The degree of nondensifiability of linear bounded operators and its applications

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ABSTRACT

In the present paper we define the degree of nondensifiability (DND for short) of a bounded linear operator T on a Banach space and analyze its properties and relations with the Hausdorff measure of noncompactness (MNC for short) of T. As an application of our results, we have obtained a formula to find the essential spectral radius of a bounded operator T on a Banach space as well as we have provided the best possible lower bound for the Hyers-Ulam stability constant of T in terms of the aforementioned DND.

1. INTRODUCTION

Given two real Banach spaces X, Y, we designate by $\mathcal{B}(X,Y)$ and $\mathcal{K}(X,Y)$ the space of all bounded, or continuous, linear operators and the space of all compact linear operators from X into Y, respectively. When $X = Y$, we will write $\mathcal{B}(X)$ and $\mathcal{K}(X)$. We assume the domain of the operators is the whole space X. For a Banach space $(X, \|\cdot\|)$, U_X denotes its closed unit ball and $\mathcal{B}_X$

the class of all non-empty bounded subsets of X. For a subset B of X, $\overline{B}$ and $\mathrm{Conv}(B)$ denote the closure and the convex hull of B, respectively.

It is a well known fact that the topological concept of compactness is crucial in Mathematics. Likewise, such a concept plays a crucial role in the development of distinct branches of Mathematical Analysis such as Fixed Point Theory, Approximation Theory, Operator Theory, etc. For a more detailed exposition and concrete results about the importance of the compactness notion, we refer to [3].

The notion of a measure of non-compactness (MNC for short) of operators [3] have been successfully applied, for instance, in the characterizations of compact operators between Banach spaces (see, for instance, [10]). For concrete results see [9, 36]. Hence, the first part of this section is devoted to recall some known facts related with MNCs.

The theory of α-dense curves [29] appeared in 1997. This theory has been developed and applied in many different directions such as optimization by reduction of variables and fixed point theory, among others. An important notion deduced from such theory is that of degree of non-densifiability (DND for short) of a bounded set providing an intrinsic quantification of its non-compactness by means of the Hausdorff distance [24] from the set to the nearest Peano continuum [22, 40] that it contains. Therefore the DND becomes an alternative to MNCs. Recently several mathematical questions (see for instance [19]) have been addressed by using the concept of DND. On a general survey on that theory and its applications we suggest to see [11, 13, 14, 15, 16, 17, 19, 27, 30, 31].

Since the definition of a MNC in a Banach space may vary slightly according to the author (see, for instance, [2, 6]), below we will use the following definition of MNC which is taken from [17]:

Definition 1.1. Let $(X, \|\cdot\|)$ be a real Banach space and $\mathcal{B}_X$ the class of all non-empty bounded subsets of X. A mapping $\mu : \mathcal{B}_X \longrightarrow [0, +\infty)$ is said to be a MNC if it satisfies the following properties:

  (i) Regularity: $\mu(B) = 0$ if, and only if, B is a precompact set.
 (ii) Invariant under closure: $\mu(B) = \mu(\overline{B})$ for all $B \in \mathcal{B}_X$.
(iii) Semi-additivity: $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all A, $B \in \mathcal{B}_X$.
 (iv) Semi-homogeneity: $\mu(\lambda B) = |\lambda|\, \mu(B)$ for all $\lambda \in \mathbb{R}$ and $B \in \mathcal{B}_X$.
  (v) Invariant under translations: $\mu(x + B) = \mu(B)$ for all $x \in X$ and $B \in \mathcal{B}_X$.

For instance, two well known $MNCs$ are those of Hausdorff and Kuratowski defined as (see, for instance, [2, 6])

$$\chi(B) := \inf\{\varepsilon > 0 : B \text{ covered by a finite number of balls of radii } \leq \varepsilon\}$$

and

$$\kappa(B) := \inf\{\varepsilon > 0 : B \text{ covered by a finite number of subsets of diameters } \leq \varepsilon\},$$

respectively.

Definition 1.2. Let μ be a MNC, $k \geq 0$ and X, Y Banach spaces. A continuous mapping $T : X \longrightarrow Y$ such that $T(B) \in \mathcal{B}_Y$ if $B \in \mathcal{B}_X$ is said to be $k - \mu$-contractive if $\mu(T(B)) \leq k\mu(B)$ for all $B \in \mathcal{B}_X$.

Although is remarkable the importance of k-μ-contractive operators on a Banach space when $k \in [0, 1)$ for the fixed point theory, here we focus on the notion of MNC of an operator $T \in \mathcal{B}(X, Y)$.

Definition 1.3. Given an operator $T \in \mathcal{B}(X, Y)$ and a MNC μ, we define the MNC of T as the number

$$\widetilde{\mu}(T) := \inf \left\{ k \geq 0 : T \text{ is } k - \mu \text{ contractive} \right\}.$$

Some relevant results related with the MNC of a linear operator can be found for instance in [2, Chapter 2], [5], [12, Chapter 1] and [36]. To facilitate the reading of the manuscript we state the following result [36, Lemma 5.3]:

Proposition 1.4. *Let X, Y, Z be Banach spaces, T, $S \in \mathcal{B}(X, Y)$, $R \in \mathcal{B}(Y, Z)$ and χ the Hausdorff MNC. Then:*
(1) $\widetilde{\chi}(T) = 0$ if and only if $T \in \mathcal{K}(X, Y)$.
(2) $\widetilde{\chi}(T) = \chi\big(T(U_X)\big)$.
(3) $\widetilde{\chi}(T + S) \leq \widetilde{\chi}(T) + \widetilde{\chi}(S)$.
(4) $\widetilde{\chi}(R \circ T) \leq \widetilde{\chi}(R)\widetilde{\chi}(T)$.

2. The degree of non-densifiability of a bounded set

The initial idea of the α-dense theory [29] was to approximate, in the Hausdorff metric [24], a non-empty and bounded set B of a metric space by means of a special class of compact sets, namely, the Peano continua [40] contained in B. Therefore, we consider it convenient to recall the notion that gave rise to such theory, namely, the notion of α-dense curve in a metric space [29].

