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Compactness in the endograph uniformity

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Abstract

Given a uniform space $(X,\mathcal{U})$, we denote by $\mathcal{F}^*(X)$ to the family of fuzzy sets u in $(X,\mathcal{U})$ such that u is normal and upper semicontinuous. Let $\mathcal{U}_E$ be the endograph uniformity on $\mathcal{F}^*(X)$. In this paper, we mainly characterize totally bounded and compact subsets in the uniform space $(\mathcal{F}^*(X),\mathcal{U}_E)$.

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1. Introduction

Compactness is a fundamental property in both theory and applications [5, 8, 14], and compactness criteria have attracted much attention. The Arzelà-Ascoli theorem(s) provide compactness criteria in classic analysis and topology (see for instance [2]). Characterizations of compactness are useful in theoretical research and practical applications. So many researches are devoted to characterizations of compactness in a variety of fuzzy set spaces endowed with different topologies (see [3] and references within).

Kloeden [9] introduced the endograph metric d_E on fuzzy sets. Given a metric space (X,d), we denote by $\mathcal{F}(X)$ to the family of fuzzy sets u in (X,d) such that u is normal, upper semicontinuous and with compact support. Let $\mathcal{F}^*(X)$ be the completion of $(\mathcal{F}(X),d_E)$. In [3], relatively compact subsets in $(\mathcal{F}^*(\mathbb{R}^n),d_E)$ (where d is the usual metric in $\mathbb{R}^n$) are characterized via the

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notion of Γ-convergence, which was introduced by Rojas-Medar and Román-Flores [13].

In [6] was introduced the endograph uniformity $\mathcal{U}_E$ on the family $\mathcal{F}^*(X)$ of fuzzy sets u in the uniform space $(X,\mathcal{U})$ such that u is normal and upper semicontinuous. In this paper, we mainly characterize totally bounded and compact subsets in the uniform space $(\mathcal{F}^*(X),\mathcal{U}_E)$ (see Theorem 3.1 and 3.6). The latter theorems generalize some results in [4].

We also study totally bounded and compact subsets in the sendograph uniformity $\mathcal{U}_S$ on the family $\mathcal{F}(X)$ of fuzzy sets u in the uniform space $(X,\mathcal{U})$ such that u is normal, upper semicontinuous and has compact support (see Theorem 4.1 and 4.2).

2. Preliminaries

Given a non-empty set X, a *fuzzy set* u on X is a function $u : X \to [0,1]$. Let $\alpha \in (0,1]$. We define the *α-level* of u as the set $[u]_\alpha = \{x \in X : u(x) \geq \alpha\}$. The *support* of u is the set $[u]_0 = \overline{\{x \in X : u(x) > 0\}}$.

Now, let (X,d) be a metric space. Denote by $\mathcal{K}(X)$ (resp. $\mathcal{C}(X)$) to the family of non-empty compact (resp. closed) subsets of X. Given $A, B \in \mathcal{K}(X)$, we put $d_\lambda(A,B) = \max\{d(a,B) : a \in A\}$, where $d(a,B) = \inf\{d(a,b) : b \in B\}$. Then d_λ is called the *Hausdorff quasi-pseudometric* on $\mathcal{K}(X)$. Note that $d_\lambda(A,B) = 0$ if and only if $A \subseteq B$. We recall that the *Hausdorff metric* on $\mathcal{K}(X)$, denoted by d_H, is defined as $d_H(A,B) = \max\{d_\lambda(A,B), d_\lambda(B,A)\}$ for each $A, B \in \mathcal{K}(X)$.

Let X be a set and let A and B be subsets of $X \times X$, i.e., relations on the set X. The inverse relation of A will be denoted by A^{-1} , and the composition of A and B will be denoted by $A \circ B$. Thus, we have

$$A^{-1} = \{(x,y) \in X \times X : (y,x) \in A\}$$

and

$$A \circ B = \{(x,y) \in X \times X : \text{there exists } z \in X \text{ such that } (x,z) \in A \text{ and } (z,y) \in B\}.$$

The symbol A^2 stands for $A \circ A$ and Δ_X for the diagonal of X, that is, the subset $\{(x,x) : x \in X\}$ of $X \times X$. Every set $A \subseteq X \times X$ that contains Δ_X is called an *entourage of the diagonal*. We will denote by $\mathcal{D}_X$ the family of all entourages of the diagonal of X.

Definition 2.1. A *uniformity* on a non-empty set X is a subfamily $\mathcal{U}$ of $\mathcal{D}_X$ which satisfies the following conditions:

- (U1) If $A \in \mathcal{U}$ and $A \subseteq B \in \mathcal{D}_X$, then $B \in \mathcal{U}$.
- (U2) If $A, B \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
- (U3) For every $A \in \mathcal{U}$, there exists $B \in \mathcal{U}$ such that $B^2 \subseteq A$.
- (U4) For every $A \in \mathcal{U}$, there exists $B \in \mathcal{U}$ such that $B^{-1} \subseteq A$.
- (U5) $\bigcap_{A \in \mathcal{U}} A = \Delta_X$.

A *uniform space* is a pair $(X,\mathcal{U})$ consisting of a set X and a uniformity $\mathcal{U}$ on the set X. Let $(X,\mathcal{U})$ be a uniform space. A family $\mathcal{B} \subseteq \mathcal{U}$ is called a *base* for the uniformity $\mathcal{U}$ if for every $A \in \mathcal{U}$, there exists $B \in \mathcal{B}$ such that $B \subseteq A$. The following result is well known and easy to prove.

Proposition 2.2. *Let X be a non-empty set. A non-empty family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity on X if and only if it satisfies the following properties:*

(BS1) *For any $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subset A \cap B$.*
(BS2) *For every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B^{-1} \subseteq A$.*
(BS3) *For every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B^2 \subseteq A$.*
(BS4) $\bigcap_{A \in \mathcal{B}} A = \Delta_X$.

As usual, a set X equipped with a topology τ is called a *topological space* and it will be denoted by (X, τ). It is a well-known fact that every uniformity $\mathcal{U}$ on a set X induces a topology $\tau(\mathcal{U})$ on X. To be precise, the topology $\tau(\mathcal{U})$ is the family $\{V \subseteq X : \text{for every } x \in V, \text{ there exists } U \in \mathcal{U} \text{ such that } U(x) \subseteq V\}$, where $U(x) = \{y \in X : (x, y) \in U\}$. In this case, the topological space $(X, \tau(\mathcal{U}))$ is a Tychonoff space (for the details we refer to the reader to Chapter 8 of the classic text [1]).

