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On topological groups of monotonic automorphisms

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Abstract

We study topological groups of monotonic automorphisms on a generalized ordered space L. We find a condition that is necessary and sufficient for the set of all monotonic automorphisms on L along with the function composition and the topology of point-wise convergence to be a topological group.

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1. Introduction

For brevity, a homeomorphism of a topological space X onto itself will be called an automorphism. In this paper we study sets of monotonic automorphisms on generalized ordered spaces endowed with the topology of point-wise convergence. Recall that a linearly ordered topological space, abbreviated as LOTS, is a linearly ordered set along with the topology generated by sets in the form $(a,b), \{x \in L : x < a\}, \{x \in L : x > a\}$ (see [5] for general facts about LOTS). A generalized ordered space, abbreviated as a GO-space, is a subspace of a linearly ordered space (see [6] for general facts about GO-spaces). Note that the Sorgenfrey Line S is an example of a GO-space, which is not a LOTS. Recall that S is the real line endowed with the topology generated by subsets in from $[a,b)$. It is a result of Čech that the topology of a generalized

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ordered space is generated by a collection of convex subsets [3]. When we discuss several linearly ordered spaces, to distinguish their intervals we will use subscription as in $[a,b]_L$ (and in other types of intervals), where a,b can be in L or in a superspace understood from the context. It is a traditional exercise that the set of all monotonic automorphisms $M(L)$ on a GO-space L along with the operation of composition is a group (for completeness, a proof is given in Proposition 2.1). We observe that this group along with the topology of point-wise convergence, denoted by $M_p(L)$, is a paratopological group. Recall that a group G along with a topology on G is a paratopological group if the group operation of G is continuous with respect to the topology of $G \times G$. The operation of inversion, however, need not be continuous in $M_p(L)$. We, therefore, identify a condition that is necessary and sufficient for $M_p(L)$ to be a topological group. In the main result of this work (Theorem 2.7), we prove that given a GO-space L, the space $M_p(L)$ is a topological group if and only any set in from $U(x;\{y\}) = \{f \in M(L) : f(x) = y\}$ is open whenever x is limit from at most one side. It is not hard to see that for a LOTS L, such sets are always open. Therefore, a corollary to our main result is the recent result of B. Sorin [7] that the group of order-preserving bijectiions of a linearly ordered space L with the operation of composition and endowed with the topology of point-wise convergence is a topological group. Sorin's argument uses the fact that the topology of point-wise convergence of the group of continuous extensions over the smallest linearly ordered compactification is generated by sets dependent on points of L only. This, however, is no longer true for a GO-space. It is also worth mentioning that the group of isometries on a metric space with the topology of point-wise convergence is a topological group too (see [2, Theorem 3.5.1]). Since monotonic maps are either order-preserving or order-reversing, we can view them as the order counterparts of isometries on metric spaces. In general, the topology of point-wise convergence need not turn a group of automorphisms on a space (even linearly ordered space) with operation of composition into a topological group. It is easy to see that neither taking the inverse nor the operation of composition are continuous with respect to this topology even for automorphisms of $\mathbb{Q}$.

Given a GO-space L, standard open sets of $M_p(L)$ are of the form

$$U = U(x_1,..,x_n; I_1,...,I_n) = \{f \in M(L) : f(x_i) \in I_i, i = 1,...,n\},$$

where x_i's are some fixed elements of L and I_k's are open convex sets of L. When introducing a set in the form $U(x_1,..,x_n; I_1,...,I_n)$, we will then refer to it by its short name U. Note that $U(x_1,..,x_n; I_1,...,I_n) = U_1(x_1; I_1) \cap ... \cap U_n(x_n; I_n)$. An unordered pair of elements is a gap in a GO-space L if the elements of the pair are the immediate neighbors of each other with respect to the order of L. In a complete linear ordering we denote by $\infty_L$the maximum and by $-\infty_L$ the minimum of L. In notations and terminology of general topological nature we will follow [4].

2. Study

Before we begin our study let us reflect on the structure of groups of monotonic automorphisms on a GO-space. First, if a GO-space under a discussion is a LOTS, then "monotonic automorphism" is equivalent to a "monotonic bijection". In this study, all maps are monotonic automorphisms. For the sake of completeness, let us start by proving the following fact.

Proposition 2.1. *Let L be a GO-space. Then the set $M(L)$ along with the function composition is a group.*

Proof. First, recall that if f and g are homeomorphic bijections on a topological space X, then so are f^{-1} and $f \circ g$. Also, observe that the identity map i_L is an increasing function, and therefore, is in $M(L)$. Therefore, it remains to show that $f \circ g$ and f^{-1} are in $M(L)$ whenever $f, g \in M(L)$.

