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On functions between generalized topological spaces

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Abstract

This paper investigates generalized topological spaces and functions between such spaces from the perspective of change of generalized topology. In particular, it considers the preservation of generalized connectedness properties by various classes of functions between generalized topological spaces.

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0. Introduction

The notion of continuity is one of the main ideas in the whole of mathematics. So much so, that in recent decades there has been the growing trend of speaking of two sorts of mathematics - continuous mathematics and discrete mathematics.

In the topological category the morphisms between the basic objects, the topological spaces, are continuous functions. So any attempt to generalize these basic objects must also provide a discussion of a property of functions that

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corresponds to continuity. In the past decade, Császár [4] and others have been considering generalized topological spaces, and developing a theory for them. In the title of his foundational paper, Császár [4] has given equal prominence to these two aspects - "Generalized topology, generalized continuity".

This paper is concerned with the adaptation of the change of topology approach from topological topics to aspects of the theory of generalized topological spaces. It shows that " change of generalized topology " exhibits some characterictics analogous to change of topology in the topological category. A common application of the change of generalized topology approach occurs where the spaces are (ordinary) topological spaces. In this case, the generalized topologies are families of distinguished subsets of a topological space which are not topologies but are generalized topologies. It appears that this case was one of the main motivations for Császár [4] to introduce and develop the concepts of generalized topology. Sections 1 and 2 provide the necessary preliminary ideas, while Section 3 uses three variations of generalized continuity between generalized topological spaces, defined by Min [12,13,14], to illustrate the change of generalized topology procedure in its most general form. Section 4 considers the preservation of generalized connectedness properties by appropriate classes of functions.

1. Preliminaries

If exp$X$ is the power set of a nonempty set X, then a subset g of exp$X$ is defined to be a generalized topology (briefly a GT) on X if $\varnothing \in g$ and any union of elements of g belongs to g, (Császár [4]) . If g is a GT on X, the members of g are called generalized open sets, or g-open sets, and their complements are called generalized closed sets, or g-closed sets. If A is a subset of X, then the generalized interior of A in (X, g), denoted by $i_g(A)$, is the union of the collection of all generalized open sets contained in A. It is the largest g-open set contained in A. The generalized closure of A in (X, g), denoted by $c_g(A)$ is the intersection of the collection of all generalized closed sets containing A. It is the smallest g-closed set containing A.

Császár [4] has pointed out some common examples of generalized topologies that are associated with a given topological space. If τ is a topology on X, we denote by iA the τ-interior of A and by cA the τ-closure of A. Then A is semi-open if $A \subset c(iA)$, A is preopen if $A \subset i(cA)$, A is β-open if $A \subset c(i(c(A))$, and A is α-open if $A \subset i(c(iA))$. We denote by $s\tau$ ($p\tau$, $\beta\tau$ and $\alpha\tau$ respectively) the collection of all semi-open sets (preopen sets, β-open sets and α-open sets respectively) of the topological space (X, τ). Each of these collections is a GT on X, and, in fact, $\alpha\tau$ is a topology. In general, the other three collections are not topologies on X.

A generalized topological space (X, g) is said to be connected (called γ-connected in [5]) if there are no nonempty disjoint sets U and V in g such that $X = U \cup V$.

Let g_X and g_Y be generalized topologies on X and Y, respectively. Then the function $f : (X, g_X) \to (Y, g_Y)$ is defined to be generalized continuous or more properly (g_X, g_Y)-continuous if $f^{-1}(V) \in g_X$ for each $V \in g_Y$ [4].

If β is an arbitrary subset of exp$X$, then the family $\vartheta \subset$ exp$X$ composed of $\varnothing$ and all sets $N \subset X$ of the form $N = \cup_{i\in I} B_i$ where $B_i \in \beta$ and I is a nonempty index set, is a *GT* on X. We say that β is a base of ϑ, and that β generates ϑ.

It is enough to check generalized continuity for each member of a base for the co-domain generalized topology, just as is the case for continuity of functions between (ordinary) topological spaces.