We turn to a brief discussion of the hyperspaces that we will consider in this paper. Given a topological space (X, τ), the symbols $\mathcal{C}(X)$ and $\mathcal{K}(X)$ denote, respectively, the hyperspaces defined by

$$\mathcal{C}(X) = \{E \subseteq X : E \text{ is closed and non-empty}\},$$
$$\mathcal{K}(X) = \{E \in \mathcal{C}(X) : E \text{ is compact}\}.$$

Thus, in the case of a uniform space $(X, \mathcal{U})$, $\mathcal{C}(X)$ (respectively, $\mathcal{K}(X)$) denotes the hyperspace of all non-empty closed (respectively, non-empty compact) subsets of $(X, \tau(\mathcal{U}))$. We will see that $\mathcal{C}(X)$ and $\mathcal{K}(X)$ can be endowed with a natural uniformity in this situation.

Let $(X, \mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$ and each $A \subset X$, let us define $U(A) = \bigcup_{x \in A} U(x)$. Now, for each $U \in \mathcal{U}$ consider the families

$$\mathcal{C}[U] = \{(A, B) \in \mathcal{C}(X) \times \mathcal{C}(X) : A \subseteq U(B),\ B \subseteq U(A)\},$$
$$\mathcal{K}[U] = \{(A, B) \in \mathcal{K}(X) \times \mathcal{K}(X) : A \subseteq U(B),\ B \subseteq U(A)\}.$$

Among the most interesting results in the theory of hyperspaces are the following three well-known results.

Proposition 2.3 ([11])**.** *If $(X, \mathcal{U})$ is a uniform space, then $\{\mathcal{K}[U] : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$.*

A remarkable result by Michael [11] allows us to describe the topology induced by the uniformity $\mathcal{K}(\mathcal{U})$. Let us recall that, for any topological space (X, τ), the topology τ induces a topology τ_V on $\mathcal{C}(X)$, the so-called *Vietoris topology*, a base for τ_V is the family of all sets of the form

$$\mathcal{V}\langle V_1, V_2, \ldots, V_k\rangle = \left\{B \in \mathcal{C}(X) : B \subset \bigcup_{i=1}^{k} V_i \text{ and } B \cap V_i \neq \varnothing \text{ for } i = 1, 2, \ldots, k\right\},$$

where $V_1, V_2, \ldots, V_n$ is a finite sequence of non-empty open sets of X.

Theorem 2.4 ([11]). *If $(X,\mathcal{U})$ is a uniform space, then the topology induced by $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$ coincides with the Vietoris topology induced by $\tau(\mathcal{U})$ on $\mathcal{K}(X)$.*

Allowing for the previous result, if no confusion can arise, $\mathcal{K}(X)$ will be denote the hyperspace of all non-empty compact subsets of $(X,\tau(\mathcal{U}))$ equipped with the Vietoris topology induced by $\tau(\mathcal{U})$. For the hyperspace $\mathcal{C}(X)$ we have the following.

Proposition 2.5 ([11]). *If $(X,\mathcal{U})$ is a uniform space, then $\{\mathcal{C}[U] : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{C}(\mathcal{U})$ on $\mathcal{C}(X)$.*

The following result is easy to prove.

Lemma 2.6. *Let $(X,\mathcal{U})$ be a uniform space. If $W \in \mathcal{U}$ and $A, B, C, D \in \mathcal{K}(X)$ satisfy $(A, C) \in \mathcal{K}[W]$ and $(B, D) \in \mathcal{K}[W]$, then $(A \cup B, C \cup D) \in \mathcal{K}[W]$.*

Let $(X,\mathcal{U})$ be a uniform space. Let us recall that a non-empty subset $A \subseteq X$ is *totally bounded* in $(X,\mathcal{U})$ if for every $U \in \mathcal{U}$, there exists a finite subset $F \subseteq A$ such that $A \subseteq U(F)$.

Proposition 2.7. *Let $(X,\mathcal{U})$ be a uniform space. Then $A \subseteq X$ is totally bounded in $(X,\mathcal{U})$ if and only if for every $U \in \mathcal{U}$, there exists a finite subset $F \subseteq X$ such that $A \subseteq U(F)$.*

Proposition 2.8. *If $(X,\mathcal{U})$ is a totally bounded uniform space, then the uniformity $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$ is totally bounded.*

Proof. Take $U \in \mathcal{U}$. Since $(X,\mathcal{U})$ is totally bounded, there exists a finite subset $A \subseteq X$ such that $X = U(A)$. Denote by F the family of all non-empty finite subsets of A. Let us show that $\mathcal{K}(X) = \mathcal{K}[U](F)$. Fix $K \in \mathcal{K}(X)$. We can find $B \in F$ such that $K \subseteq U(B)$ and $K \cap U(b) \neq \varnothing$ for each $b \in B$. The choice of B implies that $(B, K) \in \mathcal{K}[U]$. This completes the proof. □

Let (X, U) be a uniform space. Denote by $\mathcal{F}^*(X)$ the family of fuzzy sets u on $(X,\mathcal{U})$ satisfying the following conditions:

i) u is upper semicontinuous.
ii) $[u]_\alpha \in \mathcal{K}(X)$ for every $\alpha \in (0, 1]$.
iii) $u_0 = \overline{\bigcup\{[u]_\alpha : \alpha \in (0, 1]\}}$.

Theorem 2.9 ([7, Proposition 4.9]). *Let X be a Hausdorff space and $u \in \mathcal{F}^*(X)$. If $L_u : (0, 1] \to (\mathcal{K}(X), \tau_V)$ is defined by $L_u(\alpha) = [u]_\alpha$ for all $\alpha \in (0, 1]$, then L_u is left-continuous on $(0, 1]$.*

Conversely, if $\{[u]_\alpha : \alpha \in (0, 1]\} \subseteq \mathcal{K}(X)$ is a decreasing family such that the function $L : (0, 1] \to (\mathcal{K}(X), \tau_V)$ defined by $L(\alpha) = [u]_\alpha$ is left-continuous, then there exists a unique $w \in \mathcal{F}^*(X)$ such that $[w]_\alpha = [u]_\alpha$ for every $\alpha \in (0, 1]$.

Remark 2.10. Let X be a Hausdorff space and $u \in \mathcal{F}^*(X)$. If $L_u : (0, 1] \to (\mathcal{K}(X), \tau_V)$ is defined by $L_u(\alpha) = [u]_\alpha$ for all $\alpha \in (0, 1]$, then $\lim_{\alpha\to\beta^+} L_u(\alpha) = \overline{\bigcup_{\beta<\alpha}[u]_\alpha}$ for each $\beta \in (0, 1)$ and we put $\lim_{\alpha\to\beta^+} L_u(\alpha) = u_{\beta^+}$.