Definition 2.1. Let B be a non-empty bounded subset of a metric space (E, d) and $\alpha \geq 0$. A continuous mapping $\gamma : [0, 1] \longrightarrow E$ is said to be an α-dense curve in B if:

(1) $\gamma([0, 1]) \subset B$.
(2) For any $x \in B$ there exists $t \in [0, 1]$ such that $d\big(x, \gamma(t)\big) \leq \alpha$.

Definition 2.2. A non-empty subset B of a metric space (E, d) is said to be densifiable if for an arbitrary $\alpha > 0$ there exists an α-dense curve in B.

Given $\alpha \geq 0$ and a non-empty bounded subset B of a metric space (E, d), we define the class

$$\Gamma_{\alpha, B} := \left\{ \gamma : [0, 1] \longrightarrow E \text{ such that } \gamma \text{ is } \alpha\text{-dense in } B \right\}. \qquad (2.1)$$

The class $\Gamma_{\alpha, B}$ is well defined, i.e. $\Gamma_{\alpha, B} \neq \varnothing$. Indeed, as $B \neq \varnothing$, consider a point $x_0 \in B$ and define the constant mapping $\gamma(t) := x_0$ for all $t \in [0, 1]$. Then, since B is bounded, by taking $\alpha \geq \text{Diam}(B)$, the diameter of B, it is

obvious that γ is an α-dense curve in B. Therefore $\gamma \in \Gamma_{\alpha,B}$ and consequently $\Gamma_{\alpha,B} \neq \varnothing$. Now we can define the degree of non-densifiability of a bounded set [17].

Definition 2.3. Let (E,d) be a metric space and $B \subset E$ non-empty and bounded. We define the degree of non-densifiability of B as the number

$$\Phi(B) := \inf \{\alpha \geq 0 : \Gamma_{\alpha,B} \neq \varnothing\}. \qquad (2.2)$$

From above, given a non-empty and bounded set B of a metric space (E,d) there is always an $\alpha \geq 0$ such that $\Gamma_{\alpha,B} \neq \varnothing$. Therefore $\Phi(B)$ is well defined.

In [30] we can find a result on the role of the DND as an indicator of the dimension of the space.

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a Banach space. Then*

$$\Phi(U_X) = \begin{cases} 0, & \text{if } X \text{ has finite dimension} \\ \\ 1, & \text{if } X \text{ has infinite dimension} \end{cases}.$$

We need to use some properties of the DND that we can find in [16, 17]:

Proposition 2.5. *In a metric space (E,d), let Φ be defined on the class $\mathcal{B}_E$ by formula (2.2) and let $\mathcal{B}_{\mathrm{arc},E}$ be the subclass of all arc-connected sets of $\mathcal{B}_E$. Then:*

(i) If $B \in \mathcal{B}_{\mathrm{arc},E}$, $\Phi(B) = 0$ if and only if B is precompact.
(ii) $\Phi(B) = \Phi(\overline{B})$ for all $B \in \mathcal{B}_E$.
(iii) $\Phi(\lambda B) = |\lambda| \Phi(B)$ for all $\lambda \in \mathbb{R}$ and all $B \in \mathcal{B}_E$.
(iv) $\Phi(x + B) = \Phi(B)$ for all $x \in X$ and all $B \in \mathcal{B}_E$.
(v) $\Phi\big(\mathrm{Conv}(B_1 \cup B_2)\big) \leq \max\big\{\Phi\big(\mathrm{Conv}(B_1)\big), \Phi\big(\mathrm{Conv}(B_2)\big)\big\}$ for all $B_1, B_2 \in \mathcal{B}_E$.
(vi) $\Phi(B_1 + B_2) \leq \Phi(B_1) + \Phi(B_2)$ for all $B_1, B_2 \in \mathcal{B}_E$.

Definition 2.6. Let $f : X \longrightarrow Y$ be a continuous mapping such that $f(B) \in \mathcal{B}_Y$ for all $B \in \mathcal{B}_X$, and $k \geq 0$. Then, f is said to be k-DND-contractive if

$$\Phi\big(f(B)\big) \leq k\Phi(B) \quad \text{for all convex } B \in \mathcal{B}_X. \qquad (2.3)$$

Lemma 2.7. *Any $T \in \mathcal{B}(X,Y)$ is a k-DND-contractive mapping for any $k \geq \|T\|$.*

Proof. Let $B \in \mathcal{B}_X$ be a convex set of X. Since T is linear and bounded, or equivalently continuous, its norm is given by the formula $\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}$. Therefore $\|T(x)\| \leq \|T\| \|x\|$ for all $x \in X$. Furthermore, noticing that B is convex, $T(B)$ is a bounded and convex set of Y. Let $\gamma : [0,1] \longrightarrow X$ be an α-dense curve in B for some $\alpha \geq 0$. Then $T \circ \gamma : [0,1] \longrightarrow Y$ is clearly an $\alpha \|T\|$-dense curve in $T(B)$. Hence, by (2.2), $\Phi(T(B)) \leq \alpha \|T\|$. Given an

arbitrary $\alpha > \Phi(B)$, from (2.2), there exists an α-dense curve in B. Therefore $\Phi(T(B)) \leq \|T\| \Phi(B)$. Consequently (2.3) is fulfilled for all $k \geq \|T\|$ and for all convex $B \in \mathcal{B}_X$. This completes the proof. $\qquad \square$

3. The degree of nondensifiability of a bounded operator

Let us note that Lemma 2.7 allows us to justify the next definition is well done.

Definition 3.1. For every $T \in \mathcal{B}(X, Y)$ we define the DND of T as the number

$$\widetilde{\Phi}(T) := \inf \{k \geq 0 : T \text{ is } k\text{-}DND\text{- contractive}\}. \tag{3.1}$$

Some properties of the DND of an operator are given in the next result.

