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Lightly chaotic dynamical systems

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Abstract

In this paper we introduce some weak dynamical properties by using subbases for the phase space. Among them, the notion of light chaos is the most significant. Several examples, which clarify the relationships between this kind of chaos and some classical notions, are given. Particular attention is also devoted to the connections between the dynamical properties of a system and the dynamical properties of the associated functional envelope. We show, among other things, that a continuous map $f : X \to X$, where X is a metric space, is chaotic (in the sense of Devaney) if and only if the associated functional dynamical system is lightly chaotic.

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1. Introduction

When describing topological properties in terms of open sets, it suffices to restrict attention to a fixed base and, sometimes, to a fixed subbase. As an example, the *Alexander subbase Theorem* claims: "*Let X be a topological space and $\mathcal{S}$ a subbase for X. Then X is compact if and only if every open cover by members of $\mathcal{S}$ has a finite subcover.*"
However, the equivalence is not obvious in general. Except for some rare cases, expressing a given topological property by elements of a fixed subbase defines a weaker property than the previous one. In this paper, we introduce some 'subbasic' dynamical properties and a weak form of chaos, the *light chaos*,

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giving several examples in order to shed some light on the relationships between this form of chaos and the notions of transitivity and dense periodicity. Many authors investigated the chaotic behavior of a dynamical system (see for example Proposition 2.4 [2], [8], [21]). It should be rather natural to think that some chaotic behavior of a dynamical system (X, f) reflects, in some way, on its functional envelope $(S(X), F_f)$, and viceversa. Recent research deals with these kinds of relationships(see [5], [7]). We establish a clear connection between a discrete dynamical system (X, f) and the associated "functional" system $(S(X), F_f)$ regarding the chaotic behavior. We will show that a system (X, f) is chaotic (in the sense of Devaney) if and only if $(S(X), F_f)$ is *lightly chaotic*. Moreover, we will give results showing the non-chaoticity of $(S(X), F_f)$ for many dynamical systems (X, f).

Our motivation to study the connection between a chaotic dynamical system and the chaotic properties of its functional envelope also comes from interesting applications. To study the dynamical behavior of the functional envelope is, in a certain sense, equivalent to study the dynamical behavior of the solution of a hyperbolic differential equation. Research about dynamics on a space of functions has been motivated by the importance that the topic plays in the semigroup theory, in the theory of functional difference, in the dynamical systems theory, and, moreover, in the study of one dimensional wave equations (see [5], [7], [15], [20]).

A discrete dynamical system is a pair (X, f) where X is a topological space (called *phase space*) and $f : X \to X$ is a continuous map (called a *transition function*). We may associate with (X, f) a discrete dynamical system $(S(X), F_f)$ whose phase space is the set $S(X)$ of all continuous self-maps on X, endowed with a suitable topology, and the transition function $F_f : S(X) \to S(X)$ is defined by $F_f(g) = f \circ g$ for every $g \in S(X)$. In other words, the phase space of the system $(S(X), F_f)$ is formed by the transition functions of all discrete dynamical systems having X as phase space, while F_f is related in a natural way to the transition function of the dynamical system (X, f).

The first fundamental step in this topic is to define a function space dynamical system by endowing the phase set with a suitable topology ensuring continuity for the transition function. The two most quoted in the literature are the compact-open topology and the point-open topology. The second one is to investigate its dynamical properties, in connection with those satisfied by the original system. In [5], Auslander, Kolyada and Snoha first called *functional envelope* of (X, f) the induced system $(S(X), F_f)$, since it always contains an isometric copy of (X, f) (see Proposition 2.4 [5]), and denoted there by $(S(X), F_f)$, an usual notation in the semigroup theory, since $S(X)$ is a topological semigroup under the composition and the compact-open topology. Their paper *Functional envelope of a dynamical system*([5]) deals with the connections between the properties of a system and the properties of its functional envelope, with special attention to orbit closures, ω-limit sets, (non)existence of dense orbits, and topological entropy. Since (X, f) can be viewed as a subsystem of $(S(X), F_f)$, it inherits many dynamical properties from $(S(X), F_f)$, but the converse depends on the topology of $S(X)$. They showed that, when X is a compact metric space and S(X) is equipped with the compact-open topology, some properties can be carried over from (X, f) to $(S(X); F_f)$, but there are other properties, such as topological transitivity, weakly mixing, mixing, and chaoticity, that cannot be easily transferred to the functional envelope. In particular, they studied the following question, which appears in semigroup theory. Let X be a compact metric space, and S(X) be equipped with the compact-open topology. Are there two elements f and φ in $S(X)$ such that the set $O_{F_f}(\varphi)$ is dense in $S(X)$? In other words, for a given system (X, f) with X being a compact metric space, does the functional envelope $(S(X), F_f)$ have a dense orbit? They showed that the functional envelope of the full shift on $A^{\mathbb{N}}$, where A is a compact metric space, contains dense orbits, but, in general, the answer is negative ([5], Theorem 5.6 and Proposition 5.7]). So it is natural to ask if, for a compact metric space X and a continuous map $f : X \to X$, there exists a suitable, obviously coarser, topology $\mathcal{T}$ on $S(X)$ such that the functional envelope $(S(X), F_f)$ of (X, f) has at least one dense orbit. In the paper *Functional envelopes relative to the point-open topology* ([7]), the authors Chen and Huang study this question. They consider a locally compact separable topological space X and the point-open topology (denoted by $\mathcal{P}$), also known as *pointwise convergence topology*, on $S(X)$. In particular, for any continuous map $f : [0, 1] \to [0, 1]$, the functional envelope $(S([0, 1]), \mathcal{P}, F_f)$ of $([0, 1], f)$ has no dense orbits ([7], Theorem 2.1). Therefore, it is not chaotic. This leads them to restrict the point-open topology on a subset A of X, that is, to consider the topology $\mathcal{P}_A$ generated by the subbase $S_A = \{[x, U] : x \in A$ and U is an open subset of $X\}$, where $[x, U] = \{\varphi \in S(X) : \varphi(x) \in U\}$. They investigate the chaotic behavior of the $(S_A(X), F_f)$, for a countable dense subset A of X in relation to that of the dynamical system (X, f), where X is a locally compact separable metric space. Is a dynamical property, such as transitivity, minimality, or strong mixing of a system (X, f), absorbed by its functional envelope $(S_A(X), F_f)$? If not, what is a coarser property satisfied by it? If A is countable and X is a locally compact separable metric space they show that: *If (X, f) is weakly mixing and A is a countable dense subset of X, then the functional envelope $(S_A(X), F_f)$ of (X, f) has at least one dense orbit.*