3. Compactness in the endograph uniformity

Let $(X,\mathcal{U})$ be a uniform space. If $u \in \mathcal{F}^*(X)$, then the *endograph* of u is defined as $end(u) = \{(x,\alpha) \in X \times [0,1] : u(x) \geq \alpha\}$. Notice that $end(u) \in \mathcal{C}(X \times [0,1])$. Consider the uniformity $\mathcal{U}_{\mathbb{I}}$ defined on $\mathbb{I} = [0,1]$ by means of the base $\{V_\epsilon : \epsilon > 0\}$, where $V_\epsilon = \{(\alpha,\beta) \in \mathbb{I} \times \mathbb{I} : |\alpha - \beta| < \epsilon\}$. Then we can take the product uniformity $\mathcal{U} \times \mathcal{U}_{\mathbb{I}}$ on $X \times \mathbb{I}$. We have that $\{U \times V_\epsilon : U \in \mathcal{U},\ \epsilon > 0\}$ is a base for $\mathcal{U} \times \mathcal{U}_{\mathbb{I}}$. Note that $((a,\alpha),(b,\beta)) \in U \times V_\epsilon$ if and only if $(a,b) \in U$ and $|\alpha - \beta| < \epsilon$. Let $(X,\mathcal{U})$ be a uniform space. Given $U \in \mathcal{U}$ and $\epsilon > 0$, we define the following sets:

$$E[U,\epsilon] = \{(u,v) \in \mathcal{F}^*(X) \times \mathcal{F}^*(X) : (end(u), end(v)) \in \mathcal{C}[U \times V_\epsilon]\}.$$

It follows from Proposition 2.5 that the family $\{E[U,\epsilon] : U \in \mathcal{U},\ \epsilon > 0\}$ is base for a uniformity $\mathcal{U}_E$ on $\mathcal{F}^*(X)$. The uniformity $\mathcal{U}_E$ is called the *endograph uniformity*.

We start this section with a characterization of totally bounded subsets in $\mathcal{F}^*(X)$.

Theorem 3.1. *Let $(X,\mathcal{U})$ be a uniform space and a non-empty subset $A \subseteq \mathcal{F}^*(X)$. Then the following conditions are equivalent:*

i) *A is totally bounded in $(\mathcal{F}^*(X),\mathcal{U}_E)$.*
ii) *$A(\alpha) = \bigcup\{[u]_\alpha : u \in A\}$ is totally bounded in $(X,\mathcal{U})$ for each $\alpha \in (0,1]$.*
iii) *$A_\alpha = \{[u]_\alpha : u \in A\}$ is totally bounded in $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$ for each $\alpha \in (0,1]$.*

Proof. Let us show that i) implies ii). Suppose that A is a totally bounded subset in $(\mathcal{F}^*(X),\mathcal{U}_E)$. Fix $\alpha \in (0,1]$. Take $U \in \mathcal{U}$. We can find a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Put $\epsilon = \frac{\alpha}{2} < \alpha$ and $\delta = \alpha - \frac{\epsilon}{4} > 0$. Since A is totally bounded in $(\mathcal{F}^*(X),\mathcal{U}_E)$, there exist $u_1,...,u_k \in A$ such that $A \subseteq \bigcup_{i=1}^k E[V,\epsilon](u_i)$. We also put $A_\alpha(k) = \bigcup_{i=1}^k [u_i]_\alpha$ and $A_\epsilon(k) = \bigcup_{i=1}^k [u_i]_\epsilon$. Note that $A_\alpha(k) \subseteq A_\epsilon(k)$. Clearly, $A_\epsilon(k)$ is totally bounded in $(X,\mathcal{U})$. Hence, there exists a finite subset $J \subseteq A_\epsilon(k)$ such that $A_\epsilon(k) \subseteq V(J)$. Define $J' = \{b \in J : V^2(b) \cap A(\alpha) \neq \varnothing\}$.

Claim I: $A(\alpha) \subseteq U(J')$.

Take $a \in A(\alpha)$. Then $a \in [u]_\alpha$ for some $u \in A$. So $(end(u), end(u_i)) \in \mathcal{C}[V \times V_\epsilon]$ for some $i = 1,2,...,k$. Then there exists $(z_a,\beta) \in end(u_i)$ with $((a,\alpha),(z_a,\beta)) \in V \times V_\epsilon$. So $(a,z_a) \in V$ and $\alpha - \beta < \epsilon = \frac{\alpha}{2}$. Hence $\epsilon < \beta$. It follows that

$$z_a \in [u_i]_\beta \subseteq [u_i]_\epsilon \subseteq A_\epsilon(k).$$

By the choice of J, we can find $b \in J$ with $z_a \in V(b)$. Since $(a,z_a) \in V$ and $(z_a,b) \in V$, we have that $(a,b) \in V^2$. Hence $a \in V^2(b) \cap A(\alpha)$. So $b \in J'$ and $a \in V^2(b) \subseteq U(b) \subseteq U(J')$, which proves Claim I. So Proposition 2.7 and Claim I imply that $A(\alpha)$ is totally bounded in $(X,\mathcal{U})$.

Let us prove that ii) $\Rightarrow$ iii). We now assume that $A(\alpha)$ is totally bounded in $(X,\mathcal{U})$ for each $\alpha \in (0,1]$. Take $\alpha \in (0,1]$, we put $X_\alpha = A(\alpha)$ and $\mathcal{U}_\alpha = \mathcal{U}|_{X_\alpha}$.

By Proposition 2.8, the uniform space $(\mathcal{K}(X_\alpha), \mathcal{K}(\mathcal{U}_\alpha))$ is totally bounded. Note that $A_\alpha \subseteq \mathcal{K}(X_\alpha)$. It follows from [1, Theorem 8.3.2] that A_α is totally bounded in $(\mathcal{K}(X_\alpha), \mathcal{K}(\mathcal{U}_\alpha))$. Given $U \in \mathcal{U}$, there exists a finite subset $J \subseteq A_\alpha$ such that $A_\alpha \subseteq \mathcal{K}[U \cap X_\alpha^2](J) \subseteq \mathcal{K}[U](J)$. Therefore, A_α is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$.

In order to show that iii) implies i), assume that $A_\alpha = \{[u]_\alpha : u \in A\}$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$ for each $\alpha \in (0, 1]$. Let us show that A is totally bounded in $(\mathcal{F}^*(X), \mathcal{U}_E)$. Take $W \in \mathcal{U}$ and $\epsilon > 0$. We can assume that $\epsilon < 1$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Put $\alpha_i = \frac{n+1-i}{n}$ for each $i = 1, ..., n$ and $\alpha_{n+1} = 0$. Since A_{α_i} is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$ for each $i = 1, ..., n$, there exists a finite subset $I_i \subseteq A_{\alpha_i}$ such that $A_{\alpha_i} \subseteq \mathcal{K}[W](I_i)$ for each $i = 1, ..., n$. By Proposition 2.7, we can assume that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ and every I_i is closed under union. Let $\mathcal{V}$ be the family of $v \in \mathcal{F}^*(X)$ such that $[v]_\alpha = K_i \in I_i$ for each $\alpha \in (\alpha_{i+1}, \alpha_i]$ and each $i = 1, 2, ..., n$. Clearly, $\mathcal{V}$ is finite and non-empty. Let us prove the following:

$$A \subseteq E\,[W, \epsilon]\,(\mathcal{V}). \tag{3.1}$$

Take $u \in A$. Then there exists $K_i \in I_i$ such that $([u]_{\alpha_i}, K_i) \in \mathcal{K}[W]$ for each $i = 1, 2, ..., n$. By Lemma 2.6 and the fact that each I_i is closed under union, we can suppose that $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$. Let $v \in \mathcal{V}$ be such that $[v]_\alpha = K_i$ for each $\alpha \in (\alpha_{i+1}, \alpha_i]$ and each $i = 1, 2, ..., n$. Note that $v_0 = [v]_{\alpha_{n+1}} = K_n$. Pick $(x, \beta) \in end(u)$. If $\alpha_n \geq \beta \geq \alpha_{n+1}$, then

$$(x, \beta) \in [W \times V_\epsilon](x, 0) \subseteq [W \times V_\epsilon](end(v)).$$