Proposition 3.2. *Let* X, Y, Z *be Banach spaces,* T, $S \in \mathcal{B}(X, Y)$ *and* $R \in \mathcal{B}(Y, Z)$. *Then:*

(1) $\Phi\big(T(U_X)\big) \leq \widetilde{\Phi}(T) \leq \|T\|$.
(2) $\widetilde{\Phi}(T) = 0$ *if, and only if,* $T \in \mathcal{K}(X, Y)$.
(3) $\widetilde{\Phi}(R \circ S) \leq \widetilde{\Phi}(R)\widetilde{\Phi}(S)$.
(4) $\widetilde{\Phi}(\lambda T) = |\lambda| \, \widetilde{\Phi}(T)$ *for all* $\lambda \in \mathbb{R}$.

Proof. (1) Firstly assume X has finite dimension, then U_X is compact. Since T is continuous, $T(U_X)$ is compact, so precompact. On the other hand, since U_X is convex and T linear, then $T(U_X)$ is convex, so, in particular, arc-connected. Therefore $T(U_X)$ is an arc-connected and precompact set of Y. Then, by property (1) of Proposition 2.5, $\Phi(T(U_X)) = 0$. Hence the inequality $\Phi(T(U_X)) \leq \widetilde{\Phi}(T)$ follows. Regarding the inequality $\widetilde{\Phi}(T) \leq \|T\|$, by applying Lemma 2.7, T is $\|T\|$-DND-contractive. Then by (3.1), $\widetilde{\Phi}(T) \leq \|T\|$. Consequently (1) is true when X has finite dimension.

If X has infinite dimension, by Proposition 2.4, $\Phi(U_X) = 1$. Under the assumption $\Phi(T(U_X)) > \widetilde{\Phi}(T)$, determine K such that $\widetilde{\Phi}(T) < K < \Phi(T(U_X))$. Then, by (3.1), T is k-contractive for some k such that $\widetilde{\Phi}(T) \leq k < K$. Hence, by (2.3), $\Phi(T(B)) \leq k\Phi(B)$ for all convex $B \in \mathcal{B}_X$. Therefore, by taking $B = U_X$, we have $\Phi(T(U_X)) \leq k$. Then, we get

$$\Phi\big(T(U_X)\big) \leq k < K < \Phi\big(T(U_X)\big),$$

which is a contradiction. Finally, the inequality $\widetilde{\Phi}(T) \leq \|T\|$ follows from Lemma 2.7, so (1) is proved.

(2) If $T \in \mathcal{K}(X, Y)$, $T(B)$ is precompact for all bounded set B. Then, by applying property (1) of Proposition 2.5, $\Phi(T(B)) = 0$ for all convex $B \in \mathcal{B}_X$. Therefore, from (2.3), T is k-DND-contractive for all $k \geq 0$. Hence, by (3.1), $\widetilde{\Phi}(T) = 0$. Reciprocally, assume $\widetilde{\Phi}(T) = 0$, then T is k-DND-contractive for all $k \geq 0$. Therefore, from (2.3), $\Phi(T(B)) = 0$ for all convex $B \in \mathcal{B}_X$. Since

T is linear and continuous, $T(B)$ is convex, so in particular $T(B)$ is an arc-connected and bounded set of X. Then, by using property (1) of Proposition 2.5, $T(B)$ is precompact. This means that $T \in \mathcal{K}(X, Y)$.

(3) Put $k_R := \widetilde{\Phi}(R)$ and $k_S := \widetilde{\Phi}(S)$. Given $\varepsilon > 0$, by (3.1), R and S are $k_R + \varepsilon$ and $(k_S + \varepsilon)$-DND-contractive, respectively. Therefore, we have

$$\Phi\big(S(B)\big) \leq (k_S + \varepsilon)\Phi(B), \quad \text{for all convex } B \in \mathcal{B}_X \tag{3.2}$$

and

$$\Phi\big(R(C)\big) \leq (k_R + \varepsilon)\Phi(C), \quad \text{for all convex } C \in \mathcal{B}_Y. \tag{3.3}$$

Then, by taking $C = S(B)$ with B an arbitrary convex set belonging to $\mathcal{B}_X$, from (3.2), we get

$$\Phi\big(R(S(B))\big) \leq (k_R + \varepsilon)\Phi\big(S(B)\big). \tag{3.4}$$

Hence, from (3.4) and by using (3.2), we obtain

$$\Phi\big((R \circ S)(B) \leq (k_R + \varepsilon)\Phi\big(S(B)\big) \leq (k_R + \varepsilon)(k_S + \varepsilon)\Phi(B), \tag{3.5}$$

for all convex $B \in \mathcal{B}_X$. Since ε is arbitrary, the inequality (3.5) means that $R \circ S$ is $k_R k_S$-DND-contractive. Therefore, from (3.1), we have

$$\widetilde{\Phi}(R \circ S) \leq k_R k_S = \widetilde{\Phi}(R)\widetilde{\Phi}(S),$$

and (3) follows.

(4) If $\lambda = 0$, then the equality $\widetilde{\Phi}(\lambda T) = |\lambda|\,\widetilde{\Phi}(T)$ is trivial, so assume $\lambda \neq 0$. Put $k_T := \widetilde{\Phi}(T)$. Given $\varepsilon > 0$, by (3.1), T is $(k_T + \varepsilon)$-DND-contractive. Hence, $\Phi(T(B)) \leq (k_T + \varepsilon)\Phi(B)$ for all convex $B \in \mathcal{B}_X$. Therefore, since T is linear, we have

$$\Phi\big(\lambda T(B)\big) = \Phi\big(T(\lambda B)\big) \leq (k_T + \varepsilon)\Phi(\lambda B), \quad \text{for all convex } B \in \mathcal{B}_X. \tag{3.6}$$

Now, by applying property (3) of Proposition 2.5 to the last term of (3.6), we get

$$(k_T + \varepsilon)\Phi(\lambda B) = (k_T + \varepsilon)\,|\lambda|\,\Phi(B).$$

Therefore

$$\Phi\big(\lambda T(B)\big) \leq (k_T + \varepsilon)\,|\lambda|\,\Phi(B), \quad \text{for all convex } B \in \mathcal{B}_X.$$

This means that the operator λT is $(k_T + \varepsilon)\,|\lambda|$-DND-contractive and then, since ε is arbitrary, λT is $k_T\,|\lambda|$-DND-contractive. Therefore, from (3.1), it follows that $\widetilde{\Phi}(\lambda T) \leq |\lambda|\,k_T = |\lambda|\,\widetilde{\Phi}(T)$ for all $\lambda \in \mathbb{R}$. Assume $\widetilde{\Phi}(\lambda T) < |\lambda|\,\widetilde{\Phi}(T)$ for some $\lambda \neq 0$. Take $k > 0$ such that $\widetilde{\Phi}(\lambda T) < k < |\lambda|\,\widetilde{\Phi}(T)$, so $k/|\lambda| < \widetilde{\Phi}(T)$. This means that T is not $k/|\lambda|$-DND-contractive. Hence there is some convex $C \in \mathcal{B}_X$ such that $\Phi(T(C)) > (k/|\lambda|)\Phi(C)$. Then $|\lambda|\,\Phi(T(C)) > k\Phi(C)$. But by property (3) of Proposition 2.5, $|\lambda|\,\Phi(T(C)) = \Phi(\lambda T(C))$ and then $\Phi(\lambda T(C)) > k\Phi(C)$, which implies that λT is not k-DND-contractive. This is a contradiction because $\widetilde{\Phi}(\lambda T) < k$. Consequently, $\widetilde{\Phi}(\lambda T) = |\lambda|\,\widetilde{\Phi}(T)$. $\qquad\square$

The inequality $\widetilde{\Phi}(T) \leq \|T\|$ of property (1) of Proposition 3.2 is the best possible. Indeed, firstly we point out that by virtue of (2) of Proposition 3.2, if $T \in \mathcal{K}(X,Y)$ and $T \neq 0$, then $\widetilde{\Phi}(T) = 0$ but $\|T\| > 0$. Therefore, in this case the inequality $\widetilde{\Phi}(T) \leq \|T\|$ is strict for $T \neq 0$ and becomes an equality for $T = 0$. Less trivial is the following example.

Example 3.3. Let ℓ_1 be the Bancah space of absolute value summable real sequences, endowed with its usual norm $\|x\| := \sum_{n=1}^{\infty} |x_n|$, for each $x = (x_n)_n \in \ell_1$. Define the operator $T : \ell_1 \longrightarrow \ell_1$ by

$$T(x) := \Big(\frac{3}{4}x_1, \frac{1}{4}x_1, \frac{1}{4}x_2 \ldots, \frac{1}{4}x_n, \ldots\Big), \quad \text{for all } x = (x_n)_n \in \ell_1.$$

Then $\Phi(T(U_{\ell_1})) < \|T\|$.

It is obvious that T is linear. On the other hand, T is bounded on U_{ℓ_1} because

$$\|T(x)\| = \frac{3}{4}|x_1| + \frac{1}{4}\sum_{n=1}^{\infty}|x_n| = \frac{3}{4}|x_1| + \frac{1}{4}\|x\| \leq \frac{3}{4}\|x\| + \frac{1}{4}\|x\| = \|x\| \leq 1$$

for all $x \in U_{\ell_1}$. Therefore T is continuous or equivalently bounded. Moreover, for $x_0 = (1, 0, \ldots, 0 \ldots) \in U_{\ell_1}$, $T(x_0) = (3/4, 1/4, \ldots, 0 \ldots)$, so $\|T(x_0)\| = 1$ an then $\|T\| = 1$. Consider $y_0 := T(x_0)$ with $x_0 = (1, 0, \ldots, 0 \ldots) \in U_{\ell_1}$ and define the set

$$A := \{(1 - \lambda)y_0 + \lambda(-y_0) : \lambda \in [0, 1]\} = \{(1 - 2\lambda)y_0 : \lambda \in [0, 1]\}.$$

Since $T(U_{\ell_1})$ is convex and $y_0 = T(x_0)$, $-y_0 = T(-x_0) \in T(U_{\ell_1})$, we have $A \subset T(U_{\ell_1})$. Define $\gamma : [0, 1] \longrightarrow \ell_1$ as $\gamma(\lambda) := (1 - 2\lambda)y_0$. Then it is obvious that γ is continuous and $\gamma([0, 1]) = A \subset T(U_{\ell_1})$. Let $x = (x_n)_n$ be an arbitrary point of U_{ℓ_1}, then $|x_1| \leq \|x\| \leq 1$. Take $\lambda_x = \frac{1-x_1}{2}$, then

$$\|T(x) - \gamma(\lambda_x)\| = \Big\|\Big(\frac{3}{4}x_1, \frac{1}{4}x_1, \frac{1}{4}x_2, \ldots\Big) - \Big(\frac{3}{4}x_1, \frac{1}{4}x_1, 0, \ldots\Big)\Big\| =$$

$$\Big\|\Big(0, 0, \frac{1}{4}x_2, \ldots\Big)\Big\| = \frac{1}{4}|x_2| + \ldots \leq \frac{1}{4}.$$

This means that γ is $\frac{1}{4}$-dense in $T(U_{\ell_1})$, so $\Phi(T(U_{\ell_1})) \leq 1/4$. Then, noticing $\|T\| = 1$, it follows that $\Phi(T(U_{\ell_1})) < \|T\|$.

Now, we show an example where the inequality $\widetilde{\Phi}(T) \leq \|T\|$ is an equality.

Example 3.4. Let $\mathcal{C}(I)$ be the space of continuous real functions on $I = [0, 1]$ endowed with the supremum norm. Define the operator $T : \mathcal{C}(I) \longrightarrow \mathcal{C}(I)$ by $T(x(t)) = tx(t)$, $t \in I$. Then $\widetilde{\Phi}(T) = \|T\|$.

It is obvious that T is linear. Also T is bounded on $U_{\mathcal{C}(I)}$ because

$$\big\|T\big(x(t)\big)\big\| = \big\|t\big(x(t)\big)\big\| = \sup_{t \in I} |tx(t)| = \sup_{t \in I} |x(t)| = \|x(t)\| \leq 1$$

for all $x(t) \in U_{\mathcal{C}(I)}$. Therefore T is a continuous or, equivalently, a bounded operator. Moreover, for the function $x(t) := t$ that belongs to $U_{\mathcal{C}(I)}$, one has

$\|T(x)\| = \|t^2\| = 1$ and then $\|T\| = 1$. Hence, by property (1) of Proposition 3.2, $\widetilde{\Phi}(T) \leq 1$. Define the set

$$C := \big\{x(t) \in \mathcal{C}(I) : 0 = x(0) \leq x(t) \leq x(1) = 1, \, t \in I\big\}.$$

It is immediate that C is a bounded and convex set of $\mathcal{C}(I)$. In [13, Example 3.4] it was proved that $\Phi(C) = 1$ and by using the same technique it can be also proved that $\Phi(T(C)) = 1$. Then $\widetilde{\Phi}(T) = 1$ and consequently $\widetilde{\Phi}(T) = \|T\|$.

In the next example we demonstrate that the inequality $\Phi(T(U_X)) \leq \widetilde{\Phi}(T)$ that appears in (1) of Proposition 3.2 can be strict. Furthermore, such example shows that Φ is distinct from the MNC of Hausdorff χ.

Example 3.5. Let ℓ_1 be the Banach space of Example 3.3 and c the space of convergent real sequences $y = (y_n)_{n \geq 1}$ endowed with the supremum norm $\|y\|_\infty := \sup_n |y_n|$. Consider the product space $X := \ell_1 \times c$ with the norm $\|(x,y)\|^2 = \|x\|_1^2 + \|y\|_\infty^2$ and define the operator $T : X \longrightarrow X$ as $T((x,y)) := (0, x)$. Then $\Phi(T(U_X)) < \widetilde{\Phi}(T)$. Furthermore $\widetilde{\chi}(T) < \widetilde{\Phi}(T)$.

Indeed, it is obvious that T is linear. T is bounded on U_X because

$$\big\|T\big((x,y)\big)\big\| = \|(0,x)\| = \|x\|_\infty \leq 1$$

for all $(x,y) \in U_X$. Therefore T is a continuous or, equivalently, a bounded operator. Moreover, by taking $x_0 = (1, 0, \ldots)$ and $y_0 = (0, 0, \ldots)$, so $(x_0, y_0) \in U_X$, we have $\|T((x_0, y_0))\| = \|(0, x_0))\| = \|x_0\|_\infty = 1$ and then $\|T\| = 1$. Consider the bounded and convex set $C := U_{\ell_1} \times U_c$ and define the curve $\gamma : [0,1] \longrightarrow X$ as $\gamma(t) := (0,0)$ for all $t \in [0,1]$. Then it is immediate that γ is $\sqrt{2}$-dense in C and 1-dense in $T(C)$. Therefore, $\Phi(C) \leq \sqrt{2}$ and $\Phi(T(C)) \leq 1$. Take $0 < \varepsilon < 1$, then there exists a $(\Phi(T(C)) + \varepsilon)$-dense curve in $T(C)$, $\Gamma : [0,1] \longrightarrow X$. Noticing the definition of T, $\Gamma(t) = (0, \eta(t))$ for some curve $\eta : [0,1] \longrightarrow \ell_1$ with $\eta([0,1]) \subset U_{\ell_1}$. Taking into account that $\eta([0,1])$ is compact, given ε, there exists an integer $N > 1$ such that if $x = (x_n)_n \in \eta([0,1])$, then $|x_n| \leq \varepsilon$ for all $n \geq N$ (see [6, Theorem II.4.1]). For that N, define the vector $x_N := (x_{N,n})_n$ of ℓ_1 as

$$x_{N,n} := \begin{cases} 0, & \text{, if } n \neq N \\ \\ 1, & \text{if } n = N \end{cases}.$$

Then $(x_N, x_N) \in C$, so $T((x_N, x_N)) = (0, x_N) \in T(C)$. Hence there exists some $t \in [0,1]$ such $\|(0, x_N) - \Gamma(t)\| \leq \Phi(T(C)) + \varepsilon$. But $\Gamma(t) = (0, \eta(t))$ with $\eta(t) = (x_{t,n})_n$ and then

$$\Phi\big(T(C)\big) + \varepsilon \geq \|(0, x_N) - \Gamma(t)\| = \big\|(0, x_N) - (0, \eta(t))\big\| =$$

$$\big\|(0, x_N - \eta(t)))\big\| = \|x_N - \eta(t)\|_\infty \geq |1 - x_{t,N}| \geq 1 - \varepsilon$$

Since ε is arbitrary, the above inequality proves that $\Phi(T(C)) \geq 1$ and then, taking into account that $\Phi(T(C)) \leq 1$, it implies that $\Phi(T(C)) = 1$. Now we

claim that

$$\widetilde{\Phi}(T) \geq 1/\sqrt{2}. \tag{3.7}$$

Indeed, if $\widetilde{\Phi}(T) < 1/\sqrt{2}$, determine k such that $\widetilde{\Phi}(T) < k < 1/\sqrt{2}$ and then T is k-DND-contractive, i.e. $\Phi(T(B)) \leq k\Phi(B)$ for all convex and bounded non-empty set B. By taking $B = C$ we get the following contradiction

$$1 = \Phi(T(C) \leq k\Phi(C) \leq k\sqrt{2} < 1.$$

On the other hand, it was proved in [2, Example 2.4.11] (see also [5, Example 3.1]) that $T(U_X)$ has a compact that is a $\frac{1}{2}$-net. That is, there exists a compact $K \subset T(U_X)$ such that

$$T(U_X) \subset K + \frac{1}{2}U_X. \tag{3.8}$$

Then by taking $\varepsilon > 0$, because of the compactness of K, there is a finite set $\{y_i : i = 1, \ldots, m\} \subset K$ such that

$$K \subset \{y_i : i = 1, \ldots, m\} + \varepsilon U_X. \tag{3.9}$$

Since $\{y_i : i = 1, \ldots, m\} \subset K \subset T(U_X)$, and $T(U_X)$ is convex, the polygonal obtained by joining the points $\{y_i : i = 1, \ldots, m\}$ defines a curve $\gamma : I \longrightarrow X$ with $\gamma(I) \subset T(U_X)$ satisfying, by virtue of (3.8) and (3.9), the following:

(i) Given $y \in T(U_X)$ there exists $y_K \in K$ such that $\|y - y_K\| \leq \frac{1}{2}$.
(ii) Given $y_K \in K$, for some y_i with $i = 1, \ldots, m$ is $\|y_K - y_i\| \leq \varepsilon$.
(iii) For each $i = 1, \ldots, m$ there exists $t_i \in I$ such that $y_i = \gamma(t_i)$.

Then, given $y \in T(U_X)$, (i), (ii) and (iii) imply the existence of some $t_i \in I$ such that

$$\|y - \gamma(t_i)\| \leq \|y - y_K\| + \|y_K - \gamma(t_i)\| \leq \frac{1}{2} + \varepsilon.$$

This means that γ is a curve $(\frac{1}{2} + \varepsilon)$-dense in $T(U_X)$. Since ε is arbitrary, γ is a curve $\frac{1}{2}$-dense in $T(U_X)$ and then $\Phi(T(U_X)) \leq \frac{1}{2}$. Hence, from (3.7), we get

$$\Phi\big(T(U_X)\big) \leq \frac{1}{2} < \frac{1}{\sqrt{2}} \leq \widetilde{\Phi}(T).$$

Consequently the first part of the example follows. Regarding the second part, in [2, Example 2.4.11] it was proved that $\widetilde{\chi}(T) \leq \frac{1}{2}$ and then, since we have previously demonstrated that $\widetilde{\Phi}(T) \geq \frac{1}{\sqrt{2}}$, it follows $\widetilde{\chi}(T) < \widetilde{\Phi}(T)$.

In the next result, given an operator $T \in \mathcal{B}(X, Y)$, we relate the numbers $\widetilde{\Phi}(T)$ and $\widetilde{\chi}(T)$, where χ is the Hausdorff MNC.

Theorem 3.6. *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Then*

$$\widetilde{\chi}(T) \leq \widetilde{\Phi}(T) \leq 2\widetilde{\chi}(T). \tag{3.10}$$

Proof. Given $\varepsilon > 0$, by using (3.1), T is $(\widetilde{\chi}(T) + \varepsilon)$-$\chi$-contractive. Then for any bounded and convex B of X it follows

$$\chi\big(T(B)\big) \leq \big(\widetilde{\chi}(T) + \varepsilon\big)\chi(B). \tag{3.11}$$

From [17, Theorem 2.5] we have $\chi(B) \leq \Phi(B) \leq 2\chi(B)$, so from (3.11), we get

$$\Phi\big(T(B)\big) \leq 2(\chi\big(T(B)\big) \leq 2\big(\widetilde{\chi}(T) + \varepsilon\big)\chi(B) \leq 2\big(\widetilde{\chi}(T) + \varepsilon\big)\Phi(B)$$

which means that T is $2(\widetilde{\chi}(T) + \varepsilon)$-DND-contractive and then $\widetilde{\Phi}(T) \leq 2(\widetilde{\chi}(T) + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we deduce

$$\widetilde{\Phi}(T) \leq 2\widetilde{\chi}(T). \tag{3.12}$$

On the other hand, by (2) of Proposition 1.4, one has $\widetilde{\chi}(T) = \chi(T(U_X))$. Again by using [17, Theorem 2.5] and property (1) of Proposition 3.2, we get

$$\widetilde{\chi}(T) = \chi\big(T(U_X)\big) \leq \Phi\big(T(U_X)\big) \leq \widetilde{\Phi}(T),$$

that jointly with (3.12) prove the inequalities (3.10). $\qquad\square$

As an application of above theorem and the Proposition 1.4, we obtain a result on the properties of $\widetilde{\Phi}(T)$ that completes the Proposition 3.2.

Proposition 3.7. *Let X, Y be Banach spaces and $T, S \in \mathcal{B}(X, Y)$. Then*

$$\widetilde{\Phi}(T + S) \leq 2\big(\widetilde{\Phi}(T) + \widetilde{\Phi}(S)\big). \tag{3.13}$$

Proof. From (3.10), $\widetilde{\Phi}(T + S) \leq 2\widetilde{\chi}(T + S)$. Then, by using property (3) of Proposition 1.4 and again (3.10), we have

$$\widetilde{\Phi}(T + S) \leq 2\widetilde{\chi}(T + S) \leq 2\big(\widetilde{\chi}(T) + \widetilde{\chi}(S)\big) \leq 2\big(\widetilde{\Phi}(T) + \widetilde{\Phi}(S)\big).$$

$\qquad\square$

The above inequalities (3.13) allow us to complete Example 3.3.

Example 3.8. Let ℓ_1 be the Banach space of Example 3.3. Define the operator $T : \ell_1 \longrightarrow \ell_1$ by $T(x) := \big(\frac{3}{4}x_1, \frac{1}{4}x_1, \frac{1}{4}x_2 \ldots, \frac{1}{4}x_n, \ldots\big)$. Then $\widetilde{\Phi}(T) < \|T\|$.

Indeed, in Example 3.3 it was proved that $\|T\| = 1$. Observe that the operator T can be written as $T = R + S$ where $R(x) := (\frac{3}{4}x_1, 0, \ldots)$ and $S(x) := (0, \frac{1}{4}x_1, \frac{1}{4}x_2, \ldots)$. It is immediate that R is a compact operator because $R(\ell_1)$ is a finite dimensional subspace of ℓ_1, so from (2) of Proposition 3.2, $\widetilde{\Phi}(R) = 0$. On the other hand, we easily can check that $\|S\| = \frac{1}{4}$, so S is $\frac{1}{4}$-DND-contractive, i.e. it satisfies $\Phi(S(B)) \leq \frac{1}{4}\Phi(B)$ for all convex and bounded non-empty set B of ℓ_1. Therefore, from (3.1), $\widetilde{\Phi}(S) \leq \frac{1}{4}$. Then, by applying (3.13), we have

$$\widetilde{\Phi}(T) = \widetilde{\Phi}(R + S) \leq 2\big(\widetilde{\Phi}(R) + \widetilde{\Phi}(S)\big) \leq \frac{1}{2}.$$

Noticing $\|T\| = 1$, the inequality $\widetilde{\Phi}(T) < \|T\|$ follows.

Another application of Theorem 3.6 gives us an example of an operator T for which its DND is distinct from its MNC of Kuratowski $\widetilde{\kappa}(T)$.

Example 3.9. Let $\mathcal{C}(I)$ be the space of continuous real functions defined on $I = [0,1]$ endowed with the supremum norm. Define the operator $T : \mathcal{C}(I) \longrightarrow \mathcal{C}(I)$ by

$$T\big(x(t)\big) = \begin{cases} \frac{1}{2}x(2t) + \frac{1}{2}x(0) & \text{if } 0 \le t \le \frac{1}{2} \\[2mm] \frac{1}{2}x(2t-1) + \frac{1}{2}x(1) & \text{if } \frac{1}{2} < t \le 1 \end{cases}, \quad x(t) \in \mathcal{C}(I).$$

Then $\widetilde{\kappa}(T) < \widetilde{\Phi}(T)$. Indeed, in [6, Example X.2] was proved that $\widetilde{\kappa}(T) = \frac{1}{2}$ and $\widetilde{\chi}(T) = 1$. Now by applying the first inequality of (3.10), i.e. $\widetilde{\chi}(T) \le \widetilde{\Phi}(T)$, we deduce that $\widetilde{\Phi}(T) \ge 1$ and then, since $\widetilde{\kappa}(T) = \frac{1}{2}$, we have that $\widetilde{\kappa}(T) < \widetilde{\Phi}(T)$.

As an application of Theorem 3.6 we obtain a new formula to find the essential spectral radius $r_{\mathrm{e}}(T)$ of a bounded operator T on a Banach space (see [12, 36]) in terms of the DND of T.

Proposition 3.10. *Let T be a bounded operator on a Banach space $(X, \| \cdot \|)$, then*

$$r_{\mathrm{e}}(T) = \lim_{n \longrightarrow \infty} \left[\widetilde{\Phi}(T^n) \right]^{1/n}, \tag{3.14}$$

where T^n denotes the composition of T itself n-times.

Proof. We know (see [12, 36]) that $r_{\mathrm{e}}(T) = \lim_{n \longrightarrow \infty} [\widetilde{\chi}(T^n)]^{1/n}$, where χ is the Hausdorff MNC. Then, from (3.10) and taken into account that $\lim_n 2^{1/n} = 1$, the formula (3.14) follows. $\square$

4. A LOWER BOUND FOR THE HYERS-ULAM STABILITY CONSTANT OF AN OPERATOR

In 1940, Ulam [39] raised the problem that, for an approximate solution of a given functional equation, there is a solution of that equation that is close to the approximated given one. This problem was solved, in the context of Banach spaces, by Hyers [21] one year later. Thereafter this result was improved by Aoki [4], Bourgin [7] and Rassias [35] and many others authors. For a detailed exposition on that topic, we refer to the monograph [23].

Firstly we recall the notion of Hyers-Ulam stability of an operator (see for instance [38]).

Definition 4.1. An operator $T \in \mathcal{B}(X,Y)$ is said to have the Hyers-Ulam stability if there exists a constant $K > 0$, called a Hyers-Ulam stability constant (an HUS constant for T for short), such that for any $g \in T(X)$, $f \in X$ and $\varepsilon > 0$ satisfying $\|T(f) - g\| \le \varepsilon$, there exists $f_0 \in X$ with $T(f_0) = g$ and $\|f - f_0\| \le K\varepsilon$.

Definition 4.2. Given an operator $T \in \mathcal{B}(X,Y)$, the number

$$K_T := \inf \{ K : K \text{ is an HUS constant for } T \}$$

is called the Hyers-Ulam stability constant of T.

It is worth to stress that in general K_T is not necessarily a HUS constant for T such as it was demonstrated in [20].

For a detailed exposition of the Hyers-Ulam stability of a linear operator, see [8] and references therein. The Hyers-Ulam stability has been studied for the so called positive linear operators [32, 33, 34] for linear integral [41], differential [25, 26], difference [9, 33] and real and complex functional equations (on this last topic see for instance [18, 28]), among others. Hence, this topic is interesting by its many applications. In this section our goal is to relate, under suitable conditions, the numbers K_T and $\widetilde{\Phi}(T)$. To set the notation, for a given operator $T \in \mathcal{B}(X, Y)$ we write

$$\widehat{T} : X/\ker T \longrightarrow Y$$

for the one-to-one linear operator defined as $\widehat{T}(\widehat{x}) := T(x)$, where x is a representative of the equivalence class $\widehat{x}$.

The result [38, Theorem 2] is crucial for our goal:

Theorem 4.3. *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. The following statements are equivalent:*

(1) T has the Hyers-Ulam stability.

(2) T has closed range.

(3) $(\widehat{T})^{-1}$ is bounded.

Moreover, if one of (hence all of) the conditions (1), (2) and (3) is true, then we have $K_T = \|(\widehat{T})^{-1}\|$.

Assume X and Y Banach spaces. Then if $T \in \mathcal{B}(X, Y)$ has the Hyers-Ulam stability it is immediate that $1 \leq \|\widehat{T}\|\|(\widehat{T})^{-1}\|$. Therefore by the above theorem, $1 \leq \|\widehat{T}\| K_T$. Hence

$$\frac{1}{\|\widehat{T}\|} \leq K_T.$$

In the next result we obtain a sharp lower bound for K_T.

Theorem 4.4. *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$ having the Hyers-Ulam stability. If $T(X)$ is a subspace of Y of infinite dimension, then*

$$\frac{1}{\widetilde{\Phi}(\widehat{T})} \leq K_T. \tag{4.1}$$

Moreover, the above inequality is the best possible.

Proof. By (2) of Theorem 4.3, $T(X)$ is a closed subspace of Y, so $T(X)$ is a Banach space. Since $T(X)$ is a subspace of Y of infinite dimension, then we claim that $\widetilde{\Phi}(T) > 0$. Indeed, if $\widetilde{\Phi}(T) = 0$, from (2) of Proposition 3.2, T would be a compact operator. Then by applying [37, Theorem 4.18], $\dim T(X) < \infty$. This contradicts the fact that $T(X)$ has infinite dimension. Therefore $\widetilde{\Phi}(T) > 0$, as claimed. By using properties (3) and (1) of Proposition 3.2 and taking into account Theorem 4.3, we get

$$1 = \widetilde{\Phi}((\widehat{T})^{-1} \circ \widehat{T}) \leq \widetilde{\Phi}((\widehat{T})^{-1})\widetilde{\Phi}(\widehat{T}) \leq \|(\widehat{T})^{-1}\|\widetilde{\Phi}(\widehat{T}) = K_T\widetilde{\Phi}(\widehat{T})$$

and then inequality (4.1) follows.

Assume $X = Y$ and $T : X \longrightarrow X$ is the identity I_X. Then it is immediate to check that T has the Hyers-Ulam stability and the constant $K_T = 1$. Since in this case $\widehat{T} = (\widehat{T})^{-1} = T = I_X$, we have $\widetilde{\Phi}(\widehat{T}) = 1$ and then inequality (4.1) becomes an equality. Therefore it only remains to give an operator T for which inequality (4.1) to be strict. Indeed, let $X = \mathcal{C}(I)$ be as in Example 3.9. Define the bounded operator $T : X \longrightarrow X$ by

$$T\big(x(t)\big) := \big((t - \frac{1}{2})^2 + 1\big)x\big(\varphi(t)\big), \quad x(t) \in X, \, t \in I,$$

where

$$\varphi(t) := \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{2} \\ \\ 2t - 1, & \text{if } \frac{1}{2} < t \le 1 \end{cases}.$$

In [38] was proved that $K_T = 1$. On the other hand, since T is injective, $\widehat{T} = T$. So it is enough to prove that $\widetilde{\Phi}(T) \ge 5/4$ to show that inequality (4.1) is strict. Consider the set

$$C := \text{Conv}\,\{x_n(t) := t^n, \, n \ge 1, \, t \in I\}$$

and put $x_0(t) := (t - 1/2)^2 + 1$, $t \in I$. In a Banach space, for any subset A one has $\text{Diam}(A) = \text{Diam}(\text{Conv}(A))$ (see for instance [6, Remark II.2.2]). Then, since $\text{Diam}(\{x_n(t) := t^n, \, n \ge 1, \, t \in I\}) \le 1$ it follows that $\text{Diam}(C) \le 1$. Hence, noticing $\|x_0(t)\| = 5/4$, for arbitrary functions $x(t), \, y(t) \in C$, we have

$$\big\|T\big(x(t)\big) - T\big(y(t)\big)\big\| = \big\|x_0(t)\big[x\big(\varphi(t)\big) - y\big(\varphi(t)\big)\big]\big\| \le$$

$$\|x_0(t)\|\,\big\|x\big(\varphi(t)\big) - y\big(\varphi(t)\big)\big\| \le \frac{5}{4}.$$

This means that $\text{Diam}(T(C)) \le 5/4$, so $\Phi(T(C)) \le 5/4$ (see (2.1) and (2.2)). Now we claim that $\Phi(T(C)) = 5/4$. Indeed, assume $\Phi(T(C)) < 5/4$. Then there exists some curve α-dense $\gamma : I \longrightarrow X$ in $T(C)$ with $0 \le \alpha < 5/4$. Take $0 < \varepsilon < 5/8$ such that $2\varepsilon < 5/4 - \alpha$. Since $\gamma(I)$ is a compact in $T(C)$, by Ascoli theorem [37, p. 394], there exists $\delta > 0$ such that for all $y(t) \in \gamma(I)$ one has

$$|y(t) - y(t')| \le \varepsilon \quad \text{provided that } |t - t'| \le \delta. \tag{4.2}$$

Noticing $y(t) \in \gamma(I) \subset T(C)$, $y(t) = T(x(t))$ for some $x(t) \in C$. But for any $x(t) \in C$, one has $x(1) = 1$ and then

$$y(1) = T\big(x(1)\big) = \frac{5}{4}x\big(\varphi(1)\big) = \frac{5}{4}x(1) = \frac{5}{4}.$$

Therefore, by taking $t' = 1$ in (4.2), for all $y(t) \in \gamma(I)$ we get

$$\Big|y(t) - \frac{5}{4}\Big| \le \varepsilon, \quad \text{provided that } 1 - t \le \delta$$

and this implies that

$$y(t) \ge \frac{5}{4} - \varepsilon, \quad \text{for all } t \in I, \quad \text{with } 1 - t \le \delta. \tag{4.3}$$

Pick $t_0 > 1/2$ a point of I such that $1 - t_0 \leq \delta$. Determine n sufficiently large such that

$$T\big(x_n(t_0)\big) = x_0(t_0)\big(\varphi(t_0)\big)^n = ((t_0 - \frac{1}{2})^2 + 1)(2t_0 - 1)^n \leq \varepsilon. \qquad (4.4)$$

For this n, consider the function $x_n(t) = t^n \in C$, so $T(x_n(t)) \in T(C)$. By α-density of γ, there exists $y(t) \in \gamma(I)$ such that $\|y(t) - T(x_n(t))\| \leq \alpha$. However, from (4.4) and (4.3) and taking into account the choice of ε, we are led to the following contradiction

$$\alpha \geq \big\|y(t) - T\big(x_n(t)\big)\big\| \geq \big|y(t_0) - T\big(x_n(t_0)\big)\big| = y(t_0) - T\big(x_n(t_0)\big) \geq \frac{5}{4} - 2\varepsilon > \alpha.$$

Then the claim is true. Noticing $\Phi(C) = 1$ (see [13, Example 3.4]), we have

$$\Phi\big(T(C)\big) = \frac{5}{4}\Phi(C). \qquad (4.5)$$

This means that T is k-DND-contractive for $k \geq \frac{5}{4}$. Otherwise, for some $k < \frac{5}{4}$, we have $\Phi(T(B)) \leq k\Phi(B)$ for all convex $B \in \mathcal{B}_X$ which contradicts (4.5). By using (3.1), we obtain $\widetilde{\Phi}(T) \geq \frac{5}{4}$ that is the desired inequality. This proves the theorem. $\qquad \square$

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