x, y, h_{n_2}, \dots, h_{n_{m-2}}, h_{n_m})] \\ &< (m+1) \frac{\varepsilon}{m+1} = \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary  $d^h(x, y, h_{n_1}, \dots, h_{n_{m-1}}) = 0$ . Thus  $x = y$ . □

**Definition 1.16.** An  $m$ -hemi-metric space  $(H, d^h)$  is named compact if every sequence in  $H$  has a convergent subsequence.

## 2. BANACH FIXED POINT THEOREM

**Theorem 2.1.** Let  $(H, d^h)$  be an  $h$ -complete  $m$ -hemi-metric space and  $f : H \rightarrow H$  be an  $h$ -continuous self mapping satisfying

$$d^h(fh_m, \dots, fh_0) \leq \gamma d^h(h_m, \dots, h_0) \tag{2.1}$$

for all where  $h_m, \dots, h_0 \in H$   $\gamma \in [0, 1)$ . Then  $f$  has a unique fixed point

*Proof.* Let  $h_0 \in H$  be any point and  $m \in \mathbb{Z}^+$ . We define a sequence  $h_m^{(n+1)} = fh_m^{(n)}$  for  $\forall n \geq 1$ .

From (2.1),

$$\begin{aligned} d^h(h_{m+1}^{(n)}, h_m^{(n)}, \dots, h_1^{(n)}) &= d^h(fh_{m+1}^{(n)}, fh_m^{(n)}, \dots, fh_1^{(n)}) \\ &\leq \gamma d^h(h_{m+1}^{(n)}, h_m^{(n)}, \dots, h_1^{(n)}) \\ &\leq \dots \\ &\leq \gamma^n d^h(h_{m+1}^{(0)}, h_m^{(0)}, \dots, h_1^{(0)}). \end{aligned}$$

Letting limit, we have

$$\lim_{n, m \rightarrow \infty} d^h(h_{m+1}^{(n)}, h_m^{(n)}, \dots, h_1^{(n)}) = 0.$$

Hence,  $\{h_m^{(n)}\}$  is an  $h$ -Cauchy sequence. Since  $(H, d^h)$  is  $h$ -complete, there exists a  $y \in H$  with  $h_m^{(n)} \rightarrow y$ . By  $h$ -continuity of  $f$ ,

$$\lim_{n \rightarrow +\infty} d^h(fh_m^{(n)}, \dots, fh_1^{(n)}, fy) = 0.$$

And by

$$\lim_{n \rightarrow +\infty} d^h(fh_m^{(n)}, \dots, fh_1^{(n)}, fy) = 0$$

and uniqueness of limit,  $fy = y$ .

Now we show that uniqueness of fixed point. Suppose  $f$  has different fixed points  $h_1, h_2, \dots, h_m, h_{m+1}$  with  $h_i \neq h_j$  for  $1 \leq i, j \leq m + 1$ . By (2.1),

$$\begin{aligned} d^h(h_1, h_2, \dots, h_{m+1}) &= d^h(fh_1, fh_2, \dots, fh_{m+1}) \\ &\leq \gamma d^h(h_1, h_2, \dots, h_{m+1}) \end{aligned}$$

which is a contradiction with  $\gamma \in [0, 1)$ . Thus  $f$  has unique fixed point. □

**Example 2.2.** Let  $H = [0, \frac{3}{2}]$ . Define

$d^h : H^{m+1} \rightarrow \mathbb{R}$ ,  $d^h(h_1, h_2, \dots, h_{m+1}) = \min\{1, |h_i - h_j|\}$  for  $1 \leq i, j \leq m + 1$

and

$$f : H \rightarrow H, f(x) = \begin{cases} \frac{h^2}{2} & \text{if } h \in [0, 1) \\ \frac{2h-1}{2} & \text{if } h \in [1, \frac{3}{2}] \end{cases}.$$

Then  $d^h$  is an  $h$ -complete  $m$ -hemi-metric on  $H$ . Then (2.1) is satisfied. 0 is unique fixed point of  $f$ .