We now suppose that $\alpha_i \geq \beta > \alpha_{i+1}$ for some $i = 1, 2, ..., n-1$. Since $([u]_{\alpha_i}, K_i) \in \mathcal{K}[W]$ and $x \in [u]_\beta \subseteq [u]_{\alpha_{i+1}}$ for each $i = 1, 2, ..., n-1$, there exists $k \in \mathcal{K}_{i+1}$ such that $(x, k) \in W$. So $((x, \beta), (k, \alpha_{i+1})) \in W \times V_\epsilon$. Therefore, $(x, \beta) \in [W \times V_\epsilon](end(v))$ for each $(x, \beta) \in end(u)$. We have thus proved that $end(u) \subseteq [W \times V_\epsilon](end(v))$.

Using a similar argument, we can show that $end(v) \subseteq [W \times V_\epsilon](end(u))$. Hence $u \in E\,[W, \epsilon]\,(v)$. Therefore, $A \subseteq E\,[W, \epsilon]\,(\mathcal{V})$. By (3.1) and Proposition 2.7, we have that A is totally bounded in $(\mathcal{F}^*(X), \mathcal{U}_E)$. □

Corollary 3.2. *Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{D} \subseteq \mathcal{K}(X)$. Then the following conditions are equivalent:*

i) *$\boldsymbol{D} = \bigcup\{C \in \mathcal{D}\}$ is totally bounded in $(X, \mathcal{U})$.*
ii) *$\mathcal{D}$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$.*

Proof. We put $A = \{\chi_K : K \in \mathcal{D}\} \subseteq \mathcal{F}^*(X)$ and apply Theorem 3.1. □

We need the following three results in order to prove Theorem 3.6.

Lemma 3.3. *Consider a uniform space $(X, \mathcal{U})$ and $\mathcal{D} \subseteq \mathcal{K}(X)$. If $(\mathcal{D}, \mathcal{K}(\mathcal{U})|_{\mathcal{D}})$ is compact, then $\boldsymbol{D} = \bigcup\{C \in \mathcal{D}\}$ is compact with respect to the uniformity $\mathcal{U}|_{\boldsymbol{D}}$.*

Proof. We can assume that $(X, \mathcal{U})$ is complete, otherwise we can take its completion. Let $\{x_\sigma\}_{\sigma \in \Sigma}$ be a net in $\mathbf{D}$. Pick $C_\sigma \in \mathcal{D}$ such that $x_\sigma \in C_\sigma$. Since

$(\mathcal{D},\mathcal{K}(\mathcal{U})|_{\mathcal{D}})$ is compact, the net $\{C_\sigma\}_{\sigma\in\Sigma}$ has a finer net $\{C_{\sigma'}\}_{\sigma'\in\Sigma'}$ which converges to $C\in\mathcal{D}$. The set $\mathcal{E}=\{C\}\cup\{C_{\sigma'}:\sigma'\in\Sigma'\}\subseteq\mathcal{D}$ is totally bounded, since $\mathcal{D}$ is compact. By Corollary 3.2, $\mathbf{E}=\bigcup\{E\in\mathcal{E}\}$ is totally bounded in $(X,\mathcal{U})$. Then $\overline{\mathbf{E}}$ is totally bounded in $(X,\mathcal{U})$. So $\overline{\mathbf{E}}$ is compact, since $(X,\mathcal{U})$ is complete. We know that $x_{\sigma'}\in\mathbf{E}$ for each $\sigma'\in\Sigma'$. Hence there exists a net $\{x_{\sigma''}\}_{\sigma''\in\Sigma''}$ finer than $\{x_{\sigma'}\}_{\sigma'\in\Sigma'}$ which converges to $x\in\overline{\mathbf{E}}$. It is straightforward to show that $x\in C$. We have thus proved that $\{x_\sigma\}_{\sigma\in\Sigma}$ has a finer net which converges to $x\in\mathbf{D}$. Therefore, $\mathbf{D}$ is compact. □

Lemma 3.4. *Consider a uniform space $(X,\mathcal{U})$ and $\mathcal{D}\subseteq\mathcal{K}(X)$. If $\boldsymbol{D}=\bigcup\{C\in\mathcal{D}\}$ is complete with respect to the uniformity $\mathcal{U}|_{\boldsymbol{D}}$ and $\mathcal{D}$ is closed in $\mathcal{K}(X)$, then $(\mathcal{D},\mathcal{K}(\mathcal{U})|_{\mathcal{D}})$ is complete.*

Proof. If $\mathbf{D}$ is complete with respect to the uniformity $\mathcal{U}|_{\mathbf{D}}$, then $(\mathcal{K}(\mathbf{D}),\mathcal{K}(\mathcal{U})|_{\mathcal{K}(\mathbf{D})})$ is complete by [12]. Since $\mathcal{D}$ is closed in $\mathcal{K}(X)$, we have that $\mathcal{D}$ is closed in $\mathcal{K}(\mathbf{D})$. The completeness of $(\mathcal{K}(\mathbf{D}),\mathcal{K}(\mathcal{U})|_{\mathcal{K}(\mathbf{D})})$ implies that $(\mathcal{D},\mathcal{K}(\mathcal{U})|_{\mathcal{D}})$ is complete. □

Proposition 3.5. *Consider a uniform space $(X,\mathcal{U})$ and $\mathcal{D}\subseteq\mathcal{K}(X)$. Then the following conditions are equivalent:*

i) *$\mathcal{D}$ is compact in $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$.*
ii) *$\boldsymbol{D}=\bigcup\{C\in\mathcal{D}\}$ is compact in $(X,\mathcal{U})$ and $\mathcal{D}$ is closed in $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$.*

Proof. i) ⇒ ii) by Lemma 3.3. Let us show that ii) ⇒ i). If $\mathbf{D}$ is compact, then $\mathcal{D}$ is totally bounded in $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$ by Corollary 3.2. On the other hand, $\mathcal{D}$ is complete by Lemma 3.4. Therefore, $\mathcal{D}$ is compact in $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$. □