2. Preliminaries

In this section, we recall some basic definitions and results involving topological dynamics and function space topologies useful in the sequel. We refer to [1], [10], [12] and [22] for definitions and results not explicitly given. By a discrete dynamical system we mean a pair (X, f) where X is a (usually compact metrizable) topological space and $f : X \to X$ is a continuous map. Given a point $x \in X$, its orbit is the set $O_f(x) = \{x, f(x), f^2(x), ...\}$. For every $n \in \mathbb{Z}^+$, we define the iterates $f^n : X \to X$ as $f^0 = id_X$(the identity map on X) and

$f^{n+1} = f^n \circ f$. By a subsystem of a dynamical system (X, f) we mean a system (Y, g) where Y is a closed f-invariant subset of X (i.e. $f(Y) \subset Y$) and g is the restriction of f to Y. Two dynamical systems (X, f) and (Y, g)(or the maps f and g) are said to be *(topologically) conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. A dynamical system (X, f) (or the map f) is called *(topologically) transitive* if for every pair of nonempty open sets U and V there is a positive integer k such that $f^k(U) \cap V \neq \varnothing$.

Topological transitivity and the existence of a dense orbit are not equivalent for (X, f) (see [14]). It is easy to check that any T_1 point-transitive system without isolated points is transitive. Conversely, if X is separable metrizable and of second category then transitivity implies the existence of a dense orbit. A point x is *periodic* if $f^k(x) = x$ for some integer $k \geq 1$. The least k for which this happens is called *period* of x. A dynamical system (X, f) (or the map f) is called *periodically dense* if the set of periodic points of f is dense in X. Now, let (X, d) be a metric space. A continuous map $f : (X, d) \to (X, d)$ *exhibits sensitive dependence on initial conditions*, in brief is *sensitive*, if there is some $\delta > 0$ such that, for every point x and every neighborhood V of x in (X, d), there exists a $y \in V$ and an integer $k \geq 1$ such that $d(f^k(x), f^k(y)) \geq \delta$.

Transitivity, sensitivity, and periodic points' density are dynamical ingredients used to introduce various kinds of chaoticity. Among them, we consider Devaney chaoticity.

A continuous map $f : (X, d) \to (X, d)$ is chaotic (in the sense of Devaney) if it is transitive, periodically dense and sensitive (see [9]). It is worth noting that if X is infinite, a continuous map $f : (X, d) \to (X, d)$ is chaotic (in the sense of Devaney) if and only if it is transitive and periodically dense. So, it is surprising that sensitivity, a metric property that plays a central role in the definition of several kinds of chaos, is, in the definition of Devaney chaos, a redundant property (see [6]).

Moreover, if X is an interval of the real line, then f is chaotic (in the sense of Devaney) if and only if it is transitive. Several topologies can be defined on the set $Y^X = \{f : X \to Y\}$, where X, Y are topological spaces (see [3], [4], [13], [16], [17], [19]). Among them, we shall consider the compact-open topology, the point-open topology, and the uniform convergence topology. Let $\mathcal{K}[X]$ be the family of all compact subsets of X. For each $K \in \mathcal{K}[X]$ and $G \in \mathcal{T}$ let us set

$$[K, G] = \{g \in Y^X : g(K) \subset G\}.$$

The family $\mathcal{S} = \{[K, G] : K \in \mathcal{K}[X], G \in \mathcal{T}\}$ is a subbase for a topology $\mathcal{T}_k$ on Y^X, the *compact-open topology.* Moreover, given a point $x \in X$ and an open set $G \in \mathcal{T}$, let

$$[\{x\}, G] = \{g \in Y^X : g(x) \in G\}.$$

The sets $[\{x\}, G]$ form a subbase for a coarser topology $\mathcal{T}_p$ on Y^X, the *point-open topology* (or *topology of pointwise convergence*). Note that this is just the product topology on Y^X. Evidently, $\mathcal{T}_p = \mathcal{T}_k$ iff every compact subset of X is finite. In particular, this happens when X is discrete. Moreover, on

equicontinuous families of functions of Y^X the compact-open topology reduces to the point-open topology (see [22]).
Now, let (Y, d) be a metric space, and let $\mathcal{T}_d$ be the topology on Y generated by d. We may define on $B(X, Y)$, the set of all continuous bounded maps from X to Y, the *uniform metric*:

$$\widehat{d}(f, g) = sup_{x \in X} d(f(x), g(x)).$$

The topology generated by $\widehat{d}$, $\mathcal{T}_{\widehat{d}}$, is called the *uniform convergence topology.* Let $C(X, Y)$ be the set of continuous maps from X to Y. If X is compact, then $B(X, Y) = C(X, Y)$ and the topology $\mathcal{T}_{\widehat{d}}$ coincides with the compact-open topology. Moreover, this is independent of the choice of d, i.e., every metric on Y equivalent to d generates the compact-open topology. If (X, d) is a compact metric space, then $(C(X, X), \mathcal{T}_{\widehat{d}})$ is a separable complete metric space.

Let $(X, \mathcal{T})$ be a topological space. We will denote $C(X, X)$, the set of all continuous self-maps on X, by using the simpler notation $S(X)$ coming from semigroup theory ([5] and [7]). It is straightforward to check that if $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ is a continuous map, then the map

$$F_f : (S(X), \mathcal{T}_k) \to (S(X), \mathcal{T}_k)$$

defined by $F_f(g) = f \circ g$ for every $g \in S(X)$ is continuous. In this way, we associate with any discrete dynamical system (X, f) the dynamical system $(S(X), F_f)$. We will concisely denote the topological space $(S(X), \mathcal{T}_k)$ by $S_k(X)$. When X is a compact metric space, then $(S_k(X), F_f)$ (considered as a metric space with the uniform metric or with the Hausdorff metric applied to the graphs of maps) is called *functional envelope* of (X, f) (see Definition 1.1. in [5]). Let us note that one could use this name even in a more general setting.

Definition 2.1. Let (X, f) be a dynamical system given by a topological Hausdorff space X and a continuous map $f : X \to X$. If $F_f : (S(X), \mathcal{T}_{S(X)}) \to (S(X), \mathcal{T}_{S(X)})$ is continuous for some topology $\mathcal{T}_{S(X)}$ on $S(X)$, then we call the dynamical system $(S(X), F_f)$ *functional envelope* of (X, f).

Proposition 2.2. *If $\mathcal{T}_{S(X)}$ is the compact-open topology or the point- open topology then the map $F_f : (S(X), \mathcal{T}_{S(X)}) \to (S(X), \mathcal{T}_{S(X)})$ is continuous and the dynamical system $(S(X), F_f)$ contains a subsystem topologically conjugate to the original system (X, f).*

In particular, if $\mathcal{T}_{S(X)} = \mathcal{T}_p$ we will denote $(S(X), \mathcal{T}_p)$ by $S_p(X)$.
It is easy to prove that the previous result holds when $\mathcal{T}_{S(X)}$ is any $\lambda - open$ topology. The $\lambda - open$ topologies (or set-open topologies), a generalization of the compact-open and pointwise convergence topologies, were first introduced by Arens and Dugundji (see [3]).
Sometimes it is also convenient to endow $S(X)$ with coarser topologies induced by reduced subbases. For example, if (X, f) is a dynamical system given by a separable space X with metric d and a continuous map $f : X \to X$, $A =$

$\{a_1, a_2, ...\}$ is a countable dense subset of X, then $(S_A(X), F_f)$, where $S_A(X)$ is $S(X)$ endowed with the point-open topology on the set A, is a functional envelope of (X, f) (see Proposition 4 in [7]).

3. Light chaoticity

Compactness is equivalent to compactness with respect to a subbase, as the *Alexander subbase Theorem* states. However, subbases are not always sufficient to describe a topological property, and this allow to introduce a weaker property. Indeed, let P be a topological property. Let X be a topological space and $\mathcal{S}$ a subbase of X. We say that X is *lightly* $-P$ with respect to $\mathcal{S}$ if X satisfies P relatively to all subbasic open sets in $\mathcal{S}$. Evidently, any topological space P is lightly-P, but the converse is not true. Obviously, when the equivalence doesn't occur it makes sense to consider a weaker form of the property.

Our attention is devoted to some dynamical properties. We introduce in particular the *light transitivity*, the *light periodically density*, and the *light sensitivity*, and define a weak form of chaoticity, the *light chaoticity*.

Let (X, f) be a discrete dynamical system and $\mathcal{S}$ a subbase for the topological space (X, τ).

Definition 3.1. (X, f) (or the map $f : (X, \tau) \to (X, \tau)$ is said to be *lightly transitive* (with respect to $\mathcal{S}$), briefly $L_{\mathcal{S}}$*-transitive*, if for every $U, V \in \mathcal{S} - \{\varnothing\}$ there exists some positive integer k such that $f^k(U) \cap V \neq \varnothing$.

Definition 3.2. (X, f) (or the map $f : (X, \tau) \to (X, \tau)$ is said to be *lightly periodically dense* (with respect to $\mathcal{S}$), briefly $L_{\mathcal{S}}$*-periodically dense*, if every $U \in \mathcal{S} - \{\varnothing\}$ contains a periodic point of f.

Definition 3.3. (X, f) (or the map $f : (X, \tau) \to (X, \tau)$ is said to be *lightly chaotic* (with respect to $\mathcal{S}$), briefly $L_{\mathcal{S}}$*-chaotic*, if:

LC1: (X, f) is $L_{\mathcal{S}}$-transitive;
LC2: (X, f) is $L_{\mathcal{S}}$-periodically dense.

In other words, a continuous map is $L_{\mathcal{S}}$-chaotic if it satisfies transitivity and dense periodicity restricted to some subbase $\mathcal{S}$ for X. Evidently, any $L_{\mathcal{S}}$-chaotic map is a $L_{\mathcal{S}'}$-chaotic map for each $S' \subset S$. Moreover, if the topology $\tau(S')$ generated by $\mathcal{S}'$ is strictly weaker than $\tau(S)$ and $f' : (X, \tau(S') \to (X, \tau(S')$ is continuous, then the dynamical system (X, f') is lightly chaotic too.

It is clear from the definition that:

(i) Every transitive periodically dense map is $L_{\mathcal{S}}$-chaotic with respect to any subbase for X.
(ii) Every transitive interval map is chaotic and, a fortiori, lightly chaotic.
(iii) Since a transitive map $f : S^1 \to S^1$ is chaotic if and only if it has a periodic point, it follows that, for transitive self-maps on S^1, chaos and light chaos coincide.

The condition (i) is not necessary.

Example 3.4. *A lightly chaotic periodically dense map which is not transitive.* The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -x$ is a non transitive periodically dense map. Moreover, f is lightly chaotic with respect to the subbase $\mathcal{S} = \{]-\infty, a[,]b, +\infty[\}_{a,b\in\mathbb{R}}$. The map f sends any half line in its opposite and two half lines, both right(or left), are always not disjoint. So $f(]-\infty, a[)\cap]b, +\infty[\neq 0$ and $f(]b, +\infty[)\cap]-\infty, a[\neq 0$ for every $a, b \in \mathbb{R}$.

Example 3.5. *A lightly chaotic transitive map which is not periodically dense.* Consider the dynamical system $(\{0,1\}^{\mathbb{N}}, \sigma)$, where $\{0,1\}^{\mathbb{N}}$ is the Cantor set and σ is the shift map. Let S be the set of all eventually constant sequences in $\{0,1\}^{\mathbb{N}}$ and $s^* \notin S$ a transitive point. Since $\overline{O_\sigma(s^*)} = \{0,1\}^{\mathbb{N}}$ then the set $X = S \cup O_\sigma(s^*)$ is dense in $\{0,1\}^{\mathbb{N}}$. Moreover, $\sigma(X) \subset X$. Let $s \in X$. If $s = \sigma^n(s^*)$, then $\sigma(s) = \sigma^{n+1}(s^*) \in X$. If $s \in S$ (that is to say $s = (s_0, s_1, ..., s_n, ..)$ and there exists a positive integer k such that s_n is constant $\forall n \geq k$), then $\sigma(s_n)$ is eventually constant. Now, let $g = \sigma_{|X} : X \to X$. Since σ is transitive and $\overline{X} = \{0,1\}^{\mathbb{N}}$, then g is transitive (or simply $\overline{O_\sigma(s^*)} = X$). S is closed in X, the periodic points of g are the constant sequences, so g is not periodically dense. We claim that g is lightly chaotic with respect to the canonical subbase given by the sets of the form $X \cap \prod_n D_n$ where $|D_n| = 2$ $\forall n \in \omega - \{k\}$ and $|D_k| = 1$, for some k. Let $G = X \cap \prod_n D_n$ with $|D_n| = 2$ $\forall n \neq k$ and $|D_k| = 1$. We may assume that $D_k = \{0\}$, so $\overline{0} = (0, 0, ...)$ is a periodic point of g such that $\overline{0} \in G$. Therefore g is lightly chaotic.

Periodically density doesn't suffice to ensure light chaoticity as the following example shows.

Example 3.6. *A map which is not lightly periodically dense nor lightly transitive.*
Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. f is not lightly chaotic with respect to any subbase for $\mathbb{R}$. Let $\mathcal{S}$ be a subbase of $\mathbb{R}$ endowed with the usual topology. We may assume that $\mathcal{S}$ consists of intervals (if $\mathcal{S}'$ is the family of all components of members of $\mathcal{S}$, then f is lightly chaotic with respect to $\mathcal{S}$ iff it is lightly chaotic with respect to $\mathcal{S}'$). Now let $V_1, ..., V_k, S_1, ..., S_n \in \mathcal{S}$ such that $V_1 \cap ... \cap V_k =]-\infty, -1[$ and $S_1 \cap ... \cap S_n =]1, +\infty[$. We may assume that $V_i, S_j \neq \mathbb{R}$ $\forall i, j$. So $V_i =]-\infty, a_i[$, $S_j =]b_j, +\infty[$ $\forall i, j$. Then $a_i = -1$, $b_j = 1$ for some i and for some j. Therefore $]-\infty, -1[,]1, +\infty[\in \mathcal{S}$, and $f^k(]1, +\infty[)\cap]-\infty, -1[= 0$ $\forall k$. Thus f is not lightly transitive. Moreover, the set of periodic points $\mathbb{R}^+_0$ is evidently never dense.

Now, let (X, d) be a metrizable space.

Definition 3.7. A continuous map $f : (X, d) \to (X, d)$ is said to be *lightly sensitive*(with respect to a subbase $\mathcal{S}$), briefly *$L_{\mathcal{S}}$-sensitive*, if there is some $\delta > 0$ such that, for every point x and every subbasic neighborhood V of x in (X, d), there exist $y \in V$ and an integer $k \geq 1$ such that $d(f^k(x), f^k(y)) \geq \delta$.

It is clear that, since X is metrizable, any chaotic map (in the sense of Devaney) is also $L_{\mathcal{S}}$- sensitive.

Example 3.8. *A non lightly sensitive map.*
Let $f : [-1,1] \to [-1,1]$ given by $f(x) = \frac{x}{|x|+1}$. f is not lightly sensitive with respect to any subbase of $\mathbb{R}$ endowed with the usual topology. Consider $\delta > 0$, the point $x = 0$ and a subbasic neighborhood V for x such that $diam(f(V)) < \delta$. Then for every $y \in V$ and every integer $k \geq 1$ we have $d(f^k(x), f^k(y)) = d(0, \frac{y}{k|y|+1}) = \frac{|y|}{k|y|+1} \leq \frac{|y|}{|y|+1} = d(f(0), f(y)) < \delta$

Recall that a dynamical system is chaotic (in the sense of Devaney) if and only if it is transitive, sensitive, and periodically dense. If X is infinite any transitive and periodically dense map $f : (X,d) \to (X,d)$ is sensitive (see [6]). But the corresponding assertion for light properties does not remain true, as the following example shows.

Example 3.9. *A lightly chaotic map which is not lightly sensitive.*
Let $C^+ \subset \mathbb{R}^3$ be the cone whose base is the circle S^1 in the coordinate plane xy and whose vertex is $(0,0,1)$ and $C^- \subset \mathbb{R}^3$ its symmetric with respect to the coordinate plane xy. Consider the surface $X = C^+ \cup C^-$, and the map f defined by $f((e^{2\pi i\theta}), t) = (|t-1|e^{2\pi i(\theta+\alpha)}, -t)$, where $\alpha = \frac{p}{q}$ and p and q are coprime. Note that this is a glide rotation (i.e. an isometry that is a composition of a rotation around an axis with a translation parallel to the rotation axis) by an angle α. The dynamical system (X, f) is lightly chaotic but it is not lightly sensitive. All the open half-spaces in $\mathbb{R}^3$ are a subbase. Let $\mathcal{S}$ be the induced subbase on X. Evidently, any subbasic open set contains a periodic point since it contains a vertex, a fixed point, or intersects the double cone base at a periodic point of period q. Moreover, if U and V are two subbasic open sets then there is a positive integer k such that $f^k(U) \cap V$. Indeed, if U contains a vertex, then $f(U) \cap V$. If U does not contain any vertex, then there is an integer $k \leq q$ such that $f^k(U) \cap V$. However, f is not lightly sensitive. Consider any $\delta > 0$ and let $x \in X$ be a point having altitude 0. Now, if U is a subbasic neighborhood such that $diam(U) < \delta$, then $d(f^k(x), f^k(y)) = d(0, y) < \delta$ for every $y \in U$ and $k \in \mathbb{N}$.

As already noted, if X is an infinite, metrizable space, then every transitive periodically dense map is $L_{\mathcal{S}}$-chaotic and $L_{\mathcal{S}}$- sensitive with respect to any subbase for X, but there are dynamical systems satisfying both light chaoticity and light sensitivity that are not chaotic, as the following example shows.

Example 3.10. *A lightly chaotic, lightly sensitive map which is neither transitive nor periodically dense or sensitive.*
The "truncated tent map by $\frac{1}{2}$" $T_{\frac{1}{2}} : I \to I$ is defined by

$$T_{\frac{1}{2}}(x) = \begin{cases} 2x, & if \quad 0 \leq x < \frac{1}{4} \\ \frac{1}{2}, & if \quad \frac{1}{4} \leq x < \frac{3}{4} \\ -2x+2, & if \quad \frac{3}{4} \leq x \leq 1 \end{cases}$$

The map $f : I \to I$ symmetric of $T_{\frac{1}{2}} : I \to I$ with respect to the line $y - \frac{1}{2} = 0$, defined $f(x) = \begin{cases} -2x+1, & if \quad 0 \leq x < \frac{1}{4} \\ \frac{1}{2}, & if \quad \frac{1}{4} \leq x < \frac{3}{4} \\ 2x-1, & if \quad \frac{3}{4} \leq x \leq 1 \end{cases}$ is not periodically dense: $\mathcal{U} =$

$]\frac{1}{4}, \frac{1}{2}[$ is an open set such that $\forall x \in \mathcal{U}\ f^k(x) = \frac{1}{2} \neq x\ \forall k$. Moreover, f is not transitive. It suffices to observe that f is not onto. Evidently f is not sensitive. If $x = \frac{1}{2}$ and $U =]\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon[$ for some $\epsilon < \frac{1}{4}$, then $|f^k(x) - f^k(y)| = 0$ for every $y \in U$.
It is straightforward to check that f is lightly chaotic with respect to the subbase $\mathcal{S} = \{[0, a[,]b, 1]\}_{0<a\leq 1, 0\leq b<1}$. Moreover, f is lightly sensitive. Let $0 < \delta < \frac{1}{4}$, $x \in X$ and U a neighborhood of X. If $U = [0, a[$ where $0 < a \leq 1$ then there exists some positive integer $k \geq 1$ such that $|f^k(x) - f^k(0)| = |f^k(x) - 1| \geq \delta$. If $U =]b, 1]$ where $0 \leq b < 1$ then there exists some positive integer $k \geq 1$ such that $|f^k(x) - f^k(1)| = |f^k(x) - 1| \geq \delta$.

It is worth noting that, starting from the notion of topological sensitivity ([11]), it would be interesting to study $L_{\mathcal{S}}$-*topological sensitivity*, too.

4. Lightly chaotic functional envelopes

In the study of the relationships between the dynamical properties of a system and its functional envelope it is evident that the dynamical behaviour of the functional envelope is more complicated than that of the original system. In particular, the functional envelope of a chaotic dynamical system in general fails to be chaotic. It suffices to think about transitivity, even for usual spaces (see, for example, Corollary 5.5 and Theorem 5.6 in [5]). Our first result fits in the realm of these investigations. Here are some starting remarks.

Remark 4.1. For every $f : I \to I$ continuous, $F_f : S(I) \to S(I)$ is not chaotic. However, if f is chaotic then F_f is lightly chaotic.
Let $f : S^1 \to S^1$ be a transitive continuous map conjugate to an irrational rotation. Since f has no periodic points, F_f is not lightly chaotic with respect to any subbase for S^1.

It emerges a certain difficulty in obtaining that the functional envelope is chaotic, even if the original system is chaotic. So, a natural question arises: is the chaoticity of a given dynamical system equivalent to the light chaoticity of its functional envelope? The answer is given by the following.

Theorem 4.2. *Let (X, d) be a metric space, and $f : (X.d) \to (X, d)$ a continuous mapping. Then the following are equivalent*

(1) *f is chaotic*
(2) *$F_f : S_k(X) \to S_k(X)$ is lightly chaotic with respect to the canonical subbase.*
(3) *$F_f : S_p(X) \to S_p(X)$ is lightly chaotic with respect to the canonical subbase.*

Proof. Let $\mathcal{S}_k = \{[K, G] : K \in \mathcal{K}[X], G \in \mathcal{T}(d)\}$ be the canonical subbase for $(S(X), \mathcal{T}_k)$. Let $A = [F, U], B = [C, V] \in \mathcal{S} - \{\varnothing\}$. We will show that $F_f^k(A) \cap B \neq \varnothing$ for some positive integer k. Since U, V are non-empty open subsets of (X, d), and f is transitive, there is some positive integer k such that $f^k(U) \cap V \neq \varnothing$. Let $q \in U$ such that $f^k(q) = p \in V$. Now, let $g \in S(X)$ be the map defined by $g(x) = q$ for every $x \in X$. Observe that $g \in A$, so $F_f^k(g) \in F_f^k(A)$. Moreover, $(F_f^k(g))(x) = F_{f^k}(g))(x) = (f^k \circ g)(x) = f^k(g(x)) = f^k(q) = p \in V$ for every $x \in X$. Therefore ${F_f}^k(g) \in B$. Hence $F_f^k(A) \cap B \neq \varnothing$. Now let us show that every member of $\mathcal{S} - \{\varnothing\}$ contains a periodic point of F_f. Let $A = [F, U] \in \mathcal{S} - \{\varnothing\}$. Since U is a non-empty open subset of X and f is periodically dense, there is some $k > 0$ and $x_0 \in U$ such that $f^k(x_0) = x_0$. Let $g : X \to X$ be the map given by $g(x) = x_0$ for every $x \in X$. Then $g \in A$ and $(F_f^k(g))(x) = (F_{f^k}(g))(x) = (f^k \circ g)(x) = f^k(g(x)) = f^k(x_0) = x_0 \in V$ for every $x \in X$, so $F_f^k(g) = g$ and g is a periodic point of F_f contained in A. Therefore F_f is lightly chaotic. Then $(S(X), F_f)$ is lightly chaotic with respect to any subbase of the point-open topology contained in $\mathcal{S}_k$.

Conversely, we have to show that f is transitive and periodically dense. Let us first check the transitivity. Let U and V be a pair of non-empty open subsets of X, and let us pick some $x_0 \in X$. Then, denoted by $\mathcal{S}_p$ the canonical subbase for $(S(X), \mathcal{T}_p)$, $A = [\{x_0\}, U], B = [\{x_0\}, V] \in \mathcal{S}_p - \{\varnothing\}$. Since F_f is lightly chaotic with respect to the canonical subbase $\mathcal{S}_p$, there is some $k > 0$ such that $F_f^k(A) \cap B \neq \varnothing$. So there are two maps g and h such that $g \in A$, $h \in B$ and $F_f^k(g) = h$. Now $g(x_0) \in U$ and $h(x_0) \in V$, so $f^k(g(x_0)) = F_{f^k}(g)(x_0) = (F_f^k(g))(x_0) = h(x_0) \in f^k(U) \cap V \neq \varnothing$. Therefore, f is transitive.

Now let us show that f is periodically dense. Let U be a non-empty open subset of X and let $x_0 \in U$. Since F_f is lightly chaotic and $A = [\{x_0\}, U] \in \mathcal{S}_p - \{\varnothing\}$, there is some $g \in A$ and $k > 0$ such that $F_f^k(g) = g$. Thus $f^k(g(x_0)) = (F_f^k(g))(x_0) = g(x_0) \in U$, and this means that $g(x_0)$ is a periodic point of f contained in U. The map f is periodically dense, hence chaotic. □

Let us also recall that a continuous self-map f on a compact metric space (X, d) is said to be *structurally stable* if there is some $\epsilon > 0$ such that every $g \in S(X)$ with $\widehat{d}(f, g) < \epsilon$ is topologically conjugate to f. We introduce a useful notion.

Definition 4.3. Let $f : X \to X$ onto, f is *onto-stable* if there exists an open set U of $S_k(X)$ such that $f \in U$ and every $g \in U$ is onto.

Remark 4.4. Every transitive structurally stable map $f : X \to X$, where X is a compact (metric) space, is "onto-stable".

Remark 4.5. Let $f : X \to X$ be a structurally stable map where X is compact. Then f is *onto-stable*.

The next result gives some additional information about the chaotic behavior of $(S(X), F_f)$, where X is a compact metrizable space and $F_f : S_k(X) \to S_k(X)$ is a continuous map. Let $P(f)$ be the set of all periodic points for f.

Theorem 4.6. *The following hold:*

i) *Let X be a (metric) first countable continuum and let $f : X \to X$ be a continuous map. If $|\mathrm{P}(f)| < c$ then F_f is not periodically dense.*
ii) *If $f : I \to I$, where $I = [0, 1]$, is periodically dense, then F_f is not transitive.*

iii) *Let X is be a compact (metric) space and let $f : X \to X$ be a continuous map. If f is transitive structurally stable "onto-stable" then F_f is not transitive.*

iv) *Let X is be a compact (metric) space and let $f : X \to X$ be a continuous map. If f is "onto-stable" and $\mathrm{P}(f) \neq X$ then F_f is not periodically dense.*

Proof.

i) Let us suppose that g is a periodic point of F_f. We claim that g must be constant. Let us take a positive integer k such that ${F_f}^k(g) = g$, that is, $F_f^k(g)(x) = g(x)\ \forall x \in X$, then $g(x)$ is a periodic point of $f\ \forall x \in X$. So $g(X) \subset P(f)$. By hypothesis $|g(X)| < c$. Since $g(X)$ is a (metric) first countable continuum space, it follows that $|g(X)| = 1$, namely g is constant. Now, let $a, b \in X, a \neq b$ and let U and V two open sets of X such that $a \in U$, $b \in V$ and $U \cap V = \varnothing$. Then $G = [\{a\}, U] \cap [\{b\}, V]$ is a non-empty open subset of $S(X)$ (since it contains the identity map), which does not contain constant functions, so F_f is not periodically dense.

ii) Since $S(I)$ is a separable complete metric space without isolated points, it is enough to show that F_f has no dense orbit (see [14]), i.e., $\overline{O_{F_f}(g)} \neq S(I)$ for every $g \in S(I)$, where $O_{F_f}(g) = \{{F_f}^n(g) : n \in \mathbb{N}\} = \{g, f \circ g, f^2 \circ g, ...\}$. If g is a constant map, then $O_{F_f}(g)$ consists of constant functions. Now the set $V = [[0, \frac{1}{2}],]0, \frac{1}{4}[] \cap [\{1\},]\frac{2}{3}, \frac{3}{4}[]$ is a non-empty open subset of $S(I)$, equipped with the compact-open topology, which does not contain constant maps, so $V \cap O_{F_f}(g) = \varnothing$. If g is not constant, then $g(I)$ has a non-empty interior. Since f is periodically dense, $g(I)$ contains a periodic point p of f. Let $q \in I$ such that $g(q) = p$ and set $U = [I, I - O_f(p)]$. U is a non-empty open subset of $S(I)$ such that $U \cap O_{F_f}(g) = \varnothing$. In fact $(f^n \circ g)(q) = f^n(g(q)) = f^n(p) \in O_f(p)$ for every $n \in \mathbb{N}$, so $f^n \circ g \notin U\ \forall n$. Therefore f is not transitive.

iii) Any transitive structurally stable map $f : X \to X$, where X is a compact (metric) space, is onto-stable (by Definition 4.3). Let $\mathcal{C}_0$ be the subspace of $S(X)$ consisting of all onto maps. Since $f \in \mathcal{C}_0$, then $\mathcal{C}_0 \neq \varnothing$. Moreover, $\mathcal{C}_0$ is closed. Indeed, if $g \in S(X) - \mathcal{C}_0$, let $a \in X - g(X)$ and let $U = [X, X - \{a\}]$. U is open in $S(X)$, $g \in U$ and $U \cap \mathcal{C}_0 = \varnothing$, so $g \notin cl\mathcal{C}_0$. Observe that $\mathcal{C}_0 \neq X$. Moreover $\mathcal{C}_0$ is F_f-invariant: $\forall g \in \mathcal{C}_0, F_f(g) = f \circ g$ is onto, so $F_f(g) \in \mathcal{C}_0$. Since f is structurally stable then $Int\mathcal{C}_0 \neq \varnothing$. In fact, there exists $\epsilon > 0$ such that $B_\epsilon(f)$ consists only of maps conjugate to f. These are transitive maps, hence onto, so $B_\epsilon(f) \subset \mathcal{C}_0$. Therefore F_f is not transitive.

iv) Let X be a compact metric space and $f : X \to X$ an "onto-stable" map. Let us prove that if $P(f) \neq X$ then F_f is not periodically dense. Let $\varepsilon > 0$ be such that g is onto $\forall g \in B_\varepsilon(f)$. If g is a periodic point of F_f, then $\exists k > 0$ such that ${F_f}^k(g) = g$, that is $f^k(g(x)) = g(x)\ \forall x \in X$. So $g(x) \in P(f)\ \forall x \in X$, and this means that $g(X) \subset P(f)$. Therefore, by hypothesis, it follows that $g(X) \neq X$, so $g \notin B_\epsilon(f)$. Therefore $B_\epsilon(f)$ does not contain periodic points of F_f.

□

As we noted, there exists a lightly sensitive, lightly chaotic map that is neither transitive nor periodically dense. This continues to hold true in hypothesis of sensitivity.

Example 4.7. *A sensitive lightly chaotic map which is neither transitive nor periodically dense.*
Let $f : I \to I$ be any chaotic map, e.g., the tent map. Applying the previous Theorems 4.2 and 4.6, it follows that $F_f : S(I) \to S(I)$ is neither transitive nor periodically dense, but it is lightly chaotic. It remains to be shown that F_f exhibits a sensitive dependence on initial conditions. We must prove that $\exists \delta > 0$ such that $\forall g \in S(I)$ and for all open sets U_g containing g, $\exists h \in U_g$ and $n > 0$ such that $\widehat{\rho}({F_f}^n(g), {F_f}^n(h)) > \delta$. So, let $U_g = \cap_{i=1}^m [K_i, V_i]$ and take $x_0 \in K_1, g(x_0) \in V_1$. Since the map f is chaotic and therefore sensitive, called δ the sensitive constant, then $\exists y_0 \in V_1$ and $n > 0$ such that $\rho(f^n(g(x_0)), f^n(y_0)) > \delta$. Let $h : I \to I$ be a continuous map such that $h(K_1) = y_0$ and $h_{|K_j} = f_{|K_j}$ $\forall j \geq 2$. Now, $h \in U_g$. Moreover $\widehat{\rho}(\overline{f}^n(g), \overline{f}^n(h)) = \sup_{x \in I} \rho(f^n(g(x)), f^n(y)) \geq \rho(f^n(g(x_0)), f^n(y_0)) > \delta$.

5. Concluding remarks

Future investigations could have two perspectives. The first concerns the introduction of other *light* dynamical properties and the study of their interdependencies with classical dynamical properties, the second one concerns their connections with the dynamical properties of the functional envelope. It might be worth considering set-open topologies and uniform convergence topologies (see, for example [4], [13], [16], [17], [18], [19]). Moreover, analogously, it might be interesting to study the connections when the envelope is hyperspace.

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