Lemma 1.1 (6, Lemma 3.2)**.** *A function $f : (X, g_X) \to (Y, g_Y)$ is generalized continuous if and only if $f^{-1}(V) \in g_X$ for each member V of a base β of g_Y.*

2. Six classes of generalized topologies

If g is a *GT* on X, then Császár [7] has shown that $\delta(g)$ and $\theta(g)$ are also generalized topologies on X, where

1) $A \in \delta(g)$ if and only if $A \subset X$ and, if $x \in A$ then there is a g-closed set Q such that $x \in i_g Q \subset A$, and
2) $A \in \theta(g)$ if and only if $A \subset X$ and $x \in A$ implies the existence of $M \in g$ such that $x \in M \subset c_g M \subset A$.

Furthermore, $\theta(g) \subset \delta(g) \subset g$ [7, Theorem 2.3].

A subset R of (X, g) is defined to be g-regular open (or gr-open) if $R = i_g(c_g(R))$. Then the elements of $\delta(g)$ coincide with the unions of gr-open sets [7, Theorem 3.3].

To put it another way, the collection of all gr-open sets of (X, g) is a base for the generalized topology $\delta(g)$ on X.

Therefore, the relationship between a *GT* g on a set X and the *GT* $\delta(g)$ on X is analogous to that between a topological space (X, τ) and its semi-regularization (X, τ_s), where τ_s has as a base the collection of all τ-regular open subsets of X. Recall that a subset B of (X, τ) is defined to be τ-regular open if $B = i(c(B))$. Mršević, Reilly and Vamanamurthy [15] have given a systematic discussion of this relationship, especially from the point of view of change of topology. On the other hand, Veličko [18] provided the basic discussion of the properties of θ-topologies.

Császár [3] has introduced four classes of generalized open sets in a generalized topological space with the following definition.

Definition 2.1 ([3]). Let (X, g_X) be a generalized topological space and $A \subset X$. Then A is said to be
(1) g-semi-open if $A \subset c_g(i_g(A))$,
(2) g-pre-open if $A \subset i_g(c_g(A))$,
(3) g-α-open if $A \subset i_g(c_g(i_g(A)))$,
(4) g-β-open if $A \subset c_g(i_g(c_g(A)))$.

The collection of all g-semi-open sets (resp. g-pre-open sets, g-α-open sets, g-β-open sets) in (X, g_X) is denoted by $\sigma(g)$ (resp. $\pi(g), \alpha(g), \beta(g)$). Each of these families is a GT [3]. Furthermore these families satisfy the following relationships $g \subset \alpha(g) \subset \sigma(g) \subset \beta(g)$ and $\alpha(g) \subset \pi(g) \subset \beta(g)$, see [3].

3. Change of generalized topology

Three variants of generalized continuity have been recently defined and studied by Min [12, 13, 14].

If (X, g_X) and (Y, g_Y) are generalized topological spaces, then a function $f : (X, g_X) \to (Y, g_Y)$ is defined to be almost (g_X, g_Y)-continuous at $x \in X$ if for each g_Y-open set V containing $f(x)$ there exists a g_X-open set U containing x such that $f(U) \subset i_{g_Y}(c_{g_Y}(V))$ [12, Definition 3.1].

Furthermore, f is defined to be almost (g_X, g_Y)-continuous if it is almost (g_X, g_Y)-continuous at each point of X [12, Definition 3.3].

A function $f : (X, g_X) \to (Y, g_Y)$ is defined to be weakly $\theta(g_X, g_Y)$-continuous if for $x \in X$ and each $V \in \theta(g_Y)$ containing $f(x)$, there exists $U \in g_X$ such that $x \in U$ and $f(U) \subset V$ [13, Definition 3.12].

More recently, Min [14] has introduced and studied another property of functions between generalized topological spaces. Let (X, g_X) and (Y, g_Y) be generalized topological spaces and $\delta = \delta(g_X)$, $\delta' = \delta(g_Y)$. Then a function $f : (X, g_X) \to (Y, g_Y)$ is defined to be (δ, δ')-continuous at the point $x \in X$ if for each g_Y-open set V containing $f(x)$, there exists a g_X-open set U containing x such that $f(i_{g_X}(c_{g_X}U)) \subset i_{g_Y}(c_{g_Y}(V))$. Moreover, the function f is defined to be (δ, δ')-continuous if it is (δ, δ')-continuous at every point of X [14, Definition 3.4].