Theorem 3.6. *Let $(X,\mathcal{U})$ be a uniform space and a non-empty subset $A\subseteq\mathcal{F}^*(X)$. Then the following conditions are equivalent:*

i) *A is compact in $(\mathcal{F}^*(X),\mathcal{U}_E)$.*
ii) *A is closed in $(\mathcal{F}^*(X),\mathcal{U}_E)$ and $A(\alpha)=\bigcup\{[u]_\alpha:u\in A\}$ is compact in $(X,\mathcal{U})$ for each $\alpha\in(0,1]$.*

Proof. Let $(\widehat{X},\widehat{\mathcal{U}})$ the completion of $(X,\mathcal{U})$. Then $\mathcal{F}^*(X)\subseteq\mathcal{F}^*(\widehat{X})$. Let us show that i) implies ii). Clearly, A is compact in $(\mathcal{F}^*(\widehat{X}),\widehat{\mathcal{U}}_E)$. By Theorem 3.1, $A(\alpha)$ is totally bounded in $(\widehat{X},\widehat{\mathcal{U}})$ for each $\alpha\in(0,1]$. Let us show that $A(\alpha)$ is closed in $(\widehat{X},\widehat{\mathcal{U}})$ for each $\alpha\in(0,1]$. Take $\alpha\in(0,1]$ and $x\in\overline{A(\alpha)}^{\widehat{X}}$. Then there exists a net $\{x_\sigma\}_{\sigma\in\Sigma}$ in $A(\alpha)$ which converges to x. For every $\sigma\in\Sigma$, we can choose $u_\sigma\in A$ such that $x_\sigma\in[u_\sigma]_\alpha$. Since A is compact $\{u_\sigma\}_{\sigma\in\Sigma}$ has a finer net $\{u_\sigma\}_{\sigma\in\Sigma'}$ which converges to $u\in A$. We define $v\in\mathcal{F}^*(\widehat{X})$ as follows:

$$[v]_\beta=\begin{cases}[u]_\beta, & \text{if } \beta\in(\alpha,1].\\ \{x\}\cup[u]_\beta, & \text{if } \beta\in(0,\alpha].\end{cases}$$

Let us show that $\{u_\sigma\}_{\sigma\in\Sigma'}$ converges to v. Given $U\in\widehat{\mathcal{U}}$ and $\epsilon>0$, there exists $\sigma_0\in\Sigma'$ such that $(x,x_\sigma)\in U$ and $(u,u_\sigma)\in E[U,\epsilon]$ for every $\sigma\geq\sigma_0$. Take $\sigma\geq\sigma_0$. Clearly, $end(u_\sigma)\subseteq[U\times V_\epsilon](end(u))\subseteq[U\times V_\epsilon](end(v))$. We now pick $(y,\beta)\in end(v)$. If $y\neq x$, then $(y,\beta)\in end(u)\subseteq[U\times V_\epsilon](end(u_\sigma))$. On the other hand, if $y = x$, the definition of v implies that $\beta \leq \alpha$. Then $x_\sigma \in [u_\sigma]_\alpha \subseteq [u_\sigma]_\beta$. So $(x_\sigma, \beta) \in end(u_\sigma)$ and $(x, \beta) \in [U \times V_\epsilon](x_\sigma, \beta) \subseteq [U \times V_\epsilon](end(u_\sigma))$. Hence, $end(v) \subseteq [U \times V_\epsilon](end(u_\sigma))$. We have thus proved that $(v, u_\sigma) \in E[U, \epsilon]$ for every $\sigma \geq \sigma_0$. Therefore, $u = v$ and $x \in [u]_\alpha \subseteq A(\alpha)$. So $A(\alpha)$ is closed and totally bounded in $(\widehat{X}, \widehat{\mathcal{U}})$. It follows that $A(\alpha)$ is compact.

In order to show that ii) $\Rightarrow$ i), assume that A is closed in $(\mathcal{F}^*(X), \mathcal{U}_E)$ and $A(\alpha) = \bigcup\{[u]_\alpha : u \in A\}$ is compact in $(X, \mathcal{U})$ for each $\alpha \in (0, 1]$. By Theorem 3.1, A is totally bounded in $(\mathcal{F}^*(\widehat{X}), \widehat{\mathcal{U}}_E)$. We put $X_\alpha = A(\alpha)$ for each $\alpha \in (0, 1)$. Given $u \in \mathcal{F}^*(X)$ and $\alpha \in (0, 1)$, we put $end_\alpha(u) = [u_{\alpha^+} \times \{\alpha\}] \cup [end(u) \cap (X \times (\alpha, 1])]$, see Remark 2.10 for the symbol u_{α^+}. Note that $end_\alpha(u) \in \mathcal{K}(X_\alpha \times [0, 1])$. Since X_α is compact, we can conclude that $\mathcal{K}(X_\alpha \times [0, 1])$ is compact for every $\alpha \in (0, 1)$. We can argue as in the proof of [6, Theorem 5.3] to show that $E_\alpha = \{end_\alpha(u) : u \in A\}$ is closed in $\mathcal{K}(X_\alpha \times [0, 1])$. Hence E_α is compact for each $\alpha \in (0, 1)$.

Claim 1: Take $0 < \beta < \alpha < 1$. Suppose that $\{end_\alpha(u_\sigma)\}_{\sigma \in \Sigma}$ and $\{end_\beta(u_\sigma)\}_{\sigma \in \Sigma}$ have a finer net $\{end_\alpha(u_\sigma)\}_{\sigma \in \Sigma'}$ and $\{end_\beta(u_\sigma)\}_{\sigma \in \Sigma'}$ which converge to $end_\alpha(u)$ and $end_\beta(v)$, respectively. Then $[u]_\gamma = [v]_\gamma$ for each $\gamma \in (\alpha, 1]$.

Pick $\gamma \in (\alpha, 1]$. Let us show that $([u]_\gamma, [v]_\gamma) \in \mathcal{K}[W]$ for every $W \in \mathcal{U}$. Take a symmetric $U \in \mathcal{U}$ such that $U^4 \subseteq W$. Put $d = \gamma - \alpha$ and $\alpha_n = \gamma - \frac{d}{4n}$ for each $n \in \mathbb{N}$. Then the sequence $\{\alpha_n\}_n \subseteq (\alpha, \gamma)$ is increasing and converges to γ. Since $\{end_\alpha(u_\sigma)\}_{\sigma \in \Sigma'}$ and $\{end_\beta(u_\sigma)\}_{\sigma \in \Sigma'}$ converge to $end_\alpha(u)$ and $end_\beta(v)$, respectively; then for every $n \in \mathbb{N}$, there exists $\sigma_n \in \Sigma'$ such that

$$end_\alpha(u_{\sigma_n}) \subseteq [U \times V_{\frac{d}{4n}}](end_\alpha(u)) \quad \text{and} \quad end_\alpha(u) \subseteq [U \times V_{\frac{d}{4n}}](end_\alpha(u_{\sigma_n})). \tag{3.2}$$

$$end_\beta(u_{\sigma_n}) \subseteq [U \times V_{\frac{d}{4n}}](end_\beta(v)) \quad \text{and} \quad end_\beta(v) \subseteq [U \times V_{\frac{d}{4n}}](end_\beta(u_{\sigma_n})). \tag{3.3}$$