### 3. APPLICATION

Let consider the Fredholm integral equation

$$h(t) = \int_0^1 J(t, s, u(s)) ds, \quad t \in [a, b] \tag{3.1}$$

and  $C[0, 1]$  with  $m$ -hemi-metric

$$d^h(h_1, h_2, \dots, h_m, h_{m+1}) = \begin{cases} 0, & \text{if any } h_i = h_j \\ \max_{t \in [a, b]} \{|h_i(t) - h_j(t)|\}, & \text{if } h_i \neq h_j \end{cases}$$

for  $i, j = 1, 2, \dots, m + 1$  and  $h_i \in C[0, 1]$ .

**Theorem 3.1.** Consider the integral equation (3.1) and suppose

- (i)  $J : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^+$  is continuous,
- (ii) for all  $(t, s) \in [0, 1] \times [0, 1]$  and  $\gamma \in [0, 1)$  such that

$$|J(t, s, h_1(s)) - J(t, s, h_2(s))| \leq \gamma |h_1(s) - h_2(s)|$$

Then integral equation (3.1) has a unique solution.

*Proof.* Let  $h_i \neq h_j$  for all  $h_i \in C[0, 1]$ . For  $1 \leq i, j \leq m + 1$ , from definition of integral equation,

$$\begin{aligned} d^h(fh_1, fh_2, \dots, fh_{m+1}) &= \max_{t \in [0, 1]} \{|fh_i(t) - fh_j(t)|\} \\ &= \max_{t \in [0, 1]} \left| \int_0^1 (J(t, s, h_i(s)) - J(t, s, h_j(s))) ds \right| \\ &\leq \max_{t \in [0, 1]} \int_0^1 |J(t, s, h_i(s)) - J(t, s, h_j(s))| ds \\ &\leq \gamma \max_{t \in [0, 1]} \left| \int_0^1 (h_i(s) - h_j(s)) ds \right| \\ &\leq \gamma \max_{t \in [0, 1]} |h_i(s) - h_j(s)| \\ &= \gamma d^h(h_1, h_2, \dots, h_{m+1}) \end{aligned}$$

