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On traditional Menger and Rothberger variations

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Abstract

We present a comprehensive report on the relationships between variations of the Menger and Rothberger selection properties with respect to ω-covers and k-covers in the most general topological setting and address the finite productivity of some of these properties. We collect various examples that separate certain properties and we carefully identify which separation axioms simplify aspects of these properties. We finish with a consolidated list of open questions focused on topological examples.

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1. Introduction

The purpose of this paper is to bear the relationships, summarized in Figures 1 and 2, between standard variations of the Menger and Rothberger properties in the most general topological settings and to identify unresolved questions. En route to proving the implications of the above-mentioned figures, we also prove novel results about the productivity of certain properties related to selection principles using k-covers (Theorems 4.25, 4.27, and 4.30). The implications of Figure 1 are mostly established with Theorem 4.43; the implications of

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Figure 2 have been established previously, but in some cases under certain assumptions about separation axioms. Our contribution to this line is to remove those assumptions and still obtain the implications.

As an example, relationships between some variants of the Menger and Rothberger properties have originally surfaced in the context of C_p-theory, where one usually assumes every space considered is Tychonoff. One may wish to apply such equivalences to, for example, Hausdorff spaces, and consequently wonder if the Tychonoff assumption is necessary. Such equivalences, as well as others, will be proved herein without assuming any separation axioms.

We note that this work was inspired by updating the π-base [32], a database that started as a digital expansion of [44], and identifying certain gaps. As such, some examples presented here will bear the names used in [44] and their ID in the π-base, where applicable.

The paper is structured as follows. Sections 2 and 3 review the notions we'll be working with in this manuscript. Section 4 contains most of the theory which culminates in Theorem 4.43, though many results are presented for their own interest. Section 4 is developed through various subsections: Section 4.1 covers the relationships between the ω-variants and finite powers of a space, Section 4.2 discusses the k-cover analogues, and Section 4.3 summarizes some basic conclusions based on the previous two sections. Sections 4.1 and 4.2 are organized in terms of increasing strength, in a sense, and that order is motivated by the fact that traditional selection principles are the most widely studied of the levels included here and so should appear sooner rather than later. We also typically lead with the finite-selection versions before single-selection versions, a choice also motivated by *strength*. Once the implications and equivalences have been proved, Section 5 provides a list of examples that show that certain implications do not reverse. Lastly, Section 6 collects some questions the authors have not been able to answer, and is mostly a search for examples of topological spaces that separate what seem to be very similar properties.

Unless otherwise noted, no separation axioms are assumed, so when we say "for any space X," there is no implicit assumption that X is, for example, Hausdorff. When we do require separation axioms, we will use the term *regular* without assuming T_1; we will use T_3 for regular and T_1. Any terms used without being defined are to be understood as in [15].

2. Three perspectives on cover collections

The definition of $\mathcal{O}$, the collection of all open covers of a space X, is standard and used consistently through the literature (with few exceptions, where X is excluded from open covers; e.g. [30]). However, the reader should note that there are two standard definitions for Ω, one which merely requires that for each $\mathcal{W} \in \Omega$ and finite $F \subseteq X$, there exists $W \in \mathcal{W}$ with $F \subseteq W$ [19], and another which additionally disallows $X \in \mathcal{W}$ for each $\mathcal{W} \in \Omega$ [39]; i.e., the cover is not "trivial". And on occasion, authors find it necessary to consider covers as sequences rather than sets [4].

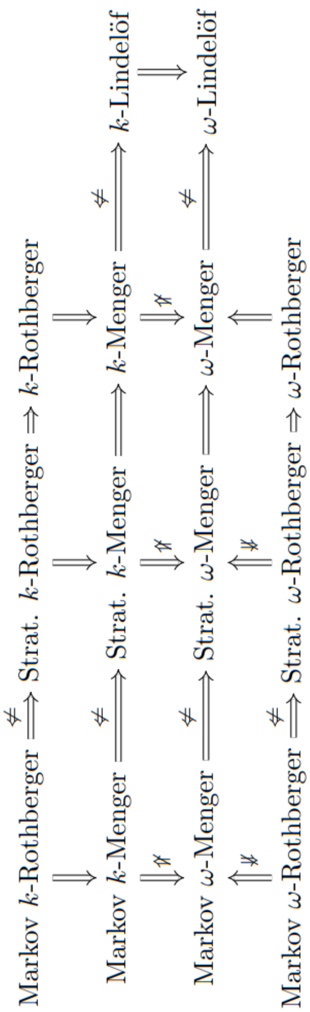

FIGURE 1. k- and ω-variants in ZFC

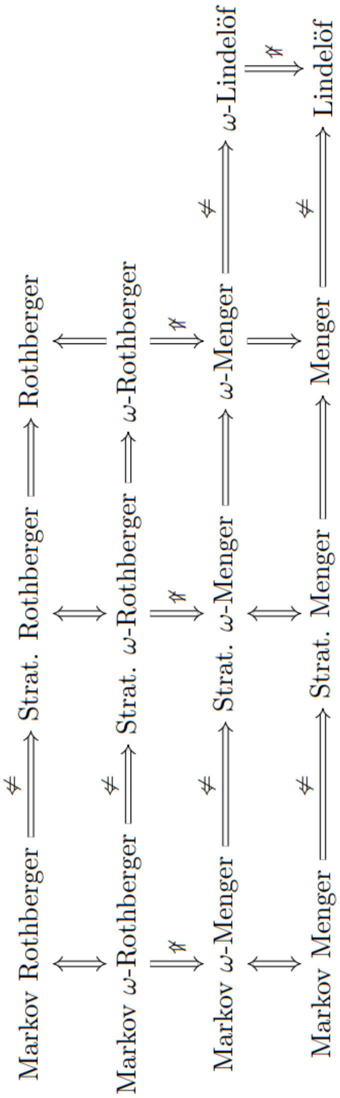

FIGURE 2. ω- and standard-variants in ZFC

This ambiguity can result in some heartburn for the careful mathematician: do results for one Ω hold for the other? And why is this discrepancy in the literature in the first place?

To understand this better, we present several characterizations used in the literature for the standard cover collections $\mathcal{O}, \Lambda, \Omega, \Gamma, \mathcal{K}$.

Definition 2.1. Let $\mathcal{O}_1 = \mathcal{O}_2$ collect all open covers of a topological space X.

Let $\mathcal{O}_3$ collect all transfinite sequences of open sets $\langle U_\beta \rangle_{\beta<\alpha}$ such that $\{U_\beta : \beta < \alpha\}$ forms an open cover of X.

Definition 2.2. Let Λ_1 collect all open covers $\mathcal{L}$ of X such that for each $x \in X$, $\{L \in \mathcal{L} : x \in L\}$ is infinite.

Let Λ_2 collect all open covers $\mathcal{L}$ of X such that either, for each $x \in X$, $\{L \in \mathcal{L} : x \in L\}$ is infinite, or $X \in \mathcal{L}$.

Let Λ_3 collect all transfinite sequences of open sets $\langle L_\beta \rangle_{\beta<\alpha}$ such that either for each $x \in X$, $\{\beta < \alpha : x \in L_\beta\}$ is infinite, or $X = L_\beta$ for some $\beta < \alpha$.

These are known as *large* or λ-covers.

Definition 2.3. Let Ω_1 collect all open covers $\mathcal{W}$ of X such that for each finite $F \subseteq X$, there exists $W \in \mathcal{W}$ with $F \subseteq W$, and $X \notin \mathcal{W}$.

Let Ω_2 collect all open covers $\mathcal{W}$ of X such that for each finite $F \subseteq X$, there exists $W \in \mathcal{W}$ with $F \subseteq W$.

Let Ω_3 collect all transfinite sequences of open sets $\langle W_\beta \rangle_{\beta<\alpha}$ such that for each finite $F \subseteq X$, there exists $\beta < \alpha$ with $F \subseteq W_\beta$.

These are known as ω-covers.

Definition 2.4. Let Γ_1 collect all open covers $\mathcal{C}$ of X such that for each $x \in X$, $\{C \in \mathcal{C} : x \in C\}$ is infinite and co-finite, and $X \notin \mathcal{C}$.

Let Γ_2 collect all open covers $\mathcal{C}$ of X such that either for each $x \in X$, $\{C \in \mathcal{C} : x \in C\}$ is infinite and co-finite, or $X \in \mathcal{C}$ and $\mathcal{C}$ is finite.

Let Γ_3 collect all transfinite sequences of open sets $\langle C_\beta \rangle_{\beta<\alpha}$ such that either for each $x \in X$, $\{\beta < \alpha : x \in C_\beta\}$ is infinite and co-finite.

These are known as γ-covers.

Definition 2.5. Let $\mathcal{K}_1$ collect all open covers $\mathcal{V}$ of X such that for each compact $K \subseteq X$, there exists $V \in \mathcal{V}$ with $K \subseteq V$, and $X \notin \mathcal{V}$.

Let $\mathcal{K}_2$ collect all open covers $\mathcal{V}$ of X such that for each compact $K \subseteq X$, there exists $V \in \mathcal{V}$ with $K \subseteq V$.

Let $\mathcal{K}_3$ collect all transfinite sequences of open sets $\langle V_\beta \rangle_{\beta<\alpha}$ such that for each compact $K \subseteq X$, there exists $\beta < \alpha$ with $K \subseteq V_\beta$.

These are known as k-covers.

We have then the following relationships.

Proposition 2.6. *For each $i \in \{1, 2, 3\}$, $\mathcal{K}_i \subseteq \Omega_i$ and $\Gamma_i \subseteq \Omega_i \subseteq \Lambda_i \subseteq \mathcal{O}_i$.*

Perhaps motivating the disqualification of X from an ω-cover as in Ω_1 is the guarantee that it disallows all finite ω-covers.

Proposition 2.7. *If $\mathcal{W} \in \Omega_2$ is finite, then $X \in \mathcal{W}$ (and thus $\mathcal{W} \notin \Omega_1$).*

Proof. We prove the contrapositive by assuming $X \notin \mathcal{W} = \{W_0, \dots, W_n\}$. Choose $x_i \in X \setminus W_i$; it follows that $\{x_0, \dots, x_n\} \not\subseteq W_i$ for any $i \leq n$; therefore $\mathcal{W} \notin \Omega_2$. □

It is the preference of the authors of this manuscript to assume $i \in \{2, 3\}$; the literature is filled with minor errata that arise when disallowing X in the open cover. (E.g., a common technique to obtain an ω-cover from an arbitrary open cover is to close it under finite unions; however, if the open cover contains a finite subcover, this would not obtain an ω-cover in the sense of $i = 1$.) Furthermore, while it's out of scope to explore in-depth here, the case where X belongs to a cover has a nice analog in C_p-theory: the γ-covers in X correspond to the sequences converging to $\mathbf{0}$ in $C_p(X)$, and when X belongs to the γ-cover, we may consider a trivial sequence in $C_p(X)$.

Regardless, all results proven in this paper can be shown true no matter what characterization is considered, and we will not specify a subscript $i \in \{1, 2, 3\}$. For references cited, while the distinctions can generally be hand-waved away, the reader should be aware that most authors consider at most one of these three characterizations.

3. Background and Preliminaries

We will use the standard definition of ω where $n \in \omega$ is $\{m \in \omega : m \in n\}$. Hence, given $A \subseteq \omega$ and $n \in \omega$, we may write $A \subseteq n$. We let $[X]^{<\omega}$ denote the set of all finite subsets of a set X. We will use $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ to denote the usual coordinate projection mappings.

We will use $\mathcal{O}_X$ to denote the collection of all open covers of X, viewing $\mathcal{O}$ as a *topological operator*. A topological operator is a class function defined on the class of all topological spaces. Another topological operator that will appear here is $\mathscr{T}$, the topological operator that produces all non-empty open subsets of a space X.

When a topological space X is given and $A \subseteq X$, we denote the neighborhood system $\{U \in \mathscr{T}_X : A \subseteq U\}$ about A with $\mathcal{N}_A$. For $x \in X$, we use the simplified notation $\mathcal{N}_x$ instead of $\mathcal{N}_{\{x\}}$, if there is no risk of confusion. Also, for a collection $\mathcal{A}$ of subsets of X, $\mathcal{N}(\mathcal{A}) = \{\mathcal{N}_A : A \in \mathcal{A}\}$.

We will consider two other kinds of open covers.

Definition 3.1. For a space X, an open cover $\mathscr{U}$ of X is said to be

- an *ω-cover* of X if every finite subset of X is contained in a member of $\mathscr{U}$.
- a *k-cover* of X if every compact subset of X is contained in a member of $\mathscr{U}$.

We will let Ω (resp. $\mathcal{K}$) be the topological operator which produces Ω_X (resp. $\mathcal{K}_X$), the set of all ω-covers (resp. k-covers) of a space X.

The notion of ω-covers is commonly attributed to [19], but they were already in use in [25] where they are refereed to as *open covers for finite sets*. The notion of k-covers appears as early as [26] in which they are referred to as *open covers for compact subsets.*

We remind the reader of a generalization of the above-mentioned cover types.

Definition 3.2. Let $\mathcal{A}$ be a collection of subsets of a space X. Then we define the $\mathcal{A}$-covers, denoted by $\mathcal{O}_X(\mathcal{A})$, to be the collection of all open covers $\mathscr{U}$ of X such that, for each $A \in \mathcal{A}$, there is some $U \in \mathscr{U}$ such that $A \subseteq U$.

We recall the usual selection principles. For more details on selection principles and relevant references, see [37, 23, 41, 42].