From (3.2) and (3.3), we have that $end_\alpha(u) \subseteq [U^2 \times V_{\frac{d}{2n}}](end_\beta(v))$ for each $n \in \mathbb{N}$. Fix $x \in [u]_\gamma$. Since $(x, \alpha_n) \in end_\alpha(u)$, we can take $(y_n, \beta_n) \in end_\beta(v)$ such that $((x, \alpha_n), (y_n, \beta_n)) \in U^2 \times V_{\frac{d}{2n}}$. Since $|\alpha_n - \beta_n| < \frac{d}{2n}$ and $\{\alpha_n\}_n$ converges to γ, we can conclude that $\{\beta_n\}_n$ converges to γ. Note that the sequence $\{(y_n, \beta_n)\}_n$ is in the compact set $end_\beta(v)$. Therefore, we can suppose that $\{(y_n, \beta_n)\}_n$ converges to (y, γ). Hence $y \in [v]_\gamma$. On the other hand, $(x, y_n) \in U^2$ for each $n \in \mathbb{N}$. The latter fact implies that $(x, y) \in \overline{U^2} \subseteq U^3$. So $x \in U^3(y) \subseteq W(y)$. Hence $[u]_\gamma \subseteq W([v]_\gamma)$.

Fix $x \in [v]_\gamma$. By (3.3) and $(x, \alpha_n) \in end_\beta(v)$, we can take $(y_n, \beta_n) \in end_\beta(u_{\sigma_n})$ such that $((x, \alpha_n), (y_n, \beta_n)) \in U \times V_{\frac{d}{4n}}$. Since $|\alpha_n - \beta_n| < \frac{d}{4n}$ for every $n \in \mathbb{N}$, we have the following:

$$\alpha = \frac{(2n-1)\alpha + \alpha}{2n} < \frac{(2n-1)\gamma + \alpha}{2n} = \gamma - \frac{d}{2n} = \alpha_n - \frac{d}{4n} < \beta_n < \alpha_n + \frac{d}{4n} = \gamma.$$

It follows that $\beta_n \in (\alpha, \gamma)$ for all $n \in \mathbb{N}$. So $(y_n, \beta_n) \in end_\alpha(u_{\sigma_n})$. By (3.2), we can take $(z_n, \delta_n) \in end_\alpha(u)$ such that $((y_n, \beta_n), (z_n, \delta_n)) \in U \times V_{\frac{d}{4n}}$. For each $n \in \mathbb{N}$, we have that

$$|\alpha_n - \delta_n| \leq |\alpha_n - \beta_n| + |\beta_n - \delta_n| < \frac{d}{2n}.$$

Since $\{\alpha_n\}_n$ converges to γ, we can conclude that $\{\delta_n\}_n$ converges to γ. Note that the sequence $\{(z_n, \delta_n)\}_n$ is in the compact set $end_\alpha(u)$. Therefore, we can suppose that $\{(z_n, \delta_n)\}_n$ converges to (z, γ). Hence $z \in [u]_\gamma$. On the other hand, $(x, z_n) \in U^2$ for each $n \in \mathbb{N}$. The latter fact implies that $(x, z) \in \overline{U^2} \subseteq U^3 \subseteq W$. So $x \in W(z)$ and $[v]_\gamma \subseteq W([u]_\gamma)$. Hence $([u]_\gamma, [v]_\gamma) \in \mathcal{K}[W]$ for every $W \in \mathcal{U}$, whence $[u]_\gamma = [v]_\gamma$ for each $\gamma \in (\alpha, 1]$. This completes the proof of **Claim 1**.

Take a net $\{u_\sigma\}_{\sigma \in \Sigma_1}$ in A. Since E_α is compact for each $(0, 1)$, the net $\{end_{\frac{1}{2}}(u_\sigma)\}_{\sigma \in \Sigma_1}$ has a finer net $\{end_{\frac{1}{2}}(u_\sigma)\}_{\sigma \in \Sigma_2}$ which converges to $end_{\frac{1}{2}}(v_2)$ with $v_2 \in A$. By induction, for every $n \in \mathbb{N}$, we can obtain a net $\{end_{\frac{1}{n+1}}(u_\sigma)\}_{\sigma \in \Sigma_{n+1}}$ which is finer than $\{end_{\frac{1}{n+1}}(u_\sigma)\}_{\sigma \in \Sigma_n}$ and $\{end_{\frac{1}{n+1}}(u_\sigma)\}_{\sigma \in \Sigma_{n+1}}$ converges to $end_{\frac{1}{n+1}}(v_{n+1})$ with $v_{n+1} \in A$.

By **Claim 1**, the set $(X \times \{0\}) \cup \bigcup_{n \geq 2} end_{\frac{1}{n}}(v_n)$ is the endograph of a fuzzy set $v \in \mathcal{F}^*(X)$. Let us show that v is an accumulation point of $\{u_\sigma\}_{\sigma \in \Sigma_1}$. Take $U \in \mathcal{U}$ and $\epsilon > 0$. We can choose $n \geq 2$ such that $\frac{1}{n} < \epsilon$. Fix $\sigma_0 \in \Sigma$. Since $\{end_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_n}$ converges to $end_{\frac{1}{n}}(v_n)$, we can find $\sigma \geq \sigma_0$ such that

$$end_{\frac{1}{n}}(u_\sigma) \subseteq [U \times V_{\frac{1}{n}}](end_{\frac{1}{n}}(v_n)) \quad \text{and} \quad end_{\frac{1}{n}}(v_n) \subseteq [U \times V_{\frac{1}{n}}](end_{\frac{1}{n}}(u_\sigma)). \tag{3.4}$$

Take $(x, \alpha) \in end(v)$ with $\alpha \in [0, \frac{1}{n}]$. Then $(x, x) \in U$ and $(\alpha, 0) \in V_\epsilon$. So $(x, \alpha) \in [U \times V_\epsilon](end(u_\sigma))$. If $\alpha > \frac{1}{n}$, (3.4) implies the following:

$$(x, \alpha) \in end_{\frac{1}{n}}(v_n) \subseteq [U \times V_{\frac{1}{n}}](end_{\frac{1}{n}}(u_\sigma)) \subseteq [U \times V_\epsilon](end(u_\sigma)).$$

We have thus proved that $end(v) \subseteq [U \times V_\epsilon](end(u_\sigma))$. Similarly, we can show that $end(u_\sigma) \subseteq [U \times V_\epsilon](end(v))$. Therefore, v is an accumulation point of $\{u_\sigma\}_{\sigma \in \Sigma_1}$. Finally, we know that A is closed in $\mathcal{F}^*(X)$, so $v \in A$. We can conclude that every net in A has an accumulation point in A, i.e., A is compact. ☐

Consider now a metric space (X, d). Define the metric d^* on $X \times [0, 1]$ as follows:

$$d^*((x, a), (y, b)) = \max\{d(x, y), |a - b|\}.$$

The *endograph metric* d_E on $\mathcal{F}^*(X)$ is the Hausdorff distance d^*_H (with respect to $X \times [0, 1]$) between $end(u)$ and $end(v)$ for each $u, v \in \mathcal{F}^*(X)$. Recall that a metric space (X, d) has a natural uniformity $\mathcal{U}_d$ determinated by the base $\{U_\epsilon : \epsilon > 0\}$, where $U_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$.

Corollary 3.7 ([4]). *Let (X, d) be a metric space and a non-empty subset $A \subseteq \mathcal{F}^*(X)$. Then the following conditions are equivalent:*

i) *A is compact in* $(\mathcal{F}^*(X), d_E)$.
ii) *A is closed in* $(\mathcal{F}^*(X), d_E)$ *and* $A(\alpha) = \bigcup\{[u]_\alpha : u \in A\}$ *is compact in* (X, d) *for each* $\alpha \in (0, 1]$.

Proof. By a result of [6], we have that $\mathcal{U}_{d_E} = (\mathcal{U}_d)_E$. It is easy to see that A is compact (closed) in $(\mathcal{F}^*(X), \mathcal{U}_{d_E})$ if and only if A is compact (closed) in $(\mathcal{F}^*(X), d_E)$ if and only if A is compact (closed) in $(\mathcal{F}^*(X), (\mathcal{U}_d)_E)$. We also have that $A(\alpha)$ is compact in $(X, \mathcal{U}_d)$ if and only if $A(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$. It remains to apply Theorem 3.6 to the uniform space $(X, \mathcal{U}_d)$. □

4. Compactness in the sendograph uniformity

Given a uniform space $(X, \mathcal{U})$, we denote by $\mathcal{F}(X)$ the elements of $\mathcal{F}^*(X)$ with compact support. If $u \in \mathcal{F}(X)$, the *sendograph* of u is defined by $send(u) = end(u) \cap (u_0 \times [0, 1])$. Observe that $send(u) \in \mathcal{K}(X \times [0, 1])$. Given $U \in \mathcal{U}$ and $\epsilon > 0$, we define the following sets:

$$S[U, \epsilon] = \{(u, v) \in \mathcal{F}(X) \times \mathcal{F}(X) : (send(u), send(v)) \in \mathcal{K}[U \times V_\epsilon]\}.$$

By Proposition 2.3, the family $\{S[U, \epsilon] : U \in \mathcal{U}, \epsilon > 0\}$ is base for a uniformity $\mathcal{U}_S$ on $\mathcal{F}(X)$. The uniformity $\mathcal{U}_S$ is called the *sendograph uniformity*.

Consider now a metric space (X, d). Define the metric d^* on $X \times [0, 1]$ as follows:

$$d^*((x, a), (y, b)) = \max\{d(x, y), |a - b|\}.$$

The *sendograph metric* d_S on $\mathcal{F}(X)$ is the Hausdorff metric d^*_H (on $\mathcal{K}(X \times [0, 1])$) between the non-empty compact subsets $send(u)$ and $send(v)$ for every $u, v \in \mathcal{F}(X)$ (see [10]).

Theorem 4.1. *Let A be a non-empty subset of a uniform space* $(X, \mathcal{U})$. *Then A is totally bounded in* $(\mathcal{F}(X), \mathcal{U}_S)$ *if and only if* $A(0) = \bigcup_{u \in A} u_0$ *is totally bounded in* $(X, \mathcal{U})$.

Proof. Suppose that A is a totally bounded subset in $(\mathcal{F}(X), \mathcal{U}_S)$. Take $U \in \mathcal{U}$. We can find a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Since A is totally bounded in $(\mathcal{F}(X), \mathcal{U}_S)$, there exist $u_1, ..., u_k \in A$ such that $A \subseteq \bigcup_{i=1}^k S[V, 1](u_i)$. We also put $A(k) = \bigcup_{i=1}^k [u_i]_0$. Clearly, $A(k)$ is totally bounded in $(X, \mathcal{U})$. Hence, there exists a finite subset $J \subseteq A(k)$ such that $A(k) \subseteq V(J)$. Define $J' = \{b \in J : V^2(b) \cap A(0) \neq \varnothing\}$.

Claim II: $A(0) \subseteq U(J')$.

Take $a \in A(0)$. Then $a \in [u]_0$ for some $u \in A$. So $(send(u), send(u_i)) \in \mathcal{K}[V \times V_1]$ for some $i = 1, 2, ..., k$. Then there exists $(z_a, \beta) \in send(u_i)$ with $((a, 0), (z_a, \beta)) \in V \times V_1$. So $(a, z_a) \in V$ and $\beta < 1$. It follows that

$$z_a \in [u_i]_\beta \subseteq [u_i]_0 \subseteq A(k).$$

By the choice of J, we can find $b \in J$ with $z_a \in V(b)$. Then $(a, z_a), (z_a, b) \in V$. So $(a, b) \in V^2$. Hence $a \in V^2(b) \cap A(0)$. So $b \in J'$ and $a \in V^2(b) \subseteq U(b) \subseteq$

$U(J')$. This completes the proof of Claim II. Proposition 2.7 and Claim II imply that $A(0)$ is totally bounded in $(X,\mathcal{U})$.

For the converse, we assume that $A(0)$ is totally bounded in $(X,\mathcal{U})$. Hence $A(\alpha)$ is totally bounded in $(X,\mathcal{U})$ for every $\alpha \in [0,1]$. For each $\alpha \in [0,1]$, we put $X_\alpha = A(\alpha)$ and $\mathcal{U}_\alpha = \mathcal{U}|_{X_\alpha}$. By Proposition 2.8, the uniform space $(\mathcal{K}(X_\alpha), \mathcal{K}(\mathcal{U}_\alpha))$ is totally bounded. Let us show that A is totally bounded in $(\mathcal{F}(X), \mathcal{U}_S)$. Take $W \in \mathcal{U}$ and $\epsilon > 0$. We can assume that $\epsilon < 1$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Put $\alpha_i = \frac{n+1-i}{n}$ for each $i = 1, ..., n$ and $\alpha_{n+1} = 0$. Since $(\mathcal{K}(X_{\alpha_i}), \mathcal{K}(\mathcal{U}_{\alpha_i}))$ is totally bounded for each $i = 1, ..., n$, there exists a finite subset $I_i \subseteq \mathcal{K}(X_{\alpha_i})$ such that $\mathcal{K}(X_{\alpha_i}) = \mathcal{K}[W \cap X^2_{\alpha_i}](I_i)$ for each $i = 1, ..., n$. By Proposition 2.7, we can assume that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ and every I_i is closed under union. Let $\mathcal{V}$ be the family of $v \in \mathcal{F}(X)$ such that $[v]_\alpha = K_i \in I_i$ for each $\alpha \in (\alpha_{i+1}, \alpha_i]$ and each $i = 1, 2, ..., n$. Clearly, $\mathcal{V}$ is finite and non-empty. Let us prove the following:

$$A \subseteq S\,[W, \epsilon]\,(\mathcal{V}). \tag{4.1}$$

Take $u \in A$. Then there exists $K_i \in I_i$ such that $([u]_{\alpha_i}, K_i) \in \mathcal{K}[W \cap X^2_{\alpha_i}]$ for each $i = 1, 2, ..., n$. By Lemma 2.6 and the fact that each I_i is closed under union, we can suppose that $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$. Let $v \in \mathcal{V}$ be such that $[v]_\alpha = K_i$ for each $\alpha \in (\alpha_{i+1}, \alpha_i]$ and each $i = 1, 2, ..., n$. Note that $v_0 = K_n$. Pick $(x, \beta) \in send(u)$. Suppose that $\alpha_i \geq \beta > \alpha_{i+1}$ for some $i = 1, 2, ..., n-1$. Since $([u]_{\alpha_i}, K_i) \in \mathcal{K}[W \cap X^2_{\alpha_i}]$ and $x \in [u]_\beta \subseteq [u]_{\alpha_{i+1}}$ for each $i = 1, 2, ..., n-1$, there exists $k \in \mathcal{K}_{i+1}$ such that $(x, k) \in W$. So $((x, \beta), (k, \alpha_{i+1})) \in W \times V_\epsilon$. Therefore, $(x, \beta) \in [W \times V_\epsilon](send(v))$. Now if $(x, \beta) \in send(u)$ and $0 \leq \beta \leq \frac{1}{n}$, then $x \in u_0 = \overline{\bigcup_{\alpha>0} [u]_\alpha}$. Hence $u_0 \cap W(x) \neq \varnothing$. So we can find $y \in [u]_\alpha$ for some $\alpha > 0$ such that $(x, y) \in W$. We can assume that $\alpha \in (0, \frac{1}{n}]$. Therefore, $(x, \beta) \in [W \times V_\epsilon](y, \alpha) \subseteq [W \times V_\epsilon](send(v))$. We have thus proved that $send(u) \subseteq [W \times V_\epsilon](send(v))$.

Using a similar argument, we can show that $send(v) \subseteq [W \times V_\epsilon](send(u))$. Hence $u \in S\,[W, \epsilon]\,(v)$. Therefore, $A \subseteq S\,[W, \epsilon]\,(\mathcal{V})$. By (4.1) and Proposition 2.7, we have that A is totally bounded in $(\mathcal{F}(X), \mathcal{U}_S)$. □

Theorem 4.2. *Let A be a non-empty subset of a uniform space $(X,\mathcal{U})$. Then A is compact in $(\mathcal{F}(X), \mathcal{U}_S)$ if and only if A is closed in $(\mathcal{F}(X), \mathcal{U}_S)$ and $A(0)$ is compact in $(X,\mathcal{U})$.*

Proof. Assume that A is compact in $(\mathcal{F}(X), \mathcal{U}_S)$. Let $(\widehat{X}, \widehat{\mathcal{U}})$ be the completion of $(X,\mathcal{U})$. Then $\mathcal{F}(X) \subseteq \mathcal{F}(\widehat{X})$. Clearly, A is compact in $(\mathcal{F}(\widehat{X}), \widehat{\mathcal{U}}_S)$. By Theorem 4.1, $A(0)$ is totally bounded in $(\widehat{X}, \widehat{\mathcal{U}})$. Let us show that $A(0)$ is closed in $(\widehat{X}, \widehat{\mathcal{U}})$. Take $x \in \overline{A(0)}^{\widehat{X}}$ and a net $\{x_\sigma\}_{\sigma\in\Sigma}$ in $A(0)$ which converges to x. For every $\sigma \in \Sigma$, we take $u_\sigma \in A$ such that $x_\sigma \in [u_\sigma]_0$. Since A is compact, the net $\{u_\sigma\}_{\sigma\in\Sigma}$ in A has a finer net $\{u_\sigma\}_{\sigma\in\Sigma'}$ which converges to $u \in A$. Let us show that $x \in u_0$. Suppose the contrary, then there exists $W \in \widehat{\mathcal{U}}$ such that $W(x) \cap u_0 = \varnothing$. Pick $V \in \widehat{\mathcal{U}}$ such that $V^2 \subseteq U$. On the other hand, there exists $\sigma_0 \in \Sigma'$ such that $(u, u_\sigma) \in S[V, 1]$ and $(x, x_\sigma) \in V$ for each $\sigma \geq \sigma_0$. Hence $(x_{\sigma_0}, 0) \in send(u_{\sigma_0}) \subseteq [V \times V_1](send(u))$. So there exists $(y, \beta) \in send(u)$ with $(x_{\sigma_0}, y) \in V$ and $\beta < 1$. Then $y \in [u]_\beta \subseteq u_0$. Since $(x, x_{\sigma_0}) \in V$ and $(x_{\sigma_0}, y) \in V$, we have that $(x, y) \in W$. So $y \in W(x)$, which contradicts that $W(x) \cap u_0 = \varnothing$. Therefore, $A(0)$ is compact in $(X,\mathcal{U})$.

We now suppose that A is closed in $(\mathcal{F}(X),\mathcal{U}_S)$ and $A(0)$ is compact in $(X,\mathcal{U})$. Put $Y = A(0)$ and $\mathcal{V} = \mathcal{U}|_Y$. We can assume that $A \subseteq \mathcal{F}(Y) \subseteq \mathcal{F}(X)$. Since $(Y, \mathcal{V})$ is compact, $(\mathcal{F}(Y), \mathcal{V}_S)$ is complete by a result of [6]. Hence A is complete, since A is closed in $(\mathcal{F}(Y), \mathcal{V}_S)$. On the other hand, A is totally bounded in $(\mathcal{F}(Y), \mathcal{V}_S)$ by Theorem 4.1. Therefore, A is compact $(\mathcal{F}(X),\mathcal{U}_S)$. □

Corollary 4.3. [4] *Let A be a non-empty subset of a metric space (X, d). Then A is compact in $(\mathcal{F}(X), d_S)$ if and only if A is closed in $(\mathcal{F}(X), d_S)$ and $A(0)$ is compact in (X, d).*

Proof. It is easy to see that A is compact (closed) in $(\mathcal{F}(X), d_S)$ if and only if A is compact (closed) in $(\mathcal{F}(X),\mathcal{U}_{d_S})$. Since $\mathcal{U}_{d_S} = (\mathcal{U}_d)_S$, we have that A is compact (closed) in $(\mathcal{F}(X), d_S)$ if and only if A is compact (closed) in $(\mathcal{F}(X), (\mathcal{U}_d)_S)$. On the othe hand, $A(0)$ is compact in (X, d) if and only if $A(0)$ is compact in $(X,\mathcal{U}_d)$. If we apply Theorem 4.2 to the uniform space $(X,\mathcal{U}_d)$, we obtain the required conclusion. □

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