Min has shown [12, Example 3.5] that a function $f : (X, g_X) \to (Y, g_Y)$ can be almost (g_X, g_Y)-continuous but not (g_X, g_Y)-continuous. It is clear from the definitions that (g_X, g_Y)-continuity implies almost (g_X, g_Y)-continuity. However, this distinction between these two concepts must be interpreted very carefully. Our next result shows that almost (g_X, g_Y)-continuity is really (g_X, g_Y)-continuity in disguise. The notion of almost generalized continuity coincides with the concept of generalized continuity when a suitable change of generalized topology is made to the co-domain of the function in question.

Proposition 3.1. *A function $f : (X, g_X) \to (Y, g_Y)$ between generalized topological space is almost (g_X, g_Y)-continuous if and only if $f : (X, g_X) \to (Y, \delta(g_Y))$ is generalized continuous.*

Proof. Min [12, Theorem 3.6 (1) and (6)] has shown that $f : (X, g_X) \to (Y, g_Y)$ is almost (g_X, g_Y)-continuous if and only if for every $g_Y r$-open subset V in Y, $f^{-1}(V)$ is g_X-open. By Lemma 1.1, this equivalent to $f : (X, g_X) \to (Y, \delta(g_Y))$ is generalized continuous. □

Note that the topological analogue of the result above is Proposition 12 (1) of Mršević, Reilly and Vamanamurthy [15].

An exactly similar situation obtains for the concept of weakly $\theta(g_X, g_Y)$-continuity defined and studied by Min [13]. Here, a different change of generalized topology on the co-domain of the function reduces the property of weak $\theta(g_X, g_Y)$-continuity to generalized continuity.

Min [13, Example 3.14] has shown that a function can be weakly $\theta(g_X, g_Y)$-continuous but not be weakly (g_X, g_Y)-continuous, and hence not be (g_X, g_Y)-continuous. However, if we make a change of generalized topology on the co-domain space of f, then weak $\theta(g_X, g_Y)$-continuity reduces to generalized continuity. Indeed, we have the following result.

Proposition 3.2. *A function $f : (X, g_X) \to (Y, g_Y)$ between generalized topological space is weakly $\theta(g_X, g_Y)$-continuous if and only if $f : (X, g_X) \to (Y, \theta(g_Y))$ is generalized continuous.*

Proof. [13, Theorem 3.15 (1) and (2)]. □

In a more recent paper Min [14] has introduced the notion of (δ, δ')-continuity of functions between generalized topological spaces. His Examples 3.5 and 3.6, his Remark 3.7 and its diagram are designed to distinguish between the generalized continuity of Császár [4] and Min's (δ, δ')-continuity. But, once again, it is necessary to be very careful in making claims about one concept being "independent" of another concept. In this case, this independence holds only if the generalized topologies on the domain and co-domain of the function in question are regarded as fixed. However, our next result shows that if appropriate changes of generalized topology are made on both the domain and co-domain of the function, then (δ, δ')-continuity of the given function is just generalized continuity in disguise. The new concept of (δ, δ')-continuity coincides with the old notion of generalized continuity when we make suitable changes of generalized topology.

Proposition 3.3. *A function $f : (X, g_X) \to (Y, g_Y)$ between generalized topological spaces is (δ, δ')-continuous if and only if $f : (X, \delta(g_X)) \to (Y, \delta(g_Y))$ is generalized continuous.*

Proof. Min [14] Theorem 3.9 (1) and (5). □

We have just considered three examples of the change of generalized topology technique for functions between generalized topological spaces. This is "change of generalized topology" in its full manifestation. This is the true analogue of the "change of topology" approach in the topological category. The fundamental paper on the change of topology technique in the topological category is Gauld, Mršević, Reilly and Vamanamurthy [9]. More recently, Gauld, Greenwood and Reilly [8] have classified more than one hundred properties of functions between (ordinary) topological spaces from this perspective.