To show that f^{-1} is in $M(L)$, assume that f is increasing and fix $a, b \in L$ with $a < b$. Since f is an increasing bijection, there exist $c, d \in L$ such that $c < d$, $f(c) = a$, and $f(d) = b$. Then $c = f^{-1}(a) < f^{-1}(b) = d$. Hence, $f^{-1} \in M(L)$. A similar argument applies if f is decreasing.

Now let us show that $f \circ g \in M(L)$ given $g, f \in M(L)$. For this, fix $a, b \in L$ with $a < b$. We have the following four cases:

Case(f and g are increasing): We have $g(a) < g(b)$ and $f(g(a)) < f(g(b))$. Hence, $f \circ g$ is increasing.

Case(f and g are decreasing): We have $g(a) > g(b)$ and $f(g(a)) < f(g(b))$. Hence, $f \circ g$ is increasing.

Case(f is increasing, g is decreasing): We have $g(a) < g(b)$ and $f(g(a)) > f(g(b))$. Hence, $f \circ g$ is decreasing.

Case(f is decreasing, g is increasing): We have $g(a) > g(b)$ and $f(g(a)) > f(g(b))$. Hence, $f \circ g$ is decreasing.

□

We will also use the following statement.

Lemma 2.2. *Let L be a GO-space. Then the set of decreasing (increasing) automorphisms on L is clopen in $M_p(L)$.*

Proof. Let f be a decreasing automorphism and $a < b$ Since $f(a) > f(b)$ there exist open intervals I and J containing $f(a)$ and $f(b)$, respectively, such that I is strictly to the right of J. Then $U = \{g \in M(L) : g(a) \in I \textit{ and } g(b) \in J\}$ contains f and does not not contain any increasing automorphisms. Hence, the set of all decreasing automorphisms on L is open in $M_p(L)$. Similarly, the set of all increasing automorphisms is open in $M_p(L)$. Since there are no other elements in $M_p(L)$ the conclusion follows. □

Note that any monotonic automorphism maps extremities to extremities. This and Lemma 2.2 imply that sets $U(\infty_L; \{y\})$ and $V(-\infty_L; \{y\})$ are open in $M_p(L)$ for any $y \in L$. We will use this fact implicitly throughout the paper. To initiate our study let us start with the following positive observation.

Lemma 2.3. *Let L be a GO-space. Then $\langle f, g\rangle \mapsto f \circ g$ is a continuous map from $M_p(L) \times M_p(L)$ to $M_p(L)$.*

Proof. Fix $f, g \in M_p(L)$. Let $z = f(g(x))$ and $y = g(x)$. Let $W_{f\circ g}$ be an arbitrary neighborhood $f \circ g$. Our goal is to find open neighborhoods V_f and U_g of f and g, respectively, such that $f_1 \circ g_1 \in W_{f\circ g}$ whenever $f_1 \in V_f$ and $g_1 \in U_g$. For our argument we will assume that f and g are increasing. Other variations are treated using very similar arguments. The structure of basic neighborhoods and Lemma 2.2 allow us to assume that $W_{f\circ g}$ is of the form $W_{f\circ g}(x; I) = \{h \in M(L) : h(x) \in I\}$ for some fixed $x \in L$ and a convex open set $I \subset L$. We have the following three cases:

Case (z is isolated): Then, x and y are isolated too. Put, $U_f = U_f(x; \{y\})$ and $V_g = V_g(y; \{z\})$.

Case (z is isolated on one side only): Without loss of generality, z is a limit point of $\{x \in L : x < z\}$. Hence, $\{x \in L : x > z\}$ is clopen in L. Therefore, there exists $z' < z$ such that $[z', z] \subset I$. Since f is onto, there exists $y' \in L$ such that $f(y') = z'$. Since f is increasing, $y' < y$. Since f is a monotonic homeomorphism, $f([y', y]) = [z', z]$. Put $V_f = V_f(y, y'; I, I)$. Clearly, V_f is an open neighborhood of f. Put $U_g = U_g(x; (y', y])$. To show that the selected neighborhoods are as desired, pick $f_1 \in V_f$ and $g_1 \in U_g$. We have $g_1(x) \in (y', y]$. Since f_1 is monotonic, we have $f_1(g_1(x))$ is between $f_1(y')$ and $f_1(y)$. By the definition of V_f and convexity of I, $f_1(g_1(x))$ is in I. Hence, $f_1(g_1(x)) \in W_{f\circ g}$.