Definition 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be sets. Then the single- and finite-selection principles are defined, respectively, to be the properties

$$\mathsf{S}_1(\mathcal{A},\mathcal{B}) \equiv (\forall A \in \mathcal{A}^\omega)\left(\exists B \in \prod_{n\in\omega} A_n\right)\ \{B_n : n \in \omega\} \in \mathcal{B}$$

and

$$\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B}) \equiv (\forall A \in \mathcal{A}^\omega)\left(\exists B \in \prod_{n\in\omega} [A_n]^{<\omega}\right)\ \bigcup\{B_n : n \in \omega\} \in \mathcal{B}.$$

Following [42], for a space X and topological operators $\mathcal{A}$ and $\mathcal{B}$, we write $X \models \mathsf{S}_\square(\mathcal{A},\mathcal{B})$, where $\square \in \{1, \mathrm{fin}\}$, to mean that X satisfies the selection principle $\mathsf{S}_\square(\mathcal{A}_X,\mathcal{B}_X)$.

Using this notation, recall that a space X is *Menger* (resp. *Rothberger*) if $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ (resp. $X \models \mathsf{S}_1(\mathcal{O},\mathcal{O})$).

Selection principles have naturally corresponding selection games, which include types of topological games. Topological games have a long history, much of which can be gathered from Telgársky's survey [47]. In this paper, we consider the traditional selection games for two players, P1 and P2, of countably infinite length.

Definition 3.4. Given sets $\mathcal{A}$ and $\mathcal{B}$, we define the *finite-selection game* $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ for $\mathcal{A}$ and $\mathcal{B}$ as follows. In round $n \in \omega$, P1 plays $A_n \in \mathcal{A}$ and P2 responds with $\mathscr{F}_n \in [A_n]^{<\omega}$. We declare P2 the winner if $\bigcup\{\mathscr{F}_n : n \in \omega\} \in \mathcal{B}$. Otherwise, P1 wins.

Definition 3.5. Given sets $\mathcal{A}$ and $\mathcal{B}$, we analogously define the *single-selection game* $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ for $\mathcal{A}$ and $\mathcal{B}$ as follows. In round $n \in \omega$, P1 plays $A_n \in \mathcal{A}$ and P2 responds with $x_n \in A_n$. We declare P2 the winner if $\{x_n : n \in \omega\} \in \mathcal{B}$. Otherwise, P1 wins.

Definition 3.6. By *selection games*, we mean the class consisting of $\mathsf{G}_\square(\mathcal{A},\mathcal{B})$ where $\square \in \{1, \mathrm{fin}\}$, and $\mathcal{A}$ and $\mathcal{B}$ are sets. So, when we say $\mathcal{G}$ is a selection game, we mean that there exist $\square \in \{1, \mathrm{fin}\}$ and sets $\mathcal{A}, \mathcal{B}$ so that $\mathcal{G} = \mathsf{G}_\square(\mathcal{A},\mathcal{B})$.

The study of games naturally inspires questions about the existence of various kinds of strategies. Infinite games and corresponding full-information strategies were both introduced in [16]. Some forms of limited-information strategies came shortly after, like positional (also known as stationary) strategies [12, 43]. For more on stationary and Markov strategies, see [18].

Definition 3.7. We define strategies of various strengths below.

- A *strategy for P1* in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is a function $\sigma : (\bigcup\mathcal{A})^{<\omega} \to \mathcal{A}$. A strategy σ for P1 is called *winning* if whenever $x_n \in \sigma\langle x_k : k < n\rangle$ for all $n \in \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If P1 has a winning strategy, we write $\mathrm{I} \uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B})$.
- A *strategy for P2* in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is a function $\sigma : \mathcal{A}^{<\omega} \to \bigcup\mathcal{A}$. A strategy σ for P2 is *winning* if whenever $A_n \in \mathcal{A}$ for all $n \in \omega$, $\{\sigma(A_0,\ldots,A_n) : n \in \omega\} \in \mathcal{B}$. If P2 has a winning strategy, we write $\mathrm{II} \uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B})$.
- A *predetermined strategy* for P1 is a strategy which only considers the current turn number. Formally it is a function $\sigma : \omega \to \mathcal{A}$. If P1 has a winning predetermined strategy, we write $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathcal{A},\mathcal{B})$.
- A *Markov strategy* for P2 is a strategy which only considers the most recent move of P1 and the current turn number. Formally it is a function $\sigma : \mathcal{A} \times \omega \to \bigcup\mathcal{A}$. If P2 has a winning Markov strategy, we write $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{A},\mathcal{B})$.
- If there is a single element $A_0 \in \mathcal{A}$ so that the constant function with value A_0 is a winning strategy for P1, we say that P1 has a *constant winning strategy*, denoted by $\mathrm{I} \underset{\text{cnst}}{\uparrow} \mathsf{G}_1(\mathcal{A},\mathcal{B})$.

These definitions can be extended to $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B})$ in the obvious way.

Note that, for any selection game $\mathcal{G}$,

$$\mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{G} \implies \mathrm{II} \uparrow \mathcal{G} \implies \mathrm{I} \not\uparrow \mathcal{G} \implies \mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathcal{G} \implies \mathrm{I} \underset{\text{cnst}}{\not\uparrow} \mathcal{G}.$$

Definition 3.8. For two selection games $\mathcal{G}$ and $\mathcal{H}$, we write $\mathcal{G} \leq_{\mathrm{II}} \mathcal{H}$ if each of the following hold:

- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{G} \implies \mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{H}$,
- $\mathrm{II} \uparrow \mathcal{G} \implies \mathrm{II} \uparrow \mathcal{H}$,
- $\mathrm{I} \not\uparrow \mathcal{G} \implies \mathrm{I} \not\uparrow \mathcal{H}$, and
- $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathcal{G} \implies \mathrm{II} \underset{\text{pre}}{\not\uparrow} \mathcal{H}$.

If, in addition,

- $\mathrm{I} \underset{\text{cnst}}{\not\uparrow} \mathcal{G} \implies \mathrm{II} \underset{\text{cnst}}{\not\uparrow} \mathcal{H}$,

we write that $\mathcal{G} \leq^+_{\mathrm{II}} \mathcal{H}$.

Note that, for any sets $\mathcal{A}$ and $\mathcal{B}$,

$$\mathsf{G}_1(\mathcal{A},\mathcal{B}) \leq^+_{\mathrm{II}} \mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B}).$$

We use the notation $\leq_{\mathrm{II}}$ to emphasize the fact that this partial order transfers winning plays for P2. As an example, note that $\mathrm{I} \not\uparrow \mathcal{G}$ means that, for any strategy that P1 employs, there exists a play by P2 that wins against that strategy. Then the implication in the definition of $\leq_{\mathrm{II}}$ would indicate that P2 can accordingly win against any strategy employed by P1 in $\mathcal{H}$.

Below, we will write statements such as $\mathsf{G}_\square(\mathcal{A},\mathcal{A}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are topological operators to mean that, for every topological space X, $\mathsf{G}_\square(\mathcal{A}_X,\mathcal{A}_X) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B}_X,\mathcal{B}_X)$.

Remark 3.9. The following are mentioned in [9, Prop. 15] and [3, Lem. 2.12] for $\square \in \{1, \mathrm{fin}\}$.

- $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B})$ is equivalent to $\mathsf{S}_\square(\mathcal{A},\mathcal{B})$.
- $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B})$ is equivalent to the property that, for every $A \in \mathcal{A}$, there is $B \in [A]^{\leq\omega}$ so that $B \in \mathcal{B}$.

Note that the property $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B})$ is a Lindelöf-like principle and falls in the category of what Scheepers [42] refers to as *Bar-Ilan selection principles.*

In particular, note that, if $\mathsf{G}_\square(\mathcal{A},\mathcal{B}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{C},\mathcal{D})$ where $\square \in \{1, \mathrm{fin}\}$, then, for any space X, $X \models \mathsf{S}_\square(\mathcal{A},\mathcal{B}) \implies X \models \mathsf{S}_\square(\mathcal{C},\mathcal{D})$.

Following [23], we will employ the following terminology. A space X is

- *ω-Lindelöf* (referred to as ϵ-spaces in [19]) if every ω-cover has a countable subset which is an ω-cover; equivalently, if $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_{\mathrm{fin}}(\Omega_X,\Omega_X)$.
- *k-Lindelöf* if every k-cover has a countable subset which is a k-cover; equivalently, if $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_{\mathrm{fin}}(\mathcal{K}_X,\mathcal{K}_X)$.

Proposition 3.10. *Every second-countable space is k-Lindelöf and ω-Lindelöf.*

Proof. Let $\mathscr{B}_0$ be a countable basis for a space X and let

$$\mathscr{B} = \left\{ \bigcup \mathscr{F} : \mathscr{F} \in [\mathscr{B}_0]^{<\omega} \right\}.$$

Notice that $\mathscr{B}$ is also countable. Since we will establish with Theorem 4.43 that every k-Lindelöf space is ω-Lindelöf, we prove here that X is k-Lindelöf, though the direct proof of ω-Lindelöfness in this case is identical in form to what follows.

So let $\mathscr{U}$ be a k-cover of X, and, for each compact $K \subseteq X$, let $U \in \mathscr{U}$ be such that $K \subseteq U$. We can then find $\mathscr{F}_K \in [\mathscr{B}_0]^{<\omega}$ such that $K \subseteq \bigcup \mathscr{F}_K \subseteq U$. Note then that $\bigcup \mathscr{F}_K \in \mathscr{B}$. Now,

$$\mathscr{V} := \left\{ \bigcup \mathscr{F}_K : K \subseteq X \text{ is compact} \right\} \subseteq \mathscr{B},$$

and is thus countable. Make a choice $U_V \in \mathscr{U}$ for each $V \in \mathscr{V}$ with $V \subseteq U_V$ and note that $\{U_V : V \in \mathscr{V}\}$ is the desired countable subset of $\mathscr{U}$. □

We recall some P1 strategy reduction theorems, the first of which are the celebrated theorems of Hurewicz [20] and Pawlikowski [31]. For more on the proofs of the theorems of Hurewicz and Pawlikowski, we refer the reader to [45]; for a pointless (that is, lattice-theoretic) approach, see [27].

Theorem 3.11 (Hurewicz [20]/Pawlikowski [31]). *For* $\square \in \{1, \mathrm{fin}\}$,

$$\mathrm{I} \uparrow \mathsf{G}_\square(\mathcal{O}, \mathcal{O}) \iff \mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_\square(\mathcal{O}, \mathcal{O}).$$

Theorem 3.12 (Scheepers [38]). *For* $\square \in \{1, \mathrm{fin}\}$,

$$\mathrm{I} \uparrow \mathsf{G}_\square(\Omega, \Omega) \iff \mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_\square(\Omega, \Omega).$$

Theorem 3.13 (Caruvana & Holshouser [3]). *For* $\square \in \{1, \mathrm{fin}\}$,

$$\mathrm{I} \uparrow \mathsf{G}_\square(\mathcal{K}, \mathcal{K}) \iff \mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_\square(\mathcal{K}, \mathcal{K}).$$

With these results, the properties considered in this paper "collapse" for P1 and we need only have terminology for the corresponding selection principles. So, following the notation above, we will say a space X is

- ω*-Menger* (resp. ω*-Rothberger*) if $X \models \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ (resp. $X \models \mathsf{S}_1(\Omega, \Omega)$).
- k*-Menger* (resp. k*-Rothberger*) if $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{K}, \mathcal{K})$ (resp. $X \models \mathsf{S}_1(\mathcal{K}, \mathcal{K})$).

We will, however, have distinguishing terminology for the situation on P2's side of things.

Definition 3.14. For the properties $\mathcal{P}$ discussed above, we will say that X is *strategically* $\mathcal{P}$ if P2 has a winning strategy in the game corresponding to $\mathcal{P}$. Analogously, we will say a space is *Markov* $\mathcal{P}$ if P2 has a winning Markov strategy in the corresponding game to $\mathcal{P}$.

4. General Results

4.1. **Characterizations using finite powers.**

An important characterization of the ω-variants is found in finite powers. We will collect these characterizations in each of their strategic levels.

Theorem 4.1 ([19, p. 156]). *A space X is ω-Lindelöf if and only if each of its finite powers is Lindelöf.*

Theorem 4.2. *For any space X,*

(1) X is ω-Menger if and only if each of its finite powers is Menger ([22, Thm. 3.9]*).*
(2) X is ω-Rothberger if and only if each of its finite powers is Rothberger ([36, p. 918]*).*

To characterize the "strategically $\mathcal{P}$" level without any separation axiom assumptions, we recall that being strategically Rothberger (resp. Menger) is finitely productive. The single-selection version of the equivalence referred to here will appear here as Theorem 4.5(2) and was originally proved in [10] by passing through a related game on $C_p(X)$ under the assumption that X is Tychonoff. We will see that one direction of the general equivalence will be the result of Theorem 4.3 and a bijection $\omega^2 \to \omega$.

Theorem 4.3. *Let X and Y be spaces.*

(1) If X and Y are strategically Menger, then $X \times Y$ is strategically Menger ([14, Prop. 5.1]).
(2) If X and Y are strategically Rothberger, then $X \times Y$ is strategically Rothberger ([14, Cor. 4.9]).

As an immediate consequence, if a space X is strategically Menger or strategically Rothberger, then each of its finite powers is, too.

To obtain the reverse direction of Theorem 4.5(2), we use a generalized idea of Galvin [17] below, which is proved by a straightforward diagonalization, and a basic unfolding argument.

Lemma 4.4. *Let X be a space, $\mathcal{A}$ be a collection of subsets of X, and suppose $\varphi : \mathcal{O}_X(\mathcal{A}) \to \mathscr{T}_X$. If $\varphi(\mathscr{U}) \in \mathscr{U}$ for every $\mathscr{U} \in \mathcal{O}_X(\mathcal{A})$, then there exists $A \in \mathcal{A}$ such that $\mathcal{N}_A \subseteq \varphi[\mathcal{O}_X(\mathcal{A})]$.*

Proof. By way of contrapositive, suppose that, for every $A \in \mathcal{A}$, there exists $U \in \mathcal{N}_A$ such that $U \notin \varphi[\mathcal{O}_X(\mathcal{A})]$. So let $U_A \in \mathcal{N}_A$ witness this property for each $A \in \mathcal{A}$. Note that $\mathscr{U} := \{U_A : A \in \mathcal{A}\} \in \mathcal{O}_X(\mathcal{A})$. It follows, by construction, that $\varphi(\mathscr{U}) \notin \mathscr{U}$. □

Theorem 4.5. *Let X be a space.*

(1) X is strategically ω-Menger if and only if it is strategically Menger.
(2) X is strategically ω-Rothberger if and only it is strategically Rothberger.

Proof. The content of (1) is [8, Thm. 35].

Now we address (2). Suppose X is strategically Rothberger. By Theorem 4.3(2), X^{n+1} is strategically Rothberger for every $n \in \omega$. So let σ_n be a winning strategy for P2 in the Rothberger game on X^{n+1} for each $n \in \omega$, and let $\beta : \omega^2 \to \omega$ be a bijection with the property that $\langle \beta(n,k) : k \in \omega \rangle$ is strictly increasing for $n \in \omega$. Now define the strategy σ in the ω-Rothberger game on X in the following way. Given $n \in \omega$ and a sequence $\langle \mathscr{U}_j : j < n \rangle$ of ω-covers of X, let $(m,k) \in \omega^2$ be such that $\beta(m,k) = n$. Note that, for each $\ell \leq k$,

$$\mathscr{U}^{(m+1)}_{\beta(m,\ell)} := \left\{ U^{m+1} : U \in \mathscr{U}_{\beta(m,\ell)} \right\} \in \mathcal{O}_{X^{m+1}}.$$

So we can define

$$\sigma(\langle \mathscr{U}_j : j \leq n \rangle) = \pi_{m+1}\left[\sigma_m \left(\left\langle \mathscr{U}^{(m+1)}_{\beta(m,\ell)} : \ell \leq k \right\rangle \right) \right],$$

where π_{m+1} is the projection mapping. From here, it's straightforward to check that σ is winning.

Now we assume that X is strategically ω-Rothberger. So let σ_0 be a winning strategy for P2 in the ω-Rothberger game. We define a strategy σ for P2 in the Rothberger game as follows.

By Lemma 4.4, we let

$$F_0 \in [X]^{<\omega}$$

be such that

$$\mathcal{N}_{F_0} \subseteq \{\sigma_0(\mathscr{W}) : \mathscr{W} \in \Omega_X\}.$$

Let $M_0 = -1 + \#F_0$ and enumerate it as $F_0 = \{x_0, \ldots, x_{M_0}\}$. Now suppose $\langle \mathscr{U}_j : j \leq M_0 \rangle$ is a given sequence of open covers of X. For each $k \leq M_0$, choose $\sigma(\langle \mathscr{U}_j : j \leq k \rangle) \in \mathscr{U}_k$ to be such that

$$x_k \in \sigma(\langle \mathscr{U}_j : j \leq k \rangle).$$

Then, we can choose $\mathscr{W}_0 \in \Omega_X$ to be such that

$$\sigma_0 (\mathscr{W}_0) = \bigcup_{k=0}^{M_0} \sigma(\langle \mathscr{U}_j : j \leq k \rangle).$$

Now let $n \in \omega$ be given and suppose we have $\langle F_j : j \leq n \rangle$, $\langle M_j : j \leq n \rangle$, $\langle \mathscr{W}_j : j \leq n \rangle$, and $\langle \mathscr{U}_j : j \leq M_n \rangle$ defined. As above, we can let $F_{n+1} \in [X]^{<\omega}$ be such that

$$\mathcal{N}_{F_{n+1}} \subseteq \{\sigma_0 (\mathscr{W}_0, \ldots, \mathscr{W}_n, \mathscr{W}) : \mathscr{W} \in \Omega_X\}.$$

Let $M_{n+1} = M_n + \#F_{n+1}$ and enumerate F_{n+1} as $\{x_{M_n+1}, \ldots, x_{M_{n+1}}\}$. Then, given a sequence

$$\langle \mathscr{U}_j : M_n < j \leq M_{n+1} \rangle$$

of open covers of X, we define, for $M_n < k \leq M_{n+1}$,

$$\sigma(\langle \mathscr{U}_j : j \leq k \rangle) \in \mathscr{U}_k$$

to be such that

$$x_k \in \sigma(\langle \mathscr{U}_j : j \leq k \rangle).$$

Then, we set $\mathscr{W}_{n+1} \in \Omega_X$ to be such that

$$\sigma_0(\mathscr{W}_0, \ldots, \mathscr{W}_{n+1}) = \bigcup_{k=M_n+1}^{M_{n+1}} \sigma(\langle \mathscr{U}_j : j \leq k \rangle).$$

This defines σ.

To see that σ is winning, consider a play $\langle \mathscr{U}_n : n \in \omega \rangle$ of the game according to σ and let $x \in X$ be arbitrary. Since σ_0 is winning in the ω-Rothberger game, there is some $n \in \omega$ such that

$$\{x\} \subseteq \sigma_0(\mathscr{W}_0, \ldots, \mathscr{W}_n) = \bigcup_{k=M_n+1}^{M_{n+1}} \sigma(\langle \mathscr{U}_j : j \leq k \rangle).$$

Hence, there is some $M_n < k \leq M_{n+1}$ such that

$$x \in \sigma(\mathscr{U}_0, \ldots, \mathscr{U}_k).$$

That is, σ is winning. □

To characterize the Markov properties of interest in this paper, we will need some other notions.

Definition 4.6. A space X is said to be *topologically countable* if there exists $\{x_n : n \in \omega\} \subseteq X$ such that $X = \bigcup_{n\in\omega} \bigcap \mathcal{N}_{x_n}$.

Lemma 4.7. *If X is a T_1 space, then, for $A \subseteq X$, $A = \bigcap \mathcal{N}_A$. Consequently, any T_1 space that is topologically countable is countable.*

Proof. Clearly $A \subseteq \bigcap \mathcal{N}_A$. On the other hand, for $x \notin A$, $X \setminus \{x\} \in \mathcal{N}_A$ since X is T_1. Hence, $x \notin \bigcap \mathcal{N}_A$. □

Example 4.8 (S42[1])**.** The right-ordered reals X is an example of an uncountable T_0 space which is topologically countable. Indeed, since the basis for X consists of intervals (a, ∞) for $a \in \mathbb{R}$, the set of integers witnesses topological countability.

Definition 4.9. For a subset A of a space X, A is said to be *relatively compact* (in X) if every open cover of X admits a finite subset which covers A. A space X is said to be *σ-relatively compact* if there exists a countable collection $\{A_n : n \in \omega\}$ of relatively compact subsets of X such that $X = \bigcup_{n\in\omega} A_n$.

The usual Tube Lemma idea applies to show that the property of relative compactness is also productive.

Lemma 4.10. *If A and B are relatively compact subsets of X and Y, respectively, then $A \times B$ is relatively compact in $X \times Y$.*

Proof. Without loss of generality, we consider only basic covers. So let $\mathscr{W}$ be an open cover of $X \times Y$ consisting of rectangles. Note that, for any $x \in X$, $\{x\} \times B$ is relatively compact in $\{x\} \times Y$ viewed as a subspace of $X \times Y$. So, for each $x \in X$, we can let $\mathscr{F}_x \in [\mathscr{W}]^{<\omega}$ be such that $\{x\} \times B \subseteq \bigcup \mathscr{F}_x$. Then let $U_x = \bigcap\{\pi_X[W] : W \in \mathscr{F}_x\}$. Now, $\{U_x : x \in X\}$ is an open cover of X so we can find $F \in [X]^{<\omega}$ such that $A \subseteq \bigcup\{U_x : x \in F\}$. Define $\mathscr{W}_0 = \bigcup_{x\in F} \mathscr{F}_x$.

To finish the proof, we need only show that $A \times B \subseteq \bigcup \mathscr{W}_0$. So let $\langle x, y\rangle \in A \times B$ be arbitrary. There is some $a \in F$ so that $x \in U_a$. Since $y \in B$, there is some $W \in \mathscr{F}_a$ so that $\langle a, y\rangle \in W$. Note then that $x \in U_a \subseteq \pi_X[W]$ which establishes that $\langle x, y\rangle \in W$. □

Lemma 4.11. *The closure of any relatively compact subset in a regular space is compact. (See* [1] *and also* [7, Prop. 4.4]*.)*

An immediate consequence is

Corollary 4.12. *Every regular σ-relatively compact space is σ-compact.*

Example 4.13 (S59)**.** The indiscrete irrational extension of the reals is σ-relatively compact but not σ-compact (see [7, Ex. 5.8]).

With eyes toward generality, we define the following types of cofinality.

Definition 4.14. Let X be a set and suppose $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of X. We say that $\mathrm{cof}_X(\mathcal{A}, \mathcal{B}) \leq \omega$ if there exists $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ such that, for every $B \in \mathcal{B}$, there exists $n \in \omega$ such that $B \subseteq A_n$.

In words, the condition $\mathrm{cof}_X(\mathcal{A}, \mathcal{B}) \leq \omega$ is asserting that $\mathcal{A}$ is of cofinality type ω relative to $\mathcal{B}$.

We also use a slightly weaker version, inspired by the notion of being topologically countable.

[1]Throughout, we will refer to spaces by their ID in the π-base [32].

Definition 4.15. Let X be a space and suppose $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of X. We say that $\widehat{\operatorname{cof}}_X(\mathcal{A},\mathcal{B}) \leq \omega$ if there exists $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ such that, for every $B \in \mathcal{B}$, there exists $n \in \omega$ such that $B \subseteq \bigcap \mathcal{N}_{A_n}$.

Note that, if we identify X with the set of singletons and let $\mathcal{A}$ be the collection of relatively compact subsets of X, then X is

- σ-relatively compact if and only if $\operatorname{cof}_X(\mathcal{A}, X) \leq \omega$.
- topologically countable if and only if $\widehat{\operatorname{cof}}_X(X, X) \leq \omega$.

Lemma 4.16. *For any space X and a collection $\mathcal{A}$ of subsets of X, $\widehat{\operatorname{cof}}_X(\mathcal{A},\mathcal{A}) \leq \omega$ if and only if* $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X(\mathcal{A}))$.

Proof. Let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ witness that $\widehat{\operatorname{cof}}_X(\mathcal{A},\mathcal{A}) \leq \omega$. For each $n \in \omega$ and $\mathscr{U} \in \mathcal{O}_X(\mathcal{A})$, choose $\sigma(\mathscr{U}, n) \in \mathscr{U}$ such that $A_n \subseteq \sigma(\mathscr{U}, n)$. To see that σ is winning, let $\{\mathscr{U}_n : n \in \omega\} \subseteq \mathcal{O}_X(\mathcal{A})$ and $A \in \mathcal{A}$. Choose $n \in \omega$ to be such that $A \subseteq \bigcap \mathcal{N}_{A_n}$. Then

$$A \subseteq \bigcap \mathcal{N}_{A_n} \subseteq \sigma(\mathscr{U}_n, n).$$

Hence, $\{\sigma(\mathscr{U}_n, n) : n \in \omega\} \in \mathcal{O}_X(\mathcal{A})$.

Now suppose that $\widehat{\operatorname{cof}}_X(\mathcal{A},\mathcal{A}) \not\leq \omega$ and let σ be a Markov strategy for P2 in $\mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X(\mathcal{A}))$. By Lemma 4.4, we can choose, for each $n \in \omega$, $A_n \in \mathcal{A}$ such that

$$\mathcal{N}_{A_n} \subseteq \{\sigma(\mathscr{U}, n) : \mathscr{U} \in \mathcal{O}_X(\mathcal{A})\}.$$

Now, by the assumption, there is some $A \in \mathcal{A}$ such that $A \setminus \bigcap \mathcal{N}_{A_n} \neq \varnothing$ for each $n \in \omega$. Hence, for each $n \in \omega$, there is some $\mathscr{U}_n \in \mathcal{O}_X(\mathcal{A})$ such that $A \setminus \sigma(\mathscr{U}_n, n) \neq \varnothing$. Note then that

$$\{\sigma(\mathscr{U}_n, n) : n \in \omega\} \notin \mathcal{O}_X(\mathcal{A}).$$

That is, σ is not winning. □

Theorem 4.17. *Let X be a space.*

(1) The following are equivalent.
  - *(a) X is Markov ω-Menger.*
  - *(b) X is Markov Menger.*
  - *(c) X is σ-relatively compact.*

(2) The following are equivalent.
  - *(a) X is Markov ω-Rothberger.*
  - *(b) X is Markov Rothberger.*
  - *(c) X is topologically countable.*

Proof. We start by addressing (1). The fact that X is σ-relatively compact if and only if X is Markov Menger is [7, Cor. 4.7]. Also note that the X being Markov ω-Menger immediately implies that X is Markov Menger by considering the closure under finite unions of open covers of X. So, to finish this portion of the proof, we show that being σ-relatively compact guarantees that the space is Markov ω-Menger. So suppose X is σ-relatively compact. By

Lemma 4.10, X^{n+1} is σ-relatively compact for each $n \in \omega$. So let $\{A_{n,k} : k \in \omega\}$ witness the σ-relative compactness of X^{n+1}. Given $\mathscr{U} \in \Omega_X$ and $j \leq n$, let $\sigma_j(\mathscr{U}, n) \in [\mathscr{U}]^{<\omega}$ be such that

$$\bigcup_{\ell=0}^{n} A_{j,\ell} \subseteq \bigcup\{U^{j+1} : U \in \sigma_j(\mathscr{U}, n)\}.$$

Then, let

$$\sigma(\mathscr{U}, n) = \bigcup_{j=0}^{n} \sigma_j(\mathscr{U}, n).$$

Observe that $\sigma(\mathscr{U}, n) \in [\mathscr{U}]^{<\omega}$. This completes the definition of σ. To see that σ is winning, let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of ω-covers of X and consider $\{x_0, \ldots, x_n\} \subseteq X$. Note that $\langle x_0, \ldots, x_n \rangle \in X^{n+1}$. Then, there is some $k \in \omega$ such that

$$\langle x_0, \ldots, x_n \rangle \in A_{n,k}.$$

Let $N = \max\{n, k\}$ and notice that

$$\langle x_0, \ldots, x_n \rangle \in A_{n,k} \subseteq \bigcup_{j=0}^{N} A_{n,j} \subseteq \bigcup\{U^{n+1} : U \in \sigma_n(\mathscr{U}_N, N)\}.$$

Hence, there must be some $U \in \sigma_n(\mathscr{U}_N, N)$ such that $\langle x_0, \ldots, x_n \rangle \in U^{n+1}$. Observe that

$$\{x_0, \ldots, x_n\} \subseteq U \in \sigma(\mathscr{U}_N, N).$$

Now for (2). Note that the equivalence of being Markov Rothberger and topologically countable follows immediately from Lemma 4.16. The equivalence of being Markov ω-Rothberger and topologically countable also follows from Lemma 4.16 since the lemma asserts that being Markov ω-Rothberger is equivalent to the condition that $\widehat{\operatorname{cof}}_X([X]^{<\omega}, [X]^{<\omega}) \leq \omega$. We need only verify this last condition is equivalent to being topologically countable.

Suppose X is topologically countable and let $\{x_n : n \in \omega\}$ be the witnessing set. Note that $F_n := \{x_k : k \leq n\}$ constructs a countable set of finite subsets of X. Consider any other $F \in [X]^{<\omega}$. For each $y \in F$, let $n_y \in \omega$ be such that $y \in \bigcap \mathcal{N}_{x_{n_y}}$. Let $M = \max\{n_y : y \in F\}$ and observe that

$$F \subseteq \bigcap \mathcal{N}_{F_M}.$$

Hence, $\widehat{\operatorname{cof}}_X([X]^{<\omega}, [X]^{<\omega}) \leq \omega$.

Finally, assume that $\widehat{\operatorname{cof}}_X([X]^{<\omega}, [X]^{<\omega}) \leq \omega$ and let $\{F_n : n \in \omega\}$ be the witnessing set of finite subsets of X. Now, since $\bigcup_{n\in\omega} F_n$ is countable, we can set $\{x_n : n \in \omega\} = \bigcup_{n\in\omega} F_n$. To see that this set satisfies the definition of topological countability, let $x \in X$ and note that $\{x\}$ is a finite subset of X. Then there is some $n \in \omega$ such that $\{x\} \subseteq \bigcap \mathcal{N}_{F_n}$. To see that there must be some $y \in F_n$ such that $x \in \bigcap \mathcal{N}_y$, suppose $z \in X$ is such that, for each $y \in F_n$, $z \notin \bigcap \mathcal{N}_y$. For each $y \in F_n$, let $U_y \in \mathcal{N}_y$ be such that $z \notin U_y$. Note then that $U := \bigcup_{y\in F_n} U_y \in \mathcal{N}_{F_n}$ and that $z \notin U$. Hence, $z \notin \bigcap \mathcal{N}_{F_n}$. It follows that there must be some $y \in F_n$ such that $x \in \bigcap \mathcal{N}_y$. Thus, X is topologically countable. □

Corollary 4.18. *Let X be a space.*

(1) *If X is T_1, then the following are equivalent:*
  (a) *X is Markov ω-Rothberger.*
  (b) *X is Markov Rothberger.*
  (c) *X is countable.*

(2) *If X is regular, then the following are equivalent:*
  (a) *X is Markov ω-Menger.*
  (b) *X is Markov Menger.*
  (c) *X is σ-compact.*

Proof. This follows immediately from Theorem 4.17, Lemma 4.7, and Corollary 4.12. □

Corollary 4.19. *Let $\mathcal{P}$ be any of the properties Markov Menger, Markov ω-Menger, Markov Rothberger, or Markov ω-Rothberger. If X and Y are both $\mathcal{P}$, then $X \times Y$ is, too.*

Proof. By Theorem 4.17, we have only two cases to consider.

First, suppose both X and Y are topologically countable. Let $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ be the witnessing sets for topological countability for X and Y, respectively. Note that $\{(x_n, y_m) : n, m \in \omega\}$ is a countable subset of $X \times Y$. For any $(x, y) \in X \times Y$, we can let $j \in \omega$ and $k \in \omega$ be such that $x \in \bigcap \mathcal{N}_{x_j}$ and $y \in \bigcap \mathcal{N}_{y_k}$. Now, consider any open neighborhood W of (x_j, y_k). There are open sets U and V of X and Y, respectively, with $x_j \in U$, $y_k \in V$, and $U \times V \subseteq W$. Note then that $(x, y) \in U \times V \subseteq W$. Since W was arbitrary, $(x, y) \in \bigcap \mathcal{N}_{(x_j, y_k)}$.

Secondly, assume both X and Y are σ-relatively compact. Let $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$ be the witnessing families of relatively compact subsets of X and Y, respectively. Note that $\{A_n \times B_m : n, m \in \omega\}$ is a countable set of relatively compact subsets of $X \times Y$ by Lemma 4.10. Now, for any $(x, y) \in X \times Y$, we can let $j \in \omega$ and $k \in \omega$ be such that $x \in A_j$ and $y \in B_k$; hence, $(x, y) \in A_j \times B_k$. □

4.2. **Considering compact subsets.** Throughout the rest of the paper, we'll use $\mathsf{K}(X)$ and $\mathsf{K}_{\mathrm{rel}}(X)$ to denote the collections of non-empty compact and non-empty relatively compact subsets of X, respectively.

We start this section by addressing the preservation of various k-variant properties under finite powers. To this end, we provide the following, as an easy application of The Wallace Theorem, which will allow us to restrict our attention to basic covers when dealing with k-covers of product spaces.

Lemma 4.20. *If $\mathscr{W}$ is a k-cover of $X \times Y$, then there is a k-cover of $X \times Y$ consisting of rectangles $U \times V$ that refines $\mathscr{W}$.*

Proof. Suppose $\mathscr{W}$ is a k-cover of $X \times Y$. For each $K \in \mathsf{K}(X \times Y)$, let $W_K \in \mathscr{W}$ be such that $\pi_X[K] \times \pi_Y[K] \subseteq W_K$. By The Wallace Theorem [15, Thm. 3.2.10], there are open sets $U_K \subseteq X$ and $V_K \subseteq Y$ such that

$$\pi_X[K] \times \pi_Y[K] \subseteq U_K \times V_K \subseteq W_K.$$

Hence, $\{U_K \times V_K : K \in \mathsf{K}(X \times Y)\}$ is a k-cover of $X \times Y$ which consists of rectangles and refines $\mathscr{W}$. □

In the case of finite powers, we can refine k-covers of the finite product space with cubes.

Lemma 4.21 ([13, Lemma 3])**.** *If $\mathscr{W}$ is a k-cover of X^n for a positive integer n, then there is a k-cover $\mathscr{U}$ of X with the property that $\{U^n : U \in \mathscr{U}\}$ is a k-cover of X^n which refines $\mathscr{W}$.*

Theorem 4.22. *Every finite power of a k-Lindelöf space is k-Lindelöf.*

Proof. Let n be a positive integer and suppose $\mathscr{W}$ is a k-cover of X^n. By Lemma 4.21, we can let $\mathscr{U} \in \mathcal{K}_X$ be such that $\{U^n : U \in \mathscr{U}\}$ is a k-cover of X^n which refines $\mathscr{W}$. Since X is assumed to be k-Lindelöf, we can choose $\{U_k : k \in \omega\} \subseteq \mathscr{U}$ such that $\{U_k : k \in \omega\} \in \mathcal{K}_X$. For each $k \in \omega$, we can choose $W_k \in \mathscr{W}$ such that $U_k^n \subseteq W_k$. To finish the proof, we verify that $\{W_k : k \in \omega\}$ is a k-cover of X^n. So let $K \subseteq X^n$ be compact and note that $L = \bigcup_{j=1}^n \pi_j[K]$ is a compact subset of X. Then there must be $k \in \omega$ such that $L \subseteq U_k$. It follows that $K \subseteq L^n \subseteq U_k^n \subseteq W_k$. □

The proof of Theorem 4.22 presented here is identical in spirit to those of the following results from [13].

Theorem 4.23. *Let X be a space.*

(1) If X is k-Menger, then every finite power of X is, too ([13, Thm. 6]*).*
(2) If X is k-Rothberger, then every finite power of X is, too ([13, Thm. 5]*).*

Before we prove the analogous results to Theorem 4.3 for k-covers, we establish a combinatorial lemma that will be useful. The generality of Lemma 4.24 is such that it may be adapted to Lemma 4.26 for use in the finite-selection context.

Throughout, we'll use $\frown$ for the ordered concatenation of words. If s is a word and x is an element of the alphabet, we'll use $s^\frown x$ in place of $s^\frown\langle x\rangle$, when there is little doubt of ambiguity.

Lemma 4.24. *Suppose $s : \omega \to \omega^{<\omega}$ is an injection where $s_0 = \langle\rangle$. Suppose further that we have a function $r : \{s_n : n \in \omega\} \to [\omega]^{<\omega}$ such that $r(s_n) \subseteq \mathrm{range}(s_n)$ and, for each $m \geq 1$, $\{n \in \omega : r(s_n) \subseteq m\}$ is infinite. Then there is a bijection $\beta : \omega^2 \to \omega$ such that, for each $n \in \omega$, $\langle \beta(n,k) : k \in \omega\rangle$ is strictly increasing and $r(s_n) \subseteq \beta(n,0)$.*

Proof. For $n \in \mathbb{Z}^+$, let p_n be the n^{th} prime and set $\beta^*(n,k) = p_n^k$. Then let $\langle \beta^*(0,k) : k \in \omega \rangle$ be the increasing enumeration of

$$\omega \setminus \{p^n : p \text{ is prime and } n \in \mathbb{Z}^+\}.$$

Note that, for each $n \in \omega$, $\langle \beta^*(n,k) : k \in \omega \rangle$ is strictly increasing. Moreover, the sequence of initial points, $\langle \beta^*(n,0) : n \in \omega \rangle$, is also strictly increasing.

Define $\gamma : \omega \to \omega$ recursively as follows. Suppose, for $n \in \omega$, that $\langle \gamma_k : k < n \rangle$ is defined. Let

$$\gamma_n = \min\{\lambda \in \omega \setminus \{\gamma_k : k < n\} : r(s_n) \subseteq \beta^*(\lambda, 0)\}.$$

The claim is that $n \mapsto \gamma_n$, $\omega \to \omega$, is a bijection. γ is clearly injective. To show surjectivity, first note that $\gamma_0 = 0$. We proceed by induction. Let $m \geq 1$ and suppose that $\{k \in \omega : k < m\} \subseteq \text{range}(\gamma)$. Then let

$$M = \min\{\lambda \in \omega : \{k \in \omega : k < m\} \subseteq \{\gamma_j : j < \lambda\}\}.$$

If $m \in \{\gamma_j : j < M\}$, we have nothing to show. Otherwise, we can consider

$$A = \{\lambda \in \omega : \lambda \geq M \wedge r(s_\lambda) \subseteq \beta^*(m, 0)\}.$$

Since $m \geq 1$, $\beta^*(m,0) > 0$. Hence, $A \neq \varnothing$ since $\{\lambda \in \omega : r(s_\lambda) \subseteq \beta^*(m,0)\}$ is infinite. Then we can let $\ell = \min A$.

If $m \in \{\gamma_j : j < \ell\}$, we are done. Otherwise, we claim that $\gamma_\ell = m$. By definition,

$$\gamma_\ell = \min\{\lambda \in \omega \setminus \{\gamma_j : j < \ell\} : r(s_\ell) \subseteq \beta^*(\lambda, 0)\}.$$

Since $\ell \in A$, we know that $r(s_\ell) \subseteq \beta^*(m, 0)$. Moreover, since $m \notin \{\gamma_j : j < \ell\}$, we see that

$$m \in \{\lambda \in \omega \setminus \{\gamma_j : j < \ell\} : r(s_\ell) \subseteq \beta^*(\lambda, 0)\}.$$

So $\gamma_\ell \leq m$.

Now, for $k < m$, notice that there exists $j < M$ so that $\gamma_j = k$ by the inductive hypothesis. Since γ is injective, $\gamma_\ell \neq k$. Since this is true for any $k < m$, we have that $\gamma_\ell = m$.

Finally, define $\beta : \omega^2 \to \omega$ by the rule $\beta(n,k) = \beta^*(\gamma_n, k)$. This completes the proof. ☐

The proof of the finite productivity of the strategically k-Rothberger property can be seen as an exercise toward the proof of the finite-selection version. The idea of the proof is identical in spirit to the analogous result in [14]; the reason it is not a direct application of the results of that paper is that $\mathbb{K}(X \times Y)$ and $\mathbb{K}(X) \times \mathbb{K}(Y)$ are, in general, topologically distinct objects. Recall that $\mathbb{K}(X)$ denotes the set $\mathsf{K}(X)$ endowed with the Vietoris topology. See [28] for more on this topological space.

Theorem 4.25. *If X and Y are strategically k-Rothberger, then $X \times Y$ is strategically k-Rothberger. In other words, the property of being strategically k-Rothberger is finitely productive.*

Proof. Let $\langle s_n : n \in \omega \rangle$ be an enumeration of $\omega^{<\omega}$ where $s_0 = \langle\rangle$. By Lemma 4.24, we can let $\beta : \omega^2 \to \omega$ be a bijection such that $\mathrm{range}(s_n) \subseteq \beta(n, 0)$ and $\langle \beta(n, k) : k \in \omega \rangle$ is strictly increasing for each $n \in \omega$. Let σ_X and σ_Y be winning strategies for P2 in the k-Rothberger game in X and Y, respectively. Without loss of generality, we consider only basic k-covers of $X \times Y$ by Lemma 4.20. In particular, let

$$\mathscr{T}_X \otimes \mathscr{T}_Y = \{U \times V : \langle U, V \rangle \in \mathscr{T}_X \times \mathscr{T}_Y\}$$

and

$$\mathbb{BK} = \{\mathscr{W} \subseteq \mathscr{T}_X \otimes \mathscr{T}_Y : \mathscr{W} \in \mathcal{K}_{X\times Y}\}.$$

We define a winning strategy σ for P2 in the k-Rothberger game on $X \times Y$ recursively as follows. Let $n \in \omega$ be given and suppose we have $\langle \mathscr{W}_p : p < n \rangle$, $\langle \mathscr{W}'_p : p < n \rangle$, $\langle \mathscr{V}_p : p < n \rangle$, $\langle W_p : p < n \rangle$, $\langle \mathscr{U}_p : p < n \rangle$, and $\langle K_p : p < n \rangle$ defined. Also set $\langle j, k \rangle \in \omega^2$ to be so that $\beta(j, k) = n$. Note that

$$\mathrm{range}(s_j) \subseteq \beta(j, 0) \subseteq \beta(j, k) = n.$$

Given $\mathscr{W}_n \in \mathbb{BK}$, define $\mathscr{W}'_n : \mathsf{K}(X) \to \wp(\mathscr{W}_n)$ by $\mathscr{W}'_n(K) = \{W \in \mathscr{W}_n : K \subseteq \pi_X[W]\}$. Then, define $\mathscr{V}_n : \mathsf{K}(X) \to \mathcal{K}_Y$ by $\mathscr{V}_n(K) = \pi_Y[\mathscr{W}'_n(K)] = \{\pi_Y[W] : W \in \mathscr{W}'_n(K)\}$. Let $W_n(K) \in \mathscr{W}'_n(K)$ be such that

$$\pi_Y[W_n(K)] = \sigma_Y\left(\left\langle \mathscr{V}_{s_j(p)}(K_{s_j(p)}) : p \in \mathrm{dom}(s_j)\right\rangle^\frown \mathscr{V}_n(K)\right).$$

Note that $\mathscr{U}_n := \{\pi_X[W_n(K)] : K \in \mathsf{K}(X)\} \in \mathcal{K}_X$ so we can let $K_n \in \mathsf{K}(X)$ be such that

$$\pi_X[W_n(K_n)] = \sigma_X\left(\left\langle \mathscr{U}_{\beta(j,p)} : p \leq k\right\rangle\right).$$

We define

$$\sigma(\langle \mathscr{W}_p : p \leq n \rangle) = W_n(K_n).$$

The final thing to show is that σ is winning. So let $A \in \mathsf{K}(X \times Y)$ be arbitrary and then let $K = \pi_X[A]$ and $L = \pi_Y[A]$. For $n \in \omega$, suppose we have $\langle \ell_p : p < n \rangle$, $\langle M_p : p < n \rangle$, and $\langle m_p : p < n \rangle$ defined. Let $m_n \in \omega$ be so that $s_{m_n} = \langle \ell_p : p < n \rangle$. Observe that

$$\left\langle \sigma_X\left(\left\langle \mathscr{U}_{\beta(m_n,p)} : p \leq N\right\rangle\right) : N \in \omega \right\rangle$$

corresponds to a play of the k-Rothberger game on X according to σ_X. Since σ_X is winning, there is some $M_n \in \omega$ so that

$$K \subseteq \sigma_X\left(\left\langle \mathscr{U}_{\beta(m_n,p)} : p \leq M_n\right\rangle\right).$$

Let $\ell_n = \beta(m_n, M_n)$.

This defines sequences $\langle \ell_n : n \in \omega \rangle$, $\langle m_n : n \in \omega \rangle$, and $\langle M_n : n \in \omega \rangle$. Observe that

$$\left\langle \sigma_Y\left(\left\langle \mathscr{V}_{\ell_p}(K_{\ell_p}) : p \leq N\right\rangle\right) : N \in \omega \right\rangle$$

corresponds to a run of the k-Rothberger game on Y corresponding to σ_Y. Since σ_Y is winning, there is some $w \in \omega$ so that

$$L \subseteq \sigma_Y\left(\left\langle \mathscr{V}_{\ell_p}(K_{\ell_p}) : p \leq w\right\rangle\right).$$

Behold that, since $s_{m_w} = \langle \ell_p : p < w\rangle$,

$$\begin{aligned} L &\subseteq \sigma_Y\left(\langle \mathscr{V}_{\ell_p}(K_{\ell_p}) : p \leq w\rangle\right) \\ &= \sigma_Y\left(\langle \mathscr{V}_{\ell_p}(K_{\ell_p}) : p < w\rangle ^\frown \mathscr{V}_{\ell_w}(K_{\ell_w})\right) \\ &= \sigma_Y\left(\langle \mathscr{V}_{s_{m_w}(p)}(K_{s_{m_w}(p)}) : p \in \operatorname{dom}(s_{m_w})\rangle ^\frown \mathscr{V}_{\beta(m_w,M_w)}(K_{\beta(m_w,M_w)})\right) \\ &= \pi_Y\left[W_{\beta(m_w,M_w)}(K_{\beta(m_w,M_w)})\right]. \end{aligned}$$

By construction,

$$K \subseteq \sigma_X\left(\langle \mathscr{U}_{\beta(m_w,p)} : p \leq M_w\rangle\right) = \pi_X\left[W_{\beta(m_w,M_w)}(K_{\beta(m_w,M_w)})\right].$$

Therefore,

$$A \subseteq K \times L \subseteq W_{\beta(m_w,M_w)}(K_{\beta(m_w,M_w)}) = \sigma\left(\langle \mathscr{W}_p : p \leq \beta(m_w, M_w)\rangle\right),$$

finishing the proof. □

Lemma 4.26. *Suppose we have an enumeration $\langle s_n : n \in \omega\rangle$ of $\bigcup_{n\in\omega} \omega^n \times \omega^n$ where $s_0 = \langle\langle\rangle, \langle\rangle\rangle$. For each $n \in \omega$, let $s_n^-, s_n^+ \in \omega^{\operatorname{len}(s_n)/2}$ be such that $s_n = s_n^- {}^\frown s_n^+$. Then there is a bijection $\beta : \omega^2 \to \omega$ such that, for each $n \in \omega$, $\langle \beta(n,k) : k \in \omega\rangle$ is strictly increasing and* $\operatorname{range}(s_n^-) \subseteq \beta(n, 0)$.

Proof. Note that $n \mapsto s_n$, $\omega \to \omega^{<\omega}$, is an injection. Then let $r : \{s_n : n \in \omega\} \to [\omega]^{<\omega}$ be defined by $r(s_n) = \operatorname{range}(s_n^-)$. For $m \geq 1$, note that $\{n \in \omega : r(s_n) \subseteq m\}$ is infinite since there are arbitrarily long sequences of 0s. Hence, Lemma 4.24 applies. □

Theorem 4.27. *If X and Y are strategically k-Menger, then $X \times Y$ is strategically k-Menger. In other words, the property of being strategically k-Menger is finitely productive.*

Proof. Let σ_X and σ_Y be winning strategies for P2 in the k-Menger game in X and Y, respectively. Without loss of generality, we consider only basic k-covers of $X \times Y$ by Lemma 4.20. So let $\mathbb{BK}$ be defined as in the proof of Theorem 4.25.

We will recursively define a strategy σ for P2 in the k-Menger game on $X \times Y$. First, fix a choice function $\vec{\cdot} : [\mathsf{K}(X)]^{<\omega} \to \mathsf{K}(X)^\omega$ to be such that $\mathbf{K} = \operatorname{range}\left(\vec{\mathbf{K}}\right)$. Also, let $\langle s_n : n \in \omega\rangle$ and $\beta : \omega^2 \to \omega$ be as in Lemma 4.26.

Now, let $n \in \omega$ be given and suppose we have $\langle \mathscr{W}_p : p < n\rangle$, $\langle \mathscr{W}_p' : p < n\rangle$, $\langle \mathscr{V}_p : p < n\rangle$, $\langle \mathscr{G}_p : p < n\rangle$, $\langle \mathscr{U}_p : p < n\rangle$, and $\langle \mathbf{F}_p : p < n\rangle$ defined. Also set $\langle j, k\rangle \in \omega^2$ such that $\beta(j, k) = n$. Note that

$$\operatorname{range}(s_j^-) \subseteq \beta(j, 0) \subseteq \beta(j, k) = n.$$

Given $\mathscr{W}_n \in \mathbb{BK}$, define $\mathscr{W}_n' : \mathsf{K}(X) \to \wp(\mathscr{W}_n)$ by $\mathscr{W}_n'(K) = \{W \in \mathscr{W}_n : K \subseteq \pi_X[W]\}$. Then, define $\mathscr{V}_n : \mathsf{K}(X) \to \mathcal{K}_Y$ by $\mathscr{V}_n(K) = \pi_Y[\mathscr{W}_n'(K)]$.

We let $\mathscr{G}_n(K) \in [\mathscr{W}_n'(K)]^{<\omega}$ be so that

$$\pi_Y[\mathscr{G}_n(K)] = \sigma_Y\left(\left\langle \mathscr{V}_{s_j^-(p)}\left(\vec{\mathbf{F}}_{s_j^-(p)}\left(s_j^+(p)\right)\right) : p \in \operatorname{dom}(s_j^-)\right\rangle ^\frown \mathscr{V}_n(K)\right).$$

Then define

$$\mathscr{U}_n = \left\{ \bigcap \pi_X[\mathscr{G}_n(K)] : K \in \mathsf{K}(X) \right\} \in \mathcal{K}_X$$

and let $\mathbf{F}_n \in [\mathsf{K}(X)]^{<\omega}$ to be such that

$$\sigma_X \left(\left\langle \mathscr{U}_{\beta(j,p)} : p \leq k \right\rangle \right) = \left\{ \bigcap \pi_X[\mathscr{G}_n(K)] : K \in \mathbf{F}_n \right\}.$$

Finally, we define

$$\sigma \left(\langle \mathscr{W}_p : p \leq n \rangle \right) = \bigcup_{K \in \mathbf{F}_n} \mathscr{G}_n(K).$$

To finish the proof, we need to show that σ is a winning strategy. So let $A \in \mathsf{K}(X \times Y)$ be arbitrary and let $K = \pi_X[A]$ and $L = \pi_Y[A]$. For $n \in \omega$, suppose we have $\langle m_p : p < n \rangle$, $\langle M_p : p < n \rangle$, $\vec{\ell} = \langle \ell_p : p < n \rangle$, $\vec{u} = \langle u_p : p < n \rangle$, and $\langle K_p : p < n \rangle$ defined. Then we can let $m_n \in \omega$ be so that $s_{m_n} = \langle \vec{\ell}, \vec{u} \rangle$. Observe that

$$\left\langle \sigma_X \left(\left\langle \mathscr{U}_{\beta(m_n,p)} : p \leq N \right\rangle \right) : N \in \omega \right\rangle$$

corresponds to a play of the k-Menger game on X according to σ_X. Since σ_X is winning, there is some $M_n \in \omega$ and $U \in \sigma_X \left(\left\langle \mathscr{U}_{\beta(m_n,p)} : p \leq M_n \right\rangle \right)$ such that $K \subseteq U$. It follows that there is some $u_n \in \omega$ such that

$$K \subseteq \bigcap \pi_X \left[\mathscr{G}_{\beta(m_n,M_n)} \left(\vec{\mathbf{F}}_{\beta(m_n,M_n)} (u_n) \right) \right].$$

Let $\ell_n = \beta(m_n, M_n)$ and $K_n = \vec{\mathbf{F}}_{\beta(m_n,M_n)} (u_n)$. Note that $K_n \in \mathbf{F}_{\beta(m_n,M_n)}$.

This defines sequences $\langle m_n : n \in \omega \rangle$, $\langle M_n : n \in \omega \rangle$, $\langle \ell_n : n \in \omega \rangle$, $\langle u_n : n \in \omega \rangle$, and $\langle K_n : n \in \omega \rangle$. Note that

$$\left\langle \sigma_Y \left(\left\langle \mathscr{V}_{\ell_p} (K_p) : p \leq N \right\rangle \right) : N \in \omega \right\rangle$$

corresponds to a play of the k-Menger game on Y according to σ_Y. Since σ_Y is winning, there is some $w \in \omega$ and $V \in \sigma_Y \left(\left\langle \mathscr{V}_{\ell_p} (K_p) : p \leq w \right\rangle \right)$ such that $L \subseteq V$.

Note that $s_{m_w} = \langle \langle \ell_p : p < w \rangle, \langle u_p : p < w \rangle \rangle$. Then

$$\begin{aligned}
&\pi_Y \left[\mathscr{G}_{\ell_w} (K_w) \right] \\
&\quad = \pi_Y \left[\mathscr{G}_{\beta(m_w,M_w)} \left(\vec{\mathbf{F}}_{\beta(m_w,M_w)}(u_w) \right) \right] \\
&\quad = \sigma_Y \left(\left\langle \mathscr{V}_{s^-_{m_w}(p)} \left(\vec{\mathbf{F}}_{s^-_{m_w}(p)} \left(s^+_{m_w}(p) \right) \right) : p \in \mathrm{dom}(s^-_{m_w}) \right\rangle ^\frown \mathscr{V}_{\beta(m_w,M_w)}(K_w) \right) \\
&\quad = \sigma_Y \left(\left\langle \mathscr{V}_{\ell_p} \left(\vec{\mathbf{F}}_{\ell_p} (u_p) \right) : p < w \right\rangle ^\frown \mathscr{V}_{\beta(m_w,M_w)} \left(\vec{\mathbf{F}}_{\beta(m_w,M_w)} (u_w) \right) \right) \\
&\quad = \sigma_Y \left(\left\langle \mathscr{V}_{\ell_p} \left(\vec{\mathbf{F}}_{\ell_p} (u_p) \right) : p \leq w \right\rangle \right).
\end{aligned}$$

Hence, there is some $V \in \pi_Y \left[\mathscr{G}_{\ell_w} (K_w) \right]$ such that $L \subseteq V$. Thus, there is some

$$W \in \mathscr{G}_{\ell_w} (K_w)$$

such that $L \subseteq \pi_Y[W]$. Observe that, since $K_w \in \mathbf{F}_{\ell_w}$,

$$W \in \bigcup_{u \in \mathbf{F}_{\ell_w}} \mathscr{G}_{\ell_w}(u) = \sigma \left(\langle \mathscr{W}_p : p \leq \ell_w \rangle \right).$$

Finally, note that

$$K \subseteq \bigcap \pi_X \left[\mathscr{G}_{\ell_w}(K_w)\right] \subseteq \pi_X[W].$$

Therefore, $A \subseteq K \times L \subseteq W$. □

When it comes to k-covers, hemicompactness plays an important role.

Proposition 4.28 ([6, Prop. 5])**.** *For any T_1 first-countable space X, the following are equivalent:*

(a) X is hemicompact.
(b) $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{K}, \mathcal{K})$.
(c) $X \models \mathsf{S}_1(\mathcal{K}, \mathcal{K})$.

The reader would observe that the T_1 assumption is expressly used in the proof of [6, Prop. 5] though it is not explicitly mentioned in the hypotheses.

Even if we relax first-countability, we still get a related characterization of hemicompactness.

Corollary 4.29. *For any space X, the following are equivalent:*

(a) X is hemicompact.
(b) $\widehat{\mathrm{cof}}_X(\mathsf{K}(X), \mathsf{K}(X)) \leq \omega$.
(c) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.

Proof. The equivalence of (b) and (c) follows from Lemma 4.16. The implication (a) $\implies$ (b) is evident, so we show that (b) $\implies$ (a). Suppose $\widehat{\mathrm{cof}}_X(\mathsf{K}(X), \mathsf{K}(X)) \leq \omega$ and let $\langle K_n : n \in \omega \rangle$ be the witnessing family of compact sets.

We finish the proof by showing that $\bigcap \mathcal{N}_{K_n}$ is compact, following an argument of [21, Lemma 31]. Let $\mathscr{U}$ be a cover of $\bigcap \mathcal{N}_{K_n}$ by sets open in X and note that $K_n \subseteq \bigcap \mathcal{N}_{K_n} \subseteq \bigcup \mathscr{U}$. Then there must be $\mathscr{F} \in [\mathscr{U}]^{<\omega}$ with $K_n \subseteq \bigcup \mathscr{F}$ by compactness of K_n. Then $\bigcup \mathscr{F} \in \mathcal{N}_{K_n}$ which implies that $\bigcap \mathcal{N}_{K_n} \subseteq \bigcup \mathscr{F}$. □

To prove Theorem 4.30(2), we can recycle, as we did for Theorem 4.27, the idea behind the proof of Theorem 4.3(1) of Dias and Scheepers [14], in which they thank L. Aurichi. In the Markov case, as one would expect, the combinatorial obstacles are significantly reduced.

We would like to compare the current proofs of parts (1) and (2) in Theorem 4.30 in light of the similarity between the proofs of Theorems 4.25 and 4.27. As can be seen, having the covering characterization of Corollary 4.29 significantly simplifies matters.

Theorem 4.30. *Let X and Y be spaces.*

(1) If X and Y are both Markov k-Rothberger, then so is $X \times Y$. In other words, the property of being Markov k-Rothberger is finitely productive.
(2) If X and Y are both Markov k-Menger, then so is $X \times Y$. In other words, the property of being Markov k-Menger is finitely productive.

Proof. Part (1) follows from Corollary 4.29. So we prove (2). Let σ_X and σ_Y be Markov strategies for P2 in the k-Menger games on X and Y, respectively. Let $\beta : \omega^2 \to \omega$ be a bijection, $\mathbb{BK}$ be as it was defined in Theorem 4.25, and $n = \beta(j,k)$. For each $\mathscr{W} \in \mathbb{BK}$ and $K \in \mathsf{K}(X)$, define

$$\mathscr{W}|_K = \{W \in \mathscr{W} : K \subseteq \pi_X[W]\}.$$

Fix a choice function $\gamma : \mathbb{BK} \times \mathsf{K}(X) \times \omega \to [\mathscr{T}_{X\times Y}]^{<\omega}$ to be such that $\gamma(\mathscr{W}, K, k) \in [\mathscr{W}|_K]^{<\omega}$ for each $(\mathscr{W}, K) \in \mathbb{BK} \times \mathsf{K}(X)$ and

$$\pi_Y[\gamma(\mathscr{W}, K, k)] = \sigma_Y\left(\pi_Y\left[\mathscr{W}|_K\right], k\right).$$

Note that

$$\mathscr{W}^{X,k} := \left\{\bigcap \pi_X[\gamma(\mathscr{W}, K, k)] : K \in \mathsf{K}(X)\right\} \in \mathcal{K}_X.$$

Let $\mathscr{F}(\mathscr{W}, j, k) \in [\mathsf{K}(X)]^{<\omega}$ be such that

$$\sigma_X(\mathscr{W}^{X,k}, j) = \left\{\bigcap \pi_X[\gamma(\mathscr{W}, K, k)] : K \in \mathscr{F}(\mathscr{W}, j, k)\right\}.$$

Then define

$$\sigma(\mathscr{W}, n) = \sigma(\mathscr{W}, \beta(j,k)) = \bigcup\{\gamma(\mathscr{W}, K, k) : K \in \mathscr{F}(\mathscr{W}, j, k)\} \in [\mathscr{W}]^{<\omega}.$$

Note that σ is a Markov strategy for P2 in the k-Menger game on $X \times Y$. We will show that it is winning.

Let $\langle \mathscr{W}_n : n \in \omega\rangle$ be a sequence of $\mathbb{BK}$ and let $E \subseteq X \times Y$ be compact. For every $k \in \omega$,

$$\bigcup\left\{\sigma_X\left(\mathscr{W}^{X,k}_{\beta(j,k)}, j\right) : j \in \omega\right\} \in \mathcal{K}_X.$$

So we can choose $j_k \in \omega$ and $K_{j_k,k} \in \mathscr{F}(\mathscr{W}_{\beta(j_k,k)}, j_k, k)$ for every $k \in \omega$ such that

$$\pi_X[E] \subseteq \bigcap \pi_X\left[\gamma\left(\mathscr{W}_{\beta(j_k,k)}, K_{j_k,k}, k\right)\right].$$

Now,

$$\bigcup\left\{\sigma_Y\left(\pi_Y\left[\mathscr{W}_{\beta(j_k,k)}|_{K_{j_k,k}}\right], k\right) : k \in \omega\right\} \in \mathcal{K}_Y,$$

so we can fix $m \in \omega$ and $W \in \gamma\left(\mathscr{W}_{\beta(j_m,m)}, K_{j_m,m}, m\right)$ such that

$$\pi_Y[E] \subseteq \pi_Y[W] \in \sigma_Y\left(\pi_Y\left[\mathscr{W}_{\beta(j_m,m)}|_{K_{j_m,m}}\right], m\right).$$

Note also that, since $W \in \gamma\left(\mathscr{W}_{\beta(j_m,m)}, K_{j_m,m}, m\right)$,

$$\pi_X[E] \subseteq \bigcap \pi_X\left[\gamma\left(\mathscr{W}_{\beta(j_m,m)}, K_{j_m,m}, m\right)\right] \subseteq \pi_X[W].$$

Hence,

$$E \subseteq \pi_X[E] \times \pi_Y[E] \subseteq \pi_X[W] \times \pi_Y[W] = W.$$

Finally, by construction $W \in \sigma(\mathscr{W}_{\beta(j_m,m)}, \beta(j_m,m))$, so σ is winning. □

Inspired by the property equivalent to being Markov ω-Menger, the property of being σ-relatively compact, we introduce modifications to hemicompactness.

Definition 4.31. A space X is *relatively hemicompact* if

$$\mathrm{cof}(\mathsf{K}_{\mathrm{rel}}(X), \mathsf{K}_{\mathrm{rel}}(X)) \leq \omega;$$

in other words, if there exists a countable set $\{A_n : n \in \omega\}$ of sets that are relatively compact in X such that, for every relatively compact E in X, there is some $n \in \omega$ such that $E \subseteq A_n$.

Definition 4.32. A space X is *weakly relatively hemicompact* if

$$\mathrm{cof}(\mathsf{K}_{\mathrm{rel}}(X), \mathsf{K}(X)) \leq \omega;$$

in other words, if there exists a countable set $\{A_n : n \in \omega\}$ of relatively compact subsets of X such that, for each compact $K \subseteq X$, there exists $n \in \omega$ with $K \subseteq A_n$.

We use the adjective "weakly" here since every compact set is relatively compact. Hence, every relatively hemicompact space is weakly relatively hemicompact, but the converse may not obtain.

Remark 4.33. In general, every hemicompact space is weakly relatively hemicompact; every relatively hemicompact space is weakly relatively hemicompact; and every weakly relatively hemicompact space is σ-relatively compact.

Lemma 4.34. *In the realm of regular spaces, the properties of being hemicompact, relatively hemicompact, and weakly relatively hemicompact are all equivalent.*

Proof. If X is regular and relatively hemicompact, we start by letting $\{A_n : n \in \omega\}$ be a sequence of relatively compact subsets witnessing relative hemicompactness for X. Since X is regular, $\mathrm{cl}_X(A_n)$ is compact for each $n \in \omega$ by Lemma 4.11. To see that X is hemicompact, consider any $K \subseteq X$ compact. Since K is also relatively compact, there is some $n \in \omega$ such that $K \subseteq A_n \subseteq \mathrm{cl}_X(A_n)$. So X is hemicompact.

Now assume X is regular and hemicompact. Let $\{K_n : n \in \omega\}$ be a set of compact subsets witnessing the hemicompactness of X. Note that each K_n is also relatively compact. So consider $A \subseteq X$ which is relatively compact. Since X is regular, $\mathrm{cl}_X(A)$ is compact, and thus, there is some $n \in \omega$ such that $A \subseteq \mathrm{cl}_X(A) \subseteq K_n$. That is, X is relatively hemicompact.

Now that we've shown that the properties of being hemicompact and relatively hemicompact are equivalent in the realm of regular spaces, we finish the proof by showing that a regular weakly relatively hemicompact space is hemicompact. This follows immediately from the fact that the closure of a relatively compact subset of a regular space is compact. □

To make some general connections to the property of being Markov k-Menger, we introduce a natural modification to the notion of k-covers relative to the family of relatively compact subsets of a space.

Definition 4.35. Let $\mathcal{K}_X^{\mathrm{rel}} = \mathcal{O}_X(\mathsf{K}_{\mathrm{rel}}(X))$, the collection of all open covers $\mathscr{U}$ such that, for every relatively compact A in X, there is some $U \in \mathscr{U}$ such that $A \subseteq U$. We will refer to these covers as *relative k-covers*; we will refer to $\mathsf{G}_{\mathrm{fin}}(\mathcal{K}_X^{\mathrm{rel}}, \mathcal{K}_X^{\mathrm{rel}})$ and $\mathsf{G}_1(\mathcal{K}_X^{\mathrm{rel}}, \mathcal{K}_X^{\mathrm{rel}})$ as the *relative k-Menger game* and the *relative k-Rothberger game* on X, respectively.

The following lemma follows immediately from the definitions.

Lemma 4.36. *If $\mathscr{U}$ is an open cover of a space X, then*

$$\mathscr{U}^{\mathrm{fin}} := \left\{ \bigcup \mathscr{F} : \mathscr{F} \in [\mathscr{U}]^{<\omega} \right\} \in \mathcal{K}_X^{\mathrm{rel}}.$$

Lemma 4.37. *For any space X, if $\varphi : \mathcal{K}_X^{\mathrm{rel}} \to [\mathscr{T}_X]^{<\omega}$ is such that $\varphi(\mathscr{U}) \in [\mathscr{U}]^{<\omega}$ for each $\mathscr{U} \in \mathcal{K}_X^{\mathrm{rel}}$, then*

$$A := \bigcap \left\{ \bigcup \varphi\left(\mathscr{U}^{\mathrm{fin}}\right) : \mathscr{U} \in \mathcal{O}_X \right\}$$

is relatively compact in X.

Proof. By Lemma 4.36, A is defined. So we need only show that A is relatively compact in X. Indeed, consider any open cover $\mathscr{U}$ of X. Then $A \subseteq \bigcup \varphi\left(\mathscr{U}^{\mathrm{fin}}\right)$. Since $\varphi\left(\mathscr{U}^{\mathrm{fin}}\right) \in \left[\mathscr{U}^{\mathrm{fin}}\right]^{<\omega}$, we see that A is covered by a finite subset of $\mathscr{U}$. □

Proposition 4.38. *For any space X, the following are equivalent:*

(a) $\mathrm{cof}_X(\mathsf{K}_{\mathrm{rel}}(X), \mathcal{B}) \leq \omega$.
(b) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_{\mathrm{fin}}(\mathcal{K}_X^{\mathrm{rel}}, \mathcal{O}_X(\mathcal{B}))$.
(c) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X^{\mathrm{rel}}, \mathcal{O}_X(\mathcal{B}))$.

Proof. (a) $\Longrightarrow$ (c): Let $\{A_n : n \in \omega\} \subseteq \mathsf{K}_{\mathrm{rel}}(X)$ witness that $\mathrm{cof}_X(\mathsf{K}_{\mathrm{rel}}(X), \mathcal{B}) \leq \omega$ and define σ in the following way. Given $\mathscr{U} \in \mathcal{K}_X^{\mathrm{rel}}$, let $\sigma(\mathscr{U}, n) \in \mathscr{U}$ be such that $A_n \subseteq \sigma(\mathscr{U}, n)$. Then σ is a winning Markov strategy. Indeed, for any sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of relative k-covers of X, if $B \in \mathcal{B}$, there is some $n \in \omega$ such that $B \subseteq A_n \subseteq \sigma(\mathscr{U}_n, n)$.

(c) $\Longrightarrow$ (b) is obvious.

To prove (b) $\Longrightarrow$ (a), we proceed by the contrapositive. Let σ be a Markov strategy for P2 in $\mathsf{G}_{\mathrm{fin}}(\mathcal{K}_X^{\mathrm{rel}}, \mathcal{O}_X(\mathcal{B}))$ and define

$$A_n = \bigcap \left\{ \bigcup \sigma\left(\mathscr{U}^{\mathrm{fin}}, n\right) : \mathscr{U} \in \mathcal{O}_X \right\}.$$

By Lemma 4.37, A_n is relatively compact for each $n \in \omega$.

By hypothesis, there is some $B \in \mathcal{B}$ that is not covered by any A_n. So let $x_n \in B \setminus A_n$ for each $n \in \omega$. In the n^{th} inning, let P1 choose $\mathscr{U}_n \in \mathcal{O}_X$ such that

$$x_n \notin \bigcup \sigma\left(\mathscr{U}_n^{\mathrm{fin}}, n\right).$$

It follows that $\langle \mathscr{U}_n^{\mathrm{fin}} : n \in \omega \rangle$ is a play by P1 that beats σ. □

Lemma 4.39. *If X is Markov k-Menger, then X is weakly relatively hemicompact.*

Proof. First, note that

$$\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_{\text{fin}}(\mathcal{K}_X, \mathcal{K}_X) \implies \mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_{\text{fin}}(\mathcal{K}_X^{\text{rel}}, \mathcal{K}_X)$$

since $\mathcal{K}_X^{\text{rel}} \subseteq \mathcal{K}_X$ (which follows from the fact that every compact set is relatively compact). So then Proposition 4.38 applies to assert that $\operatorname{cof}_X(\mathsf{K}_{\text{rel}}(X), \mathsf{K}(X)) \leq \omega$; that is, X is weakly relatively hemicompact. □

We now collect some particular applications of Proposition 4.38.

Corollary 4.40. *Let X be a space.*

(1) *The following are equivalent:*
  (a) *X is relatively hemicompact.*
  (b) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_{\text{fin}}(\mathcal{K}_X^{\text{rel}}, \mathcal{K}_X^{\text{rel}})$.
  (c) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X^{\text{rel}}, \mathcal{K}_X^{\text{rel}})$.
(2) *The following are equivalent:*
  (a) *X is weakly relatively hemicompact.*
  (b) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_{\text{fin}}(\mathcal{K}_X^{\text{rel}}, \mathcal{K}_X)$.
  (c) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X^{\text{rel}}, \mathcal{K}_X)$.
(3) *The following are equivalent:*
  (a) *X is σ-relatively compact.*
  (b) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_{\text{fin}}(\mathcal{K}_X^{\text{rel}}, \mathcal{O}_X)$.
  (c) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X^{\text{rel}}, \mathcal{O}_X)$.

Corollary 4.41. *For any regular space X, the following are equivalent:*

(a) *X is hemicompact.*
(b) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_{\text{fin}}(\mathcal{K}_X, \mathcal{K}_X)$.
(c) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.

Proof. By Corollary 4.40(1) and Lemma 4.34, it suffices to show that $\mathcal{K}_X = \mathcal{K}_X^{\text{rel}}$ when X is regular.

Since every compact set is relatively compact, $\mathcal{K}_X^{\text{rel}} \subseteq \mathcal{K}_X$. We show that $\mathcal{K}_X \subseteq \mathcal{K}_X^{\text{rel}}$ under the assumption that X is regular. So let $\mathscr{U} \in \mathcal{K}_X$ and suppose $A \subseteq X$ is relatively compact. By Lemma 4.11, $\operatorname{cl}_X(A)$ is compact, so there is some $U \in \mathscr{U}$ such that $\operatorname{cl}_X(A) \subseteq U$. That is, $\mathscr{U} \in \mathcal{K}_X^{\text{rel}}$. □

4.3. **Relating the various games.** The following proposition appears as [3, Lemma 3.7], but we offer it here as a consequence of the investigations of this paper without any separation axiom assumptions; it also summarizes many of the implications in Figure 2.

Theorem 4.42. *In general,*

$$\mathsf{G}_1(\Omega, \Omega) \leq^+_{\mathrm{II}} \mathsf{G}_1(\mathcal{O}, \mathcal{O}).$$

Proof. This follows immediately from Theorems 4.1, 4.2(2), 4.5(2), and 4.17(2). □

As will be established with Example 5.3,

$$\mathsf{G}_1(\mathcal{K},\mathcal{K}) \not\leq^+_{\mathrm{II}} \mathsf{G}_1(\Omega,\Omega),$$

generally. However, for finite selections, we have the following theorem which proves the "k-Menger $\implies$ ω-Menger" variations that appear in Figure 1.

Theorem 4.43. *In general,*

$$\mathsf{G}_{\mathrm{fin}}(\mathcal{K},\mathcal{K}) \leq^+_{\mathrm{II}} \mathsf{G}_{\mathrm{fin}}(\Omega,\Omega) \leq^+_{\mathrm{II}} \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}).$$

Proof. The

$$\mathsf{G}_{\mathrm{fin}}(\Omega,\Omega) \leq^+_{\mathrm{II}} \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$$

portion is [3, Lemma 3.4], but also follows immediately from Theorems 4.1, 4.2(1), 4.5(1), and 4.17(1).

Now we show that

$$\mathsf{G}_{\mathrm{fin}}(\mathcal{K},\mathcal{K}) \leq^+_{\mathrm{II}} \mathsf{G}_{\mathrm{fin}}(\Omega,\Omega).$$

First, assume that X is Markov k-Menger. By Lemma 4.39, X is weakly relatively hemicompact which implies that X is σ-relatively compact (Remark 4.33). Hence, by Theorem 4.17, X is Markov ω-Menger.

Assume X is strategically k-Menger. By Theorem 4.5, it suffices to show that X is strategically Menger. To see this, just take the closure of open covers under finite unions to create k-covers. When you apply P2's strategies to these k-covers, P2's selections can be seen as finite selections from the original open covers. Since P2's selections win in the k-Menger game, the decomposed collections in the Menger game must still cover the space.

Assume X is k-Menger. Note that any space which is k-Menger is Menger. This follows from similar reasoning as above by closing open covers under finite unions. Then, by Theorem 4.23(1), every finite power of X is Menger. So, by Theorem 4.2(1), X is ω-Menger.

Lastly, assume X is k-Lindelöf. Note that any space which is k-Lindelöf is Lindelöf by taking an open cover and closing it under finite unions, just as above. Then, by Theorem 4.22, every finite power of X is Lindelöf. Hence, by Theorem 4.1, X is ω-Lindelöf. □

5. Examples

In this section, we provide some examples that separate some of the properties in Figures 1 and 2; we also answer a question posed in [2].

Note that $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$ is a variant of the compact-open game, the game in which P1 chooses a compact set and P2 responds with an open set containing that compact set, where P2 is trying to avoid forming a k-cover of X.

Recall also that the finite-open game on X, which corresponds to the game $\mathsf{G}_1(\mathcal{N}([X]^{<\omega}), \neg\mathcal{O}_X)$ in our notation, is the game in which P1 chooses a finite set and P2 responds with an open set containing that finite set, where P2 is trying to avoid forming a cover of X.

In [2], it was asked if there is a space X for which P1 has a winning strategy in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$, but P1 doesn't have a winning strategy in the finite-open game on X and P1 doesn't have a predetermined winning strategy in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$. We can provide a straightforward answer in the affirmative.

Example 5.1. There is a T_5 space for which P1 has a winning strategy in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$, but P1 doesn't have a winning strategy in the finite-open game on X and P1 doesn't have a predetermined winning strategy in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$.

Proof. Let X_0 be the Fortissimo space on the reals (S22). Recall that this topology adds a point $\infty \notin \mathbb{R}$ to $\mathbb{R}_d$, the set of the real numbers with the discrete topology, so that the neighborhood basis at ∞ consists of co-countable subsets of $\mathbb{R}$. Finally, let X be the disjoint union, $[0,1] \sqcup X_0$, of the closed unit interval $[0,1]$ with X_0. As the disjoint union of two T_5 spaces (see [44]), X is T_5.

The winning strategy for P1 in the game $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(Y)), \neg\mathcal{K}_Y)$ where Y is the Fortissimo space on discrete ω_1 is described in [2, Ex. 3.24]. We will verify that a modification to this idea works for X. Throughout, we will use the identification $A \mapsto \mathcal{N}_A$, $\mathcal{A} \to \mathcal{N}(\mathcal{A})$.

P1 starts by playing

$$\sigma(\varnothing) = K_0 = \{\infty\} \cup [0,1].$$

Then, for $n \in \omega$, suppose we have $\langle K_j : j \leq n\rangle$ and $\langle A_j : j < n\rangle$ defined where $K_0 \subseteq K_j$ for each $j \leq n$, and $A_j = \{x_{j,k} : k \in \omega\} \subseteq X \setminus K_j$ for each $j < n$. P2 must respond to K_n with some open set that covers K_n. Since $\{\infty\} \cup [0,1] \subseteq K_n$, P2's move can be written as $X \setminus A_n$ where

$$A_n = \{x_{n,j} : j \in \omega\} \subseteq X \setminus K_n \subseteq X_0.$$

P1 responds to $X \setminus A_n$ with

$$\sigma(\langle X \setminus A_j : j \leq n\rangle) = K_{n+1} = K_0 \cup \{x_{j,k} : j, k \leq n\}.$$

This defines the strategy σ for P1.

We now show that σ is winning. So consider any run of the game according to σ, as coded above with $\langle K_n : n \in \omega\rangle$ and $\langle A_n : n \in \omega\rangle$ where $A_n = \{x_{n,k} : k \in \omega\} \subseteq X_0$. Let $A = \bigcup\{A_n : n \in \omega\}$ and note that A is countable. Let K be a compact subset of X. Since X_0 is anticompact (every compact subset of X_0 is finite), there must be some $n \in \omega$ for which $K \cap A \subseteq K_n \subseteq U_n := X \setminus A_n$.

We show that $K \subseteq U_n$. We already have that $K \cap A \subseteq U_n$. Also, observe that $K \cap [0,1] \subseteq K_n \subseteq U_n$. Now, for any $x \in (K \cap X_0) \setminus A$,

$$x \in X_0 \setminus \bigcup_{j\in\omega} A_j = \bigcap_{j\in\omega} X_0 \setminus A_j \subseteq X_0 \setminus A_n \subseteq U_n.$$

Hence, σ is a winning strategy for P1 in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$.

We now argue that P1 cannot win $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$ with a pre-determined strategy. By the results of [9], $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$ is dual to $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$; a particular application of that duality is that P1 has a pre-determined winning strategy in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$ if and only if P2 has a winning Markov strategy in $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$. Since X_0 is anticompact and uncountable, X is not hemicompact. Hence, by Corollary 4.29, P2 does not have a winning Markov strategy in the k-Rothberger game on X. Thus, P1 does not have a pre-determined winning strategy in $\mathsf{G}_1(\mathcal{N}(\mathsf{K}(X)), \neg\mathcal{K}_X)$.

Lastly, we argue that P1 doesn't have a winning strategy in the finite-open game on X. We do this by showing that P2 actually has a winning Markov strategy in this game. Suppose P1 has played $F_n \in [X]^{<\omega}$. Let $V_n = X_0$ and $W_n \subseteq [0,1]$ be an open set with $F_n \cap [0,1] \subseteq W_n$ such that the Lebesgue measure of W_n is less than $\frac{1}{2^{n+2}}$. Then let P2 respond to F_n with $U_n := V_n \cup W_n$. Since the Lebesgue measure of $\bigcup_{n\in\omega} W_n$ is less than $1/2$, P2 has won the finite-open game. □

Example 5.2 (S43)**.** The Sorgenfrey line is an example of a space which is Lindelöf but not ω-Lindelöf. This follows from the fact that the Sorgenfrey plane is not Lindelöf; see [44] and Theorem 4.1.

Example 5.3 (S25)**.** The space of reals $\mathbb{R}$ is an example of a space which is

(1) Markov ω-Menger and not Markov ω-Rothberger (see Theorems 4.17 and 4.2 and the fact the reals are not even Rothberger, which is witnessed by any sequence of open covers consisting of intervals with exponentially decreasing diameters),
(2) Strategically ω-Menger and not strategically ω-Rothberger (same as in (1)),
(3) ω-Menger and not ω-Rothberger (same as in (1)),
(4) Markov k-Rothberger and not Markov ω-Rothberger (see Corollary 4.41, Theorem 4.2 and the fact that the reals are not even Rothberger),
(5) Strategically k-Rothberger and not strategically ω-Rothberger (same as in (4)), and
(6) k-Rothberger and not ω-Rothberger (same as in (4)).

Example 5.4 (S27)**.** The space of rationals $\mathbb{Q}$ is an example of a space which is

(1) Markov ω-Menger and not Markov k-Menger,
(2) Strategically ω-Menger and not strategically k-Menger,
(3) ω-Menger and not k-Menger,
(4) Markov ω-Rothberger and not Markov k-Rothberger,
(5) Strategically ω-Rothberger and not strategically k-Rothberger, and
(6) ω-Rothberger and not k-Rothberger.

Proof. See Theorem 4.17, Proposition 4.28, and the fact that $\mathbb{Q}$ is not hemicompact [44]. □

The following is well-known, but we record a proof of it here for inclusion in the literature.

Proposition 5.5. *The space of irrationals* $\mathbb{R} \setminus \mathbb{Q}$ *is not Menger.*

Proof. Using continued fraction expansions, the irrationals are homeomorphic to the Baire space ω^ω. Let $\mathscr{U}_n = \{[t] : t \in \omega^{<\omega}\}$, where $[t] = \{f \in \omega^\omega : f \text{ extends } t\}$. Suppose that $F_n \subseteq \mathscr{U}_n$ are finite. We can find an $f \in \omega^\omega$ so that $f \notin \bigcup_n F_n$, which shows that the irrationals are not Menger. First pick $f(0)$ so that $[\langle f(0)\rangle] \notin F_0$. Then if $f(0), \cdots, f(n)$ have been chosen so that $[\langle f(0), \cdots, f(k)\rangle] \notin F_k$ for $k \leq n$, we can choose $f(n+1)$ so that $[\langle f(0), \cdots, f(n+1)\rangle] \notin F_{n+1}$. Continue recursively in this way to produce f as desired. □

Example 5.6 (S28)**.** The space of irrationals $\mathbb{R} \setminus \mathbb{Q}$ is an example of a space which is

(1) ω-Lindelöf and not ω-Menger (the irrationals are second-countable, so by Proposition 3.10 are ω-Lindelöf; from Proposition 5.5 the irrationals are not Menger, so by Theorem 4.2, cannot be ω-Menger), and
(2) k-Lindelöf and not k-Menger (the irrationals are second-countable, so by Proposition 3.10 are k-Lindelöf, and they are not Menger as before).

Example 5.7 (S22)**.** The Fortissimo space on the reals is an example of a space which is

(1) Strategically k-Menger and not Markov k-Menger,
(2) Strategically k-Rothberger and not Markov k-Rothberger,
(3) Strategically ω-Menger and not Markov ω-Menger, and
(4) Strategically ω-Rothberger and not Markov ω-Rothberger.

Proof. By similar reasoning to the argument in Example 5.1, X is strategically k-Rothberger and strategically ω-Rothberger. These in turn, imply that it is strategically k-Menger and strategically ω-Menger. However, the Fortissimo space is not hemicompact and is not countable, so by Corollaries 4.18 and 4.41 it is not Markov for any of those games. □

Let $\mathbb{K}(X)$ be the set $\mathsf{K}(X)$ endowed with the Vietoris topology (see [28]), as mentioned above. Let $\mathcal{P}_{\text{fin}}(X)$ denote the set $[X]^{<\omega}$ viewed as a subspace of $\mathbb{K}(X)$. There are many equivalences between properties studied in this paper and these hyperspaces in [3]. For example, a space X is ω-Rothberger if and only if $\mathcal{P}_{\text{fin}}(X)$ is Rothberger. Analogous equivalences hold at the Lindelöf level and also for finite-selections. Notably, the equivalence that fails is that of X being k-Rothberger and $\mathbb{K}(X)$ being Rothberger, as witnessed by $\mathbb{R}$.

From results discussed in this work, every k-Lindelöf space is ω-Lindelöf. As a consequence, if $\mathbb{K}(X)$ is Lindelöf, then the dense subspace $\mathcal{P}_{\text{fin}}(X)$ is also Lindelöf. However,

Example 5.8. There is a T_5 space X such that $\mathbb{K}(X)$ is strategically k-Rothberger and strategically ω-Rothberger, but not hereditarily Lindelöf. Consequently, none of the Menger or Rothberger variants discussed in this paper, excluding the Markov levels, are necessarily hereditary for $\mathbb{K}(X)$.

Proof. Let X be the Fortissimo space on the reals and let $\mathbb{R}_d$ be the reals with the discrete topology. As asserted in [44], X is T_5. Since X is anticompact, $\mathbb{K}(X) = \mathcal{P}_{\text{fin}}(X)$. As mentioned in Example 5.7, X is strategically k-Rothberger and strategically ω-Rothberger. By [3, Thm. 4.8], $\mathbb{K}(X) = \mathcal{P}_{\text{fin}}(X)$ is strategically ω-Rothberger; consequently, by anticompactness, $\mathbb{K}(X)$ is also strategically k-Rothberger. Indeed, if $\mathbf{K} \subseteq \mathbb{K}(X)$ is compact, $\bigcup \mathbf{K} \subseteq X$ is compact (see [28]), hence finite. That means that $\mathbf{K}$ is finite, as well. Lastly, $[\mathbb{R}_d]^{<\omega}$ is an uncountable relatively discrete subspace of $\mathbb{K}(X)$, so $\mathbb{K}(X)$ is not hereditarily Lindelöf; moreover, $\mathbb{K}(X)$ is neither hereditarily Rothberger nor hereditarily Menger. □

Recall that a Luzin subset of $\mathbb{R}$ is an uncountable set X such that $X \cap F$ is countable for every closed and nowhere dense F. Evidently, no Luzin set is meager.

Rothberger [34] showed that every Luzin set is Rothberger. Hence, if a Luzin set exists, it is a non-meager Rothberger subset of $\mathbb{R}$.

As a response to a question of Galvin, Recław [33] showed that every Luzin set is undetermined in the point-open game. By duality (see [17, 9]), Luzin sets are undetermined for the Rothberger game, as well. We now collect some facts about Luzin sets related to other properties investigated in this work.

Lemma 5.9. *No Luzin set is strategically Menger or k-Rothberger. Consequently, every Luzin set is an example of a Menger space which is not strategically Menger; an example of a Rothberger space which is not strategically Rothberger.*

Proof. We first argue that a Luzin set cannot be σ-compact. So let $X \subseteq \mathbb{R}$ be a Luzin set and note that X cannot have a non-empty interior since any open subset of $\mathbb{R}$ contains a copy of the Cantor set. Hence, every compact subset of X is nowhere dense. By definition, it follows that every compact subset of X must be countable. Hence, X is not σ-compact.

As a subspace of $\mathbb{R}$, X is second-countable and regular. Note that [7, Lemma 4.10] asserts that any second-countable strategically Menger space is Markov Menger. By Corollary 4.18, a regular Markov Menger space is σ-compact. Hence, since X is not σ-compact, it cannot be Markov Menger and hence, X cannot be strategically Menger.

Now, since X is not σ-compact, it is certainly not hemicompact. As a subspace of $\mathbb{R}$, X is T_1 and first-countable, so X cannot be k-Rothberger by Proposition 4.28.

Lastly, by Rothberger's result [34], every Luzin set is Rothberger, hence Menger. So the final assertion holds. □

In fact, a bit more is true about Luzin sets. As discussed in [29], every Luzin set is universally null, which means that they have outer measure zero with respect to every continuous Borel probability measure on $\mathbb{R}$. Combining this with [5, Thm. 3.9], no Luzin set can even have the Baire property.

In a separate line of investigation, Scheepers [40] discusses selection principle properties of Luzin sets involving families of open sets with the property that their closures cover the space in question.

Example 5.10. Assuming CH, there is a Luzin set which is not ω-Rothberger. Indeed, [22, Lemma 2.6 and Thm. 2.8] shows that there is a Luzin subset L of the reals such that L does not satisfy a selection principle $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Omega)$. According to [22, Fig. 3], $\mathsf{S}_1(\Omega, \Omega) \implies \mathsf{U}_{\mathrm{fin}}(\Gamma, \Omega)$. Hence, L is not ω-Rothberger. Now, since every Luzin set is Rothberger, L is Rothberger without being ω-Rothberger. Note, moreover, by Theorem 4.2(2), L has some finite power which fails to be Rothberger.

Example 5.11. Assuming CH, [22, Thm. 2.13] proves the existence of a Luzin set which is ω-Rothberger. Observe that, by Theorem 4.5 and Lemma 5.9, this Luzin set is not strategically ω-Rothberger; by the same results, this Luzin set is ω-Menger but not strategically ω-Menger.

Example 5.12. In [11, Ex. 6.4], under MA, a countable space is constructed which is not k-Lindelöf. Since every countable space is trivially ω-Lindelöf, this is an example of a space which is ω-Lindelöf but not k-Lindelöf.

6. Open Questions

We finish with a list of questions.

Question 6.1. *Are there spaces X for which the games $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$ and $\mathsf{G}_{\mathrm{fin}}(\mathcal{K}_X, \mathcal{K}_X)$ are undetermined (that is, that neither player has a winning strategy)?*

This two-fold question may reduce to a single question if the single- and finite-selection versions turn out to be the same.

Question 6.2. *Does (strategically) k-Menger imply (strategically) k-Rothberger?*

Note that, by Proposition 4.28 and Corollary 4.41, no metrizable space can be an affirmative response to Question 6.1.

It is shown in [46, Thm. 2.16] that, consistently, there are two sets X and Y of reals which are ω-Menger, but that $X \times Y$ is not even Menger. That is, it is consistent with ZFC that the property of being ω-Menger is not finitely productive.

Question 6.3. *Is it consistent with ZFC that any of the properties ω-Lindelöf, ω-Menger, or ω-Rothberger are (finitely) productive?*

Question 6.4. *Are any of the properties k-Lindelöf, k-Menger, or k-Rothberger (finitely) productive?*

Corollary 4.41 asserts that the properties of being Markov k-Menger and being Markov k-Rothberger are equivalent for regular spaces.

Question 6.5. *Are the properties of being Markov k-Menger and Markov k-Rothberger equivalent for all topological spaces? Alternatively, is there an example of a (preferably T_1 or T_2) space which is Markov k-Menger but not Markov k-Rothberger (equivalently, not hemicompact, by Corollary 4.29)?*

As a related group of questions,

Question 6.6. *Is there an example of a space which is hemicompact, but not relatively hemicompact? Is there an example of a space which is relatively hemicompact, but not hemicompact?*

As pointed out in Example 5.12, [11, Ex. 6.4] offers an example of a space, assuming MA, that is ω-Lindelöf but not k-Lindelöf.

Question 6.7. *Is there a ZFC example of an ω-Lindelöf space which is not k-Lindelöf?*

As Example 5.10 demonstrates, it is consistent with ZFC that there exists a Rothberger subset of the reals which is not ω-Rothberger. As a contrast to this, it is well-known that, in Laver's model [24], every Rothberger subset of the reals is countable, and hence ω-Rothberger (see [49] and [35]). Since these assertions are about subsets of reals, one can ask about more general settings.

Question 6.8. *Is it consistent with ZFC that every Rothberger space is ω-Rothberger?*

In [48], it is shown to be consistent with ZFC that, for all metrizable spaces, the properties of being Menger and ω-Menger are equivalent. Of course, every ω-Menger space is Menger without any separation axiom assumptions. So we are left with

Question 6.9. *Is it consistent with ZFC that every Menger space is ω-Menger?*

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