Thus, by Theorem 2.1, integral equation (3.1) has unique solution. □

**ACKNOWLEDGEMENTS.** We would like to thank the reviewer for his/her suggestions.

## REFERENCES

- [1] M. Abtahi, Z. Kadelburg and S. Radenovic, Fixed points and coupled fixed points in partially ordered  $v$ -generalized metric spaces, *Appl. General Topology* 19, no. 2 (2018), 189–201.
- [2] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.* 30 (1989), 26–37.
- [3] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math.* 57 (2000), 31–37.
- [4] S. Chandok, V. Ozturk and S. Radenovic, On fixed points in context of  $b$ -metric spaces, *Matematicki Vesnik* 71, no. 1-2 (2019), 23–30.
- [5] V. V. Chistyakov, Modular metric spaces, I: basic concepts, *Nonlinear Anal.* 72 (2010), 1–14.
- [6] K. P. Chi and H. T. Thuy, A fixed point theorem in 2-metric spaces for a class of maps that satisfy a contractive condition dependent on an another function, *Lobachevskii Journal of Mathematics* 31 (2010), 338–346.
- [7] H. Choia, S. Kim and S. Y. Yang, Structure for  $g$ -metric spaces and related fixed point theorems, [arxiv:1804.03651](https://arxiv.org/abs/1804.03651).
- [8] P. Das, A fixed point theorem in a generalized metric space, *Soochow J. Math.* 33, no. 1 (2007), 33–39.
- [9] E. Deza and M. Deza, *Dictionary of Distances*, Elsevier, Amsterdam, 2006.
- [10] M. Deza and E. Deza, *Encyclopedia of Distances*, Springer, Berlin, 2009.
- [11] M. Deza and I. G. Rosenberg, Small cones of  $m$ -hemimetrics, *Discrete Mathematics* 291 (2005), 81–97.
- [12] B. C. Dhage, Generalized metric space and mapping with fixed point, *Bull. Cal. Math. Soc.* 84 (1992), 329–336.
- [13] H. S. Ding, V. Ozturk and S. Radenovic, On some new fixed point results in  $b$ -rectangular metric spaces, *Journal of Nonlinear Sciences and Applications* 8, no. 4 (2015), 378–386.
- [14] Z. M. Fadail, A. G. B. Ahmad, V. Ozturk and S. Radenovic, Some remarks on fixed point results of  $b_2$ -metric spaces, *Far East J. Math. Sci.* 97 (2015), 533–548.
- [15] S. Gähler, 2-metriche raume and ihre topologische strucktur, *Mathematische Nachrichten* 26 (1963), 115–148.
- [16] R. George, S. Radenovic, K. P. Reshma and S. Shukla, Rectangular  $b$ -metric space and contraction principles, *J. Nonlinear Sci. Appl.* 8 (2015), 1005–1013.
- [17] F. Jahangir, P. Haghmaram and K. Nourouzi, A note on  $F$ -metric spaces, *J. Fixed Point Theory Appl.* 23 (2021): 2.
- [18] M. Jleli and B. Samet, On a new generalizations of metric spaces theorems, *J. Fixed Point Theory Appl.* 20 (2018): 128.
- [19] M. Jleli and B. Samet, Remarks on  $G$ -metric spaces and fixed point theorems, *J. Fixed Point Theory Appl.* 2012 (2012): 210.
- [20] M. Khamsi, Generalized metric spaces: A survey, *J. Fixed Point Theory Appl.* 17 (2015), 455–475.
- [21] B. K. Lahiri, P. Das and L. K. Dey, Cantor’s theorem in 2-Metric spaces and its applications to fixed point problems, *Taiwanese Journal of Mathametics* 15, no. 1 (2011), 337–352.
- [22] S. N. Lal and A. K. Singh, An analogue of Banach’s contraction principle for 2-metric spaces, *Bulletin of the Australian Mathematical Society* 18, no. 1 (1978), 137–143.
- [23] D. Lateefa, and J. Ahmad, Dass and Gupta’s fixed point theorem in  $F$ -metric spaces, *J. Nonlinear Sci. Appl.* 12 (2019), 405–411.
- [24] S. G. Matthews, Partial metric topology. Research Report 212. Department of Computer Science, University of Warwick (1992).
- [25] S. G. Matthews, Partial metric topology, general topology and its applications, *Proceedings of the 8th Summer Conference, Queen’s College* (1992). *Ann NY Acad Sci*, no.728 (1994), 183–197.

- [26] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis* 7, no. 2 (2006), 289–297.
- [27] G. Nallaselli, A. J. Gnanaprakasam, G. Mani, Z. D. Mitrović, A. Aloqaily and N. Mlaiki, Integral equation via fixed point theorems on a new type of convex contraction in b-Metric and 2-Metric Spaces, *Mathematics* 11, no. 2 (2023): 344.
- [28] V. Ozturk, Some results for Ciric Presic type contractions in F-Metric Spaces, *Symmetry* 15, no. (2023): 1521.
- [29] V. Ozturk, Fixed Point Theorems in b-Rectangular Metric Spaces, *Universal Journal of Mathematics and Applications* 3, no. 1 (2020), 28–32.
- [30] M. Warrens, If  $d$  is super-metric, then  $d/(1+d)$  is super-metric, *International Mathematical Forum* 12, no. 18 (2017), 861–868.
- [31] W. A. Wilson, On semi-metric Spaces, *American Journal of Mathematics* 53, no. 2 (1931), 361–373.