However, there is another way of thinking about the change of generalized topology approach. In this application of the approach we do not have full generality. Rather, we consider functions between (ordinary) topological spaces. Can some properties of such functions be considered in the setting of generalized continuity? To do so, it may be necessary to consider the (ordinary) topological spaces as examples of generalized topological spaces. It is clear from the work of Császár [4] that this is one of the main reasons for introducing the notion of generalized topology.

Now let g_X and g_Y be generalized topologies on X and Y, respectively. Császár [4] has pointed out how many variations of continuity in the topological case can be regarded as examples of generalized continuity of functions between generalized topological spaces. Specifically, if τ_X and τ_Y are (ordinary) topologies on X and Y, then (τ_X, τ_Y)-continuity is classical topological continuity. Furthermore, $(s\tau_X, \tau_Y)$-continuity is the semi-continuity of [10], $(p\tau_X, \tau_Y)$-continuity is precontinuity in the sense of [11], $(s\tau_X, s\tau_Y)$-continuous functions are defined to be irresolute in [2], while $(p\tau_X, p\tau_Y)$-continuous functions are called preirresolute in [16]. So these are all examples of the use of the change of generalized topology technique to reduce a particular property of functions between ordinary topological spaces to generalized continuity. The corresponding change of topology technique has a much more limited breadth of application (it is restricted to the topological category).

An interesting question can be posed.

Question 3.4. *Is there a variant of continuity between (ordinary) topological spaces that cannot be reduced to generalized continuity by applying the change of generalized topology technique ?*

We recall that a subset A of a topological space (X, τ_X) is defined to be locally closed if $A = U \cap F$ where U is τ_X-open and F is τ_X-closed. Moreover, a function $f : (X, \tau_X) \to (Y, \tau_Y)$ is defined to be LC-continuous if $f^{-1}(U)$ is locally closed in (X, τ_X) for each τ_Y-open U in Y. Let $LC(X, \tau_X)$ denote the collection of all locally closed subsets of (X, τ_X). The next example shows that $LC(X, \tau_X)$ is not always a GT.

Example 3.5. Let X be the set $\mathbb{R}$ of real numbers, and take $\tau_X = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\varnothing\}$. Then (X, τ_X) is a topological space, and the τ_X-closed sets are $\{(-\infty, b] : b \in \mathbb{R}\} \cup \{X\}$. If $a < b$ then $(a, b] \in LC(X, \tau_X)$. Hence, for each positive integer n, $(a, b - \frac{1}{n}] \in LC(X, \tau_X)$. But $\displaystyle\bigcup_{n=1}^{\infty}(a, b - \frac{1}{n}] = (a, b) \notin LC(X, \tau_X)$. Thus $LC(X, \tau_X)$ is not a GT.

It is clear from Example 3.5 that LC-continuity cannot be characterized as an example of generalized continuity, thereby providing an affirmative answer to the Question above.

4. Preservation of generalized connectedness

In a recent paper [1], Bai and Zuo have introduced the class of g-α-irresolute functions between generalized topological spaces. Bai and Zuo [1, Definition 3.1] provide the following definition.

Definition 4.1. Let (X, g_X) and (Y, g_Y) be generalized topological spaces. Then a function $f : (X, g_X) \to (Y, g_Y)$ is g-α-irresolute if $f^{-1}(V)$ is g-α-open in X for each g-α-open subset V of Y.

The change of generalized topology approach has an immediate application here. The next result follows from Definition 4.1.

Proposition 4.2. *$f : (X, g_X) \to (Y, g_Y)$ is g-α-irresolute if and only if $f : (X, \alpha(g_X)) \to (Y, \alpha(g_Y))$ is generalized continuous.*

Bai and Zuo [1, Examples 3.9.1 and 3.9.2] distinguish between g-α-irresolute functions and (g_X, g_Y)-continuous functions. Indeed, they claim "g-α-irresolute and (g_X, g_Y)-continuous are independent of each other in generalized topological spaces". This distinction between these two concepts requires very careful interpretation. From Proposition 4.2 it is clear that the concept of g-α-irresoluteness coincides with the standard notion of generalized continuity when an appropriate change of generalized topology is made to each of the domain and co-domain of the function. The independence of these concepts, claimed by Bai and Zou [1], holds only when the generalized topologies on the domain and co-domain spaces are taken as fixed. In fact, the concept of g-α-irresoluteness is a disguised form of generalized continuity.

To see that this observation is more than a philosophical point, consider the discussion of α-connectedness in [1, page 387]. We recall that a generalized topological space (X, g_X) is α-connected if $(X, \alpha(g_X))$ is connected [17, Definition 2.1]. Now our Proposition 4.2, together with the preservation of connectedness by generalized continuous functions (see Császár [5], Theorem 2.2]), provides an elegant alternative proof of one of the main results of [1].

Proposition 4.3 ([1, Theorem 3.16])**.** *Let f be a g-α-irresolute function from (X, g_X) onto (Y, g_Y). If (X, g_X) is α-connected then (Y, g_Y) is α-connected.*

Proof. We have that $f : (X, \alpha(g_X)) \to (Y, \alpha(g_Y))$ is generalized continuous, by Proposition 4.2, and that $(X, \alpha(g_X))$ is connected. Hence $(Y, \alpha(g_Y))$ is connected, and thus (Y, g_Y) is α-connected. □

A similar discussion can be provided for the Theorem 3.18 of [1] after noting that (X, g_X) is α-compact if and only if $(X, \alpha(g_X))$ is compact.

Császár [5, page 276] has given the following definitions.

Let (X, g_X) and (Y, g_Y) be generalized topological spaces. Then a function $f : (X, g_X) \to (Y, g_Y)$ is said to be

(1) (α, g_Y)-continuous if $f^{-1}(V)$ is g-α-open in X for every g-open set V in Y.
(2) (σ, g_Y)-continuous if $f^{-1}(V)$ is g-semi-open in X for every g-open set V in Y.
(3) (π, g_Y)-continuous if $f^{-1}(V)$ is g-pre-open in X for every g-open set V in Y.
(4) (β, g_Y)-continuous if $f^{-1}(V)$ is g-β-open in X for every g-open set V in Y.

Now it is clear that the change of generalized topology approach can be applied to each of these variants of generalized continuity. In each case, a change of generalized topology on the domain of the function reduces the variant of generalized continuity to generalized continuity.

Proposition 4.4.

(1) $f : (X, g_X) \to (Y, g_Y)$ *is* (α, g_Y)*-continuous if and only if* $f : (X, \alpha(g_X)) \to (Y, g_Y)$ *is generalized continuous.*
(2) $f : (X, g_X) \to (Y, g_Y)$ *is* (σ, g_Y)*-continuous if and only if* $f : (X, \sigma(g_X)) \to (Y, g_Y)$ *is generalized continuous.*
(3) $f : (X, g_X) \to (Y, g_Y)$ *is* (π, g_Y)*-continuous if and only if* $f : (X, \pi(g_X)) \to (Y, g_Y)$ *is generalized continuous.*
(4) $f : (X, g_X) \to (Y, g_Y)$ *is* (β, g_Y)*-continuous if and only if* $f : (X, \beta(g_X)) \to (Y, g_Y)$ *is generalized continuous.*

The change of generalized topology approach now provides proofs of four results that are similar to Proposition 4.3. Shen [17, Definition 2.1] defined a generalized topological space (X, g_X) to be σ-connected (π-connected, β-connected) if $(X, \sigma(g_X))$ (respectively $(X, \pi(g_X)), (X, \beta(g_X))$) is connected.

Proposition 4.5.

(1) *Let* f *be* (α, g_Y)*-continuous from* (X, g_X) *onto* (Y, g_Y)*. If* (X, g_X) *is* α*-connected then* (Y, g_Y) *is connected.*
(2) *Let* f *be* (σ, g_Y)*-continuous from* (X, g_X) *onto* (Y, g_Y)*. If* (X, g_X) *is* σ*-connected then* (Y, g_Y) *is connected.*
(3) *Let* f *be* (π, g_Y)*-continuous from* (X, g_X) *onto* (Y, g_Y)*. If* (X, g_X) *is* π*-connected then* (Y, g_Y) *is connected.*
(4) *Let* f *be* (β, g_Y)*-continuous from* (X, g_X) *onto* (Y, g_Y)*. If* (X, g_X) *is* β*-connected then* (Y, g_Y) *is connected.*

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