Case (z is a limit point on both sides): Since z is limit on both sides, so are x and y. Fix $y_1, y_2 \in L$ such that $y_1 < y < y_2$ and $f(y_1), f(y_2) \in I$ Next, fix $x_1, x_2 \in L$ such that $x_1 < x < x_2$ and $g(x_1), g(x_2) \in (y_1, y_2)$. By monotonicity, $g(x)$ is between $g(x_1)$ and $g(x_2)$ while $f(y)$ is between $f(y_1)$ and $f(y_2)$. Put $U_g = U_g(x_1, x_2; (y_1, y_2), (y_1, y_2)\}$ and $V_f = V_f(y_1, y_2; I, I)$. Clearly, the sets contain g and f, respectively. Fix $g_1 \in U_g$ and $f_1 \in V_f$. Then $g_1(x)$ is between $g_1(x_1)$ and $g_1(x_2)$, and therefore, $g_1(x) \in (y_1, y_2)$. Since $f_1 \in V_f$, $f_1((y_1, y_2)) \subset I$. Hence, $f_1(g_1(x)) \in I$. Therefore, $f \circ g(x) \in I$.

□

In connection with our observation, it must be mentioned that the operation of inversion need not be continuous on $M_p(L)$ when L is a GO-space.

Example 2.4. The operation of inversion is not continuous on $M_p(S)$, where S is the Sorgenfrey Line.

Proof. Let f be the identity map on S. Put $V_{f^{-1}} = \{h^{-1} : h \in M(S), h^{-1}(0) \in [0, 1)\}$. Clearly, $f^{-1} = f$ and $f^{-1} \in V_{f^{-1}}$. Our goal is to show that any neighborhood U_f of f contains g such that g^{-1} is not in $V_{f^{-1}}$. We may assume that U_f is of the form $U_f(x_1 = 0, x_2, ..., x_n; [0, 1), I_2, ..., I_n\}$. Clearly, we can can find an increasing $g \in M(S)$ such that $g \in U_f$ and $g(0) > 0$. Then $g^{-1}(0) < 0$, meaning that $g^{-1} \notin V_{f^{-1}}$. □

Lemma 2.5. *Let L be a GO-space. If the operation of inversion is continuous on $M_p(L)$, then any set in form $W(x;\{y\})$ is open in $M_p(L)$ whenever x is isolated from at least one side.*

Proof. Fix an arbitrary $x \in L$, which is isolated from at least one side and any $y \in L$. Without loss of generality we may assume that x isolated from the right. To show that $W = W(x;\{y\})$ is open in $M_p(L)$, fix an arbitrary $f \in W$. We may assume that f is increasing. We need to find an open neighborhood of f which is a subset of W. Since $f \in W$, we have $f(x) = y$. Since x is isolated on the right, the set $I = \{z \in L : z \leq x\}$ is open. Put $V_{f^{-1}} = \{h^{-1} : h \in M(L), h^{-1}(y) \in I\}$. Since the operator of inversion is continuous on $M_p(L)$, there exists an open neighborhood U_f of f such that $(U_f)^{-1} \subset V_{f^{-1}}$. We may assume that there exist $x_2, ..., x_n$ and open convex mutually disjoint sets $I_1, ..., I_n \subset L$ such that U_f is the set of all increasing functions of $U(x_1 = x, x_2, ...x_n; I_1, ..I_n)$. Since x is isolated from the right and f is increasing we may assume that $\max I_1 = y$. It remains to show that $U_f \subset W$. For this fix $g \in U_f$. We already know that $g(x) \leq y$. Assume that $g(x) < y$. Since g is increasing, we conclude that $g^{-1}(y) > x$. This contradicts the fact that $(U_f)^{-1} \subset V_{f^{-1}}$. Therefore, $g(x) = y$. Hence, $U_f \subset W$. □

Lemma 2.6. *Let L be a GO-space. If any set in the form $W(x;\{y\})$ is open in $M_p(L)$ whenever x is isolated from at least one side, then the operation of inversion is continuous in $M_p(L)$.*

Proof. Let $W_{f^{-1}} = \{h^{-1} : h \in M(L), h^{-1}(y) \in I\}$ for some fixed $y \in L$ and convex open I in L. Let $x = f^{-1}(y)$. We need to find U_f an open neighborhood of f such that $g^{-1} \in W_{f^{-1}}$ for every $g \in U_f$. We have the following two cases:

- Case (x is isolated on at least one side): Then $U_f = \{h \in M(L) : h(x) \in \{y\}\}$ is open by hypothesis. Clearly, $f \in U_f$. Pick any $g \in U_f$. Then, $g^{-1}(y) = x \in I$. Hence, $g^{-1} \in W_{f^{-1}}$.
- Case (x is a limit point on both sides): Fix $x_1, x_2, x'_1, x'_2 \in I$ such that $x_1 < x'_1 < x < x'_2 < x_2$. Let $(y_1, y_2) = f((x_1, x_2))$. Without loss of generality, f is increasing. Put $U_f = \{h \in M(L) : h(x'_1) \in (y_1, y), h(x'_2) \in (y, y_2)\}$. Clearly, $f \in U_f$. Fix $h \in U_f$. Then $h(x'_1) < y < h(x'_2)$. Hence, $h^{-1} \in (x'_1, x'_2) \subset I$.

□

Lemmas 2.3, 2.5, and 2.6 form the following criterion.

Theorem 2.7. *Let L be a GO-space. Then, $M_p(L)$ is a topological group if and only if any set in the form $W(x;\{y\})$ is open in $M_p(L)$ whenever x is isolated from at least one side.*

We already know that the space of monotonic automorphisms of the Alexandroff Arrow is not a topological group. Let us next discuss some positive cases. Firstly, it was proved by Sorin in [7] that $M_p(L)$ is a topological group if L is a LOTS. Sorin stated his result for the space of order-preserving bijections but the argument is valid for the space of all monotonic bijections. To derive Sorin's result from our criterion we need the following lemma.

Lemma 2.8. *Let L be a GO-space, $\{a_l, a_r\} \subset L$ a gap, and $b \in L$. Then $U(a_l;\{b\})$ and $V(a_r;\{b\})$ are open in $M_p(L)$.*

Proof. Put $S = \{f \in M(L) : f(a_l) = b,\ f \textit{ is increasing}\}$. By Lemma 2.2, it suffices to shows that S is open. If there is no increasing f in $M_p(L)$ that maps a_l to b, then S is empty, and therefore, open. Otherwise, fix $h \in S$. There exists a gap $\{b_l, b_r\}$ such that $b = b_l$, $h(a_l) = b$ and $h(a_r) = b_r$. We have

$$\{f \in M(L) : f(a_l) = b, f(a_r) = b_r\}$$

$$=$$

$$\{f \in M(L) : f(a_l) \in [-\infty_L, b]_L, f(a_r) \in [b_r, \infty_L)_L\}$$

Since $\{b, b_r\}$ is a gap, the intervals in the right side of the equality are open in L. Hence, the sides represent an open subset of $M_p(L)$. Next, observe that $S = \{f \in M(L) : f(a_l) = b, f(a_r) = b_r\}$. □

Lemma 2.8 and Theorem 2.7 imply Sorin's result.

Corollary 2.9 (Sorin [7])**.** *Let L be a LOTS. Then $M_p(L)$ is a topological group.*

Let x be isolated from one side in a GO-space L and let x not belong to a gap. Suppose that there exists an open neighborhood I of x in L that has no other points of this kind other than x. Let us show that $U = U(x;\{y\})$ is open in $M_p(L)$ for any y. For this pick $f \in U$. Since f is a homeomorphism, y is also isolated from one side and is not a member of a gap. Since f is a homeomorphism, $J = f(I)$ is an open neighborhood of y that has no points with the properties of y other than y itself . Put $V_f = \{h \in M(L) : h(x) \in J\}$. Since the inclusion $h(x) \in J$ implies that $h(x) = y$, we conclude that $V_f = U$. This observation leads to the following statement.

Theorem 2.10. *Let L be a GO-space. If L is a disjoint union of clopen sets each of which is a LOTS, then $M_p(L)$ is a topological group.*

It is obvious that if two LOTS are order-isomorphic, then their spaces of monotonic bijections are homeomorphic and even topologically isomorphic (as topological groups). If, however, two LOTS are simply homeomorphic, their spaces of monotonic bijections need not be homeomorphic. For example, $M_p(\mathbb{N})$ contains only the identity map, while $M_p(\mathbb{Z})$ is infinite. This is a rather cheap example but it gives a route for exploration. Let $\mathcal{M}_p[L] = \{M_p(L') : L' \textit{ is a } GO-\textit{space and is homeomorphic to } L\}$.

Problem 2.11. *Identify nice classes $\mathcal{P}$ of GO-spaces within which two GO-spaces L and L' are homemorphic if and only if $\mathcal{M}_p[L] = \mathcal{M}_p[L']